CHAPTER TWO

Operators of Fractional Integration and A Generalized Hankel Transform

"Watch a man do his most common
actions. Those are indeed the
things which will tell you the
real character of a great man."

... VIVEKANAND

CHAPTER - II

OPERATORS OF FRACTIONAL INTEGRATION AND A GENERALIZED HANKEL TRANSFORM

2. In this Chapter, the Erdelyi-Kober operators of fractional integration are applied to develop the theory of the generalized Hankel transform

$$(2.1)\quad g(x) = 2\gamma\beta^{1/2\gamma}\int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma}\,\middle|\,\begin{matrix}(a_p+\frac{2\gamma-1}{4\gamma}\alpha_p,\alpha_p),\ (1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\ (b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q),\ (1-b_q-\frac{2\gamma+1}{4\gamma}\beta_q,\beta_q)\end{matrix}\right] f(y)dy,$$

which may be called as a generalization of the Hankel transform (1.41) in Tricomi's form, because on putting $\gamma=\frac{1}{2}$, $\beta = 1$, $p = o$, $q = 1$, $\beta_1 = 1$, $b_1 = \nu/2$, (2.1) yields

$$g(x) = \int_0^\infty J_\nu(2\sqrt{xy})f(y)dy.$$

2.1 Analytic functions - A necessary and sufficient condition

In this section we have determined the conditions under which a function f(x) is self-reciprocal in (2.1) and these will be used in the subsequent sections.

<u>THEOREM 1</u> A necessary and sufficient condition that a function f(x) of $A(\alpha,a)$ should be $R_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$ i.e. self-reciprocal in (2.1) is that it should be of the form

$$(2.2)\quad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \beta^{-s/2\gamma}\ \frac{\prod_{j=1}^{q}\Gamma\left(b_j+\frac{2\gamma-1}{4\gamma}\beta_j+\frac{s}{2\gamma}\beta_j\right)}{\prod_{j=1}^{p}\Gamma\left(1-a_j-\frac{2\gamma+1}{4\gamma}\alpha_j+\frac{s}{2\gamma}\alpha_j\right)}\ \mathrm{x}$$

$$\mathrm{x}\ \psi(s)\, x^{-s}\, ds$$

where $\psi(s)$ is regular and satisfies the condition

$$(2.3)\quad \psi(s) = \psi(1-s),\ s = \sigma + it,$$

in the strip

$$(2.4)\quad a < \sigma < 1 - a$$

and

$$\psi(s) = O\left(e^{\left[(Q-P)\pi/4\gamma - \alpha + \varepsilon\right]|t|}\right)$$

where

$$P = \sum_{j=1}^{p}\alpha_j,\quad Q = \sum_{j=1}^{q}\beta_j,$$

for every positive ε, and uniformly in any strip interior to (2.4) and c is any value of σ in (2.4).

<u>PROOF</u> Let us now investigate the form of the function f(x), which satisfies the integral equation

$$(2.5)\quad f(x) = 2\,\gamma\beta^{1/2\gamma} \int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2\,(xy)^{2\gamma}\left|\begin{matrix}(a_p + \frac{2\gamma-1}{4\gamma}\alpha_p,\alpha_p), & (1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\ (b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q), & (1-b_q-\frac{2\gamma+1}{4\gamma}\beta_q,\beta_q)\end{matrix}\right.\right] f(y)dy,$$

i.e. the function $f(x)$ is $R_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$. If $F(s)$ is the Mellin transform of $f(x)$, then

$$(2.6)\quad F(s) = \int_0^\infty x^{s-1} f(x)dx,\quad R(s) \geq s_0 > 0$$

$$(2.7) = 2\,\gamma\beta^{1/2\gamma} \int_0^\infty x^{s-1}\,dx \int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma}\left|\begin{matrix}(a_p+\frac{2\gamma-1}{4\gamma}\alpha_p,\alpha_p), & (1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\ (b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q), & (1-b_q-\frac{2\gamma+1}{4\gamma}\beta_q,\beta_q)\end{matrix}\right.\right] f(y)\,dy$$

$$= \beta^{1/\gamma\,(\frac{1}{2}-s)} \int_0^\infty y^{-s} f(y)dy \int_0^\infty x^{\frac{s}{2\gamma}-1} \times$$

$$H^{q,p}_{2p,2q}\left[x\left|\begin{matrix}(a_p+\frac{2\gamma-1}{4\gamma}\alpha_p,\alpha_p), & (1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\ (b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q), & (1-b_q-\frac{2\gamma+1}{4\gamma}\beta_q,\beta_q)\end{matrix}\right.\right] dx$$

On changing the order of integration and replacing $\beta^2(xy)^{2\gamma}$ by x.

Hence

$$(2.8)\quad F(s) = \beta^{1/\gamma[\frac{1}{2}-s)]}\ \frac{\prod_{j=1}^{q}\Gamma(b_j+\frac{2\gamma-1}{4\gamma}\beta_j+\frac{s}{2\gamma}\beta_j)}{\prod_{j=1}^{q}\Gamma(b_j+\frac{2\gamma+1}{4\gamma}\beta_j-\frac{s}{2\gamma}\beta_j)}\ \times$$

$$\times\ \frac{\prod_{j=1}^{p}\Gamma(1-a_j-\frac{2\gamma-1}{4\gamma}\alpha_j-\frac{s}{2\gamma}\alpha_j)}{\prod_{j=1}^{p}\Gamma(1-a_j-\frac{2\gamma+1}{4\gamma}\alpha_j+\frac{s}{2\gamma}\alpha_j)}\ F(1-s).$$

The inversion of the order of integration in (2.7) can easily be justified by de la Vallee Poussin's theorem [11, P. 504], if the integral, involved in (2.1), is absolutely convergent and Mellin transform of $|f(x)|$ exists.

If now we suppose that

$$(2.9)\quad F(s) = \beta^{-\frac{s}{2\gamma}}\ \frac{\prod_{j=1}^{q}\Gamma(b_j+\frac{2\gamma-1}{4\gamma}\beta_j+\frac{s}{2\gamma}\beta_j)}{\prod_{j=1}^{p}\Gamma(1-a_j-\frac{2\gamma+1}{4\gamma}\alpha_j+\frac{s}{2\gamma}\alpha_j)}\ \psi(s),$$

then we see that $\psi(s)$ satisfies the functional equation (2.3) and therefore, by applying Mellin's inversion formula [66, P.7] to (2.9), we obtain

$$(2.10)\quad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \beta^{-s/2\gamma}\, \frac{\prod_{j=1}^{q}\Gamma\left(b_j + \frac{2\gamma-1}{4\gamma}\beta_j + \frac{s}{2\gamma}\beta_j\right)}{\prod_{j=1}^{p}\Gamma\left(1-a_j - \frac{2\gamma+1}{4\gamma}\alpha_j + \frac{s}{2\gamma}\alpha_j\right)} \times$$

$$\times\ \psi(s)\, x^{-s} ds.$$

The rest of the proof follows as in the corresponding theorem of the Hankel transform [66, P. 252]

2.11 COROLLARIES

(i) With $\alpha_j = 1$ (j= 1,2, ... , p) and $\beta_j = 1$ (j = 1, 2, ...,q) the above theorem reduces to the result given by Sharma [61, p. 31] .

(ii) By putting $\beta = \frac{1}{2}$, $\gamma=1$, $\alpha_j = 1$ (j=1,2, ..., p), $\beta_j = 1$ (j = 1,2, ...,q) and replacing a_j by $(a_j - \frac{1}{4})$ (j = 1, 2, ..., p) and b_j by $(b_j - \frac{1}{4})$ (j = 1,2, ...,q), the above theorem reduces to a result of Sharma [62, P. 117].

(iii) When $\beta = \frac{1}{2}$, $\gamma = 1$, p = 1, q = 2, $a_1 = k - m - \frac{1}{2} - \nu/2$, $b_1 = \nu/2$, $b_2 = \nu/2 + 2m$, and $\alpha_1 = \beta_1 = \beta_2 = 1$, the theorem yields a known result given by R.Narain

[47, p. 283].

(iv) On having $\beta = \frac{1}{2}$, $\delta = 1$, $p = 0$, $q = 1$, $\beta_1 = 1$ and $b_1 = \nu/2$ we arrive at a known result [66, p. 252].

2.12 ILLUSTRATION

The Mellin transform of

$$(2.11) \quad H^{\lambda+q,\lambda}_{r+p,r+q}\left[x \left| \begin{matrix} (c_r + \frac{2\gamma-1}{4\gamma}\gamma r, \gamma r), (1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p) \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q), (dr + \frac{2\gamma-1}{4\gamma}\delta r, \delta r) \end{matrix} \right. \right]$$

is

$$\frac{\prod_{j=1}^{q}\Gamma(b_j + \frac{2\gamma-1}{4\gamma}\beta_j + s.\beta_j)\ \prod_{j=1}^{\lambda}\Gamma(d_j + \frac{2\gamma-1}{4\gamma}\delta_j + s.\delta_j)}{\prod_{j=1}^{p}\Gamma(1-a_j - \frac{2\gamma+1}{4\gamma}\alpha_j + s.\alpha_j)\ \prod_{j=\lambda+1}^{r}\Gamma(1-d_j - \frac{2\gamma-1}{4\gamma}\delta_j - s.\delta_j)} \times$$

$$\times \frac{\prod_{j=1}^{\lambda}\Gamma(1-c_j - \frac{2\gamma-1}{4\gamma}\gamma_j - s.\gamma_j)}{\prod_{j=\lambda+1}^{r}\Gamma(c_j + \frac{2\gamma-1}{4\gamma}\gamma_j + s\gamma_j)}, \quad 0 \leq 2\lambda \leq 2r < 4\lambda + q - p.$$

Hence, using Mellin's inversion formula and replacing x by $\beta x^{2\gamma}$ and sby $s/(2\gamma)$, we get

$$(2.12) \quad 2\gamma\, H^{\lambda+q,\lambda}_{r+p,r+q}\left[\beta x^{2\gamma} \left| \begin{matrix} (c_r + \frac{2\gamma-1}{4\gamma}\gamma_r, \gamma_r), \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q), \end{matrix} \right. \right.$$

$$\left.\begin{matrix}(1 - a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p)\\ (d_r + \frac{2\gamma-1}{4\gamma}\delta_r, \delta_r)\end{matrix}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \beta^{-s/2\gamma} \times$$

$$\times \frac{\prod_{j=1}^{q}\Gamma(\frac{2\gamma-1}{4\gamma}\beta_j + b_j + \frac{s}{2\gamma}\beta_j)}{\prod_{j=1}^{p}\Gamma(1-a_j - \frac{2\gamma+1}{4\gamma}\alpha_j + \frac{s}{2\gamma}\alpha_j)} \times$$

$$\times \frac{\prod_{j=1}^{\ell}\Gamma(d_j + \frac{2\gamma-1}{4\gamma}\delta_j + \frac{s}{2\gamma}\delta_j)\prod_{j=1}^{\ell}\Gamma(1-c_j - \frac{2\gamma-1}{4\gamma}\gamma_j - \frac{s}{2\gamma}\gamma_j)}{\prod_{j=\ell+1}^{r}\Gamma(1-d_j - \frac{2\gamma-1}{4\gamma}\delta_j - \frac{s}{2\gamma}\delta_j)\prod_{j=\ell+1}^{r}\Gamma(c_j + \frac{2\gamma-1}{4\gamma}\gamma_j + \frac{s}{2\gamma}\gamma_j)} x^{-s}ds$$

Thus in (2.12) the right hand side is of the same form as that of (2.2) with

$$\psi(s) = \frac{\prod_{j=1}^{\ell}\Gamma(d_j + \frac{2\gamma-1}{4\gamma}\delta_j + \frac{s}{2\gamma}\delta_j)\prod_{j=1}^{\ell}\Gamma(1-e_j - \frac{2\gamma-1}{4\gamma}\gamma_j - \frac{s}{2\gamma}\gamma_j)}{\prod_{j=\ell+1}^{r}\Gamma(1-d_j - \frac{2\gamma-1}{4\gamma}\delta_j - \frac{s}{2\gamma}\delta_j)\prod_{j=\ell+1}^{r}\Gamma(c_j + \frac{2\gamma-1}{4\gamma}\gamma_j + \frac{s}{2\gamma}\gamma_j)},$$

which satisfies the functional relation (2.3) if $\delta_j = \gamma_j$ $(j = 1, 2, \ldots, r)$ and $c_j + d_j + \gamma_j = 1$ $(j = 1, 2, \ldots, r)$.

Therefore we see that

$$(2.13) \quad 2\gamma\, H^{\lambda+q,\lambda}_{r+p,r+q}\left[\beta x^{2\gamma}\left|\begin{matrix}(c_r+\frac{2\gamma-1}{4\gamma}\gamma_r,\gamma_r),(1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\(b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q),(1-c_r-\frac{2\gamma+1}{4\gamma}\gamma_r,\gamma_r)\end{matrix}\right.\right]$$

is $R_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$, provided that

$$0 \leq \lambda \leq r \leq 2\lambda - \tfrac{1}{2}(p-q),$$

$$\frac{1-2\gamma}{2} - 2\gamma\left(\min_{1\leq j\leq q} R(b_j/\beta_j)\right) < R(s) < \frac{1-2\gamma}{2}$$

$$-2\gamma\left(\max_{1\leq j\leq \lambda} R\left(\frac{c_j-1}{\gamma_j}\right)\right),$$

and

$$\frac{1+2\gamma}{2} + 2\gamma\left(\min_{1\leq j\leq \lambda} R\left(\frac{c_j-1}{\gamma_j}\right)\right) < R(s) < \frac{1-2\gamma}{2}$$

$$-2\gamma\left(\max_{1\leq j\leq \lambda} R\left(\frac{c_j-1}{\gamma_j}\right)\right).$$

2.121 <u>PARTICULAR CASES</u>

Many known and unknown self-reciprocal functions can be derived as special cases of (2.13) under various generalizations of the Hankel transform. Here we mention some of the known cases :

(i) With $\alpha_j = 1 (j = 1, 2, \ldots, p)$, $\beta_j = 1\ (j = 1,2,\ldots,q)$, and $\gamma_j = 1\ (j = 1, 2, \ldots, r)$ we obtain a function [61, p. 34] , which is self-reciprocal in the transform

(1.47)

(ii) When $\gamma = 1$, $\beta = \frac{1}{2}$ and $\alpha_j = 1$ $(j = 1,2 \ldots, p)$, $\beta_j = 1$ $(j = 1,2, \ldots, q)$, $\gamma_j = 1$ $(j = 1,2,\ldots,r)$ and replacing c_j by $c_j - \frac{1}{4}$ $(j = 1, 2, \ldots, r)$, a_j by $a_j - \frac{1}{4}$ $(j = 1, , \ldots, p)$ and b_j by $b_j - \frac{1}{4}(j=1,2,\ldots,q)$ we obtain a function [62, p. 118], which is self-reciprocal in the transform (1.48).

(iii) If $\gamma = 1$, $\beta = \frac{1}{2}$, $p = 1$, $q = 2$, $r = 0$, $\alpha_1 = 1$, $\beta_1 = \beta_2 = 1$, $a_1 = k - m - \nu/2 \div \frac{1}{2}$, $b_1 = \nu/2$ and $b_2 = \nu/2 + 2m$, we have a known function [47, p.286], which is $R\ \nu, k, m$, i.e. self-reciprocal in (1.49).

2.2 <u>A theorem on the Generalized Hankel transform.</u>

We now establish a theorem, on the generalized Hankel transform, which transforms one $R_T\left[(a_p, \alpha_p); (b_q,\beta_q)\right]$ in to another $R_T\left[(a_p, \alpha_p); (b_q, \beta_q)\right]$.

<u>THEOREM 2</u> If $I^+_{\eta,\alpha}$ and $K^-_{\eta,\alpha}$ belong to L_2 and $(I^+_{\eta,\alpha})^m \times (K^-_{\eta,\alpha})^n$ stands for the operator to perform m and n times the operations of the operators $I^+_{\eta,\alpha}$ and $K^-_{\eta,\alpha}$ respectively in any order, then the operator $\left[(I^+_{\eta,\alpha})^m(K^-_{\eta,\alpha})^n + (I^+_{\eta,\alpha})^n(K^-_{\eta,\alpha})^m\right]$ transforms an $R_T\left[(a_p,\alpha_p);(b_q,\beta_q)\right]$ function in to another

$R_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$ function. Similar property holds good for the operator

$$\left[(I^-_{\eta,\alpha})^m\ (K^+_{\eta,\alpha})^n + (I^-_{\eta,\alpha})^n\ (K^+_{n,\alpha})^m\right].$$

PROOF Let $f(x) \in L_2$ be $R_T\left[(a_p,\ \alpha_p);\ \ (b_q,\ \beta_q)\right]$, then from Theorem 1, we get

$$f(x) = \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} p(\tfrac{1}{2} + it)\ x^{-\frac{1}{2} - it}\, dt,$$

where

$$p(\tfrac{1}{2} + it) = \frac{1}{2\pi}\ \beta^{\frac{-\frac{1}{2} + it}{2\gamma}}\ \frac{\prod_{j=1}^{q} \Gamma(b_j + \frac{2\gamma-1}{4\gamma}\beta_j + \frac{\frac{1}{2}+it}{2\gamma}\beta_j)}{\prod_{j=1}^{p} \Gamma(1-a_j - \frac{2\gamma+1}{4\gamma}\alpha_j + \frac{\frac{1}{2}+it}{2\gamma}\alpha_j)} \times$$

$$\times\ \Psi(\tfrac{1}{2} + it)$$

with

$$\Psi(\tfrac{1}{2} + it) = \Psi(\tfrac{1}{2} - it)$$

and

$$\Psi(\tfrac{1}{2} + it) = O\left(e^{\left[(Q - P)\pi/4\gamma - \alpha + \epsilon\right].\ |t|}\right)$$

where $Q = \sum_{j=1}^{q} \beta_j,\quad P = \sum_{j=1}^{p} \alpha_j$, Therefore,

$$(2.14)\qquad I^{+}_{\eta,\alpha}\, f(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1}\, u^{\eta} du \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} p(\tfrac{1}{2}+it)\, u^{-\frac{1}{2}-it}\, dt$$

$$= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} p(\tfrac{1}{2}+it) dt \int_0^x (x-u)^{\alpha-1}\, u^{\eta-\frac{1}{2}-it}\, du,$$

On changing the order of integration. Now evaluating the u-integral with the help of the known result [4,p.185(7)], we obtain

$$I^{+}_{\eta,\alpha}\, f(x) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} p(\tfrac{1}{2}+it)\, \frac{\Gamma(\eta+\frac{1}{2}-it)}{\Gamma(\eta+\alpha+\frac{1}{2}-it)}\, x^{-\frac{1}{2}-it}\, dt,$$

$$R(\eta) > -\tfrac{1}{2},\quad R(\alpha) > 0.$$

The change in the order of integration in (2.14) is valid due to absolute convergence of both the u - and t - integrals.

Repeating the above operation m times, we see that

$$(2.15)\qquad (I^{+}_{\eta,\alpha})^{m} f(x) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} p(\tfrac{1}{2}+it) \left[\frac{\Gamma(\eta+\frac{1}{2}-it)}{\Gamma(\eta+\alpha+\frac{1}{2}-it)}\right]^{m} \times$$

$$\times\; x^{-\frac{1}{2}-it}\, dt.$$

Similarly

$$(2.16)\quad K^{-}_{\eta,\alpha} f(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_{x}^{\infty} (u-x)^{\alpha-1} u^{-\eta-\alpha}\, du \times$$

$$\times \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} p(\tfrac{1}{2}+it)\, u^{-\frac{1}{2}-it}\, dt$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} p(\tfrac{1}{2}+it)\, x^{-\frac{1}{2}-it} \frac{\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}+it)}\, dt,$$

On changing the order of integration and then evaluating the u-integral with the help of known result {4, p. 201(6)}.

The inversion of the order of integration in (2.16) is justified because of the absolute convergence of both the u - and t - integrals.

By repeating this type of operation of $K^{-}_{\eta,\alpha}$, as shown in (2.16), n-times over (2.15), we can easily obtain

$$(I^{+}_{\eta,\alpha})^{m} (K^{-}_{\eta,\alpha})^{n} f(x) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} p(\tfrac{1}{2}+it) x^{-\frac{1}{2}-it}\, dt \times$$

$$\times \frac{\left[\Gamma(\eta+\frac{1}{2}-it)\right]^{m} \left[\Gamma(\eta+\frac{1}{2}+it)\right]^{n}}{\left[\Gamma(\eta+\alpha+\frac{1}{2}-it)\right]^{m} \left[\Gamma(\eta+\alpha+\frac{1}{2}+it)\right]^{n}}\, dt,$$

whenever $R(\alpha) > 0$, $R(\eta) > -\frac{1}{2}$.

Hence, obviously

$$\left[(I^{+}_{\eta,\alpha})^{m}\,(K^{-}_{\eta,\alpha})^{n} + (I^{+}_{\eta,\alpha})^{n}(K^{-}_{\eta,\alpha})^{m}\right] f(x)$$

$$= \frac{1}{2\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \beta^{-\frac{\frac{1}{2}+it}{2\gamma}}\; \frac{\prod_{j=1}^{q}\Gamma\left(b_j + \frac{2\gamma-1}{4\gamma}\beta_j + \frac{\frac{1}{2}+it}{2\gamma}\beta_j\right)}{\prod_{j=1}^{p}\Gamma\left(1-a_j - \frac{2\gamma+1}{4\gamma}\alpha_j + \frac{\frac{1}{2}+it}{2\gamma}\alpha_j\right)} \times$$

$$\times\ \psi_1(\tfrac{1}{2}+it) \times x^{-\frac{1}{2}-it}\,dt,$$

where

$$\psi_1(\tfrac{1}{2}+it) = \left\{ \frac{\left[\Gamma(\eta+\frac{1}{2}-it)\right]^{m}\left[\Gamma(\eta+\frac{1}{2}+it)\right]^{n}}{\left[\Gamma(\eta+\alpha+\frac{1}{2}-it)\right]^{m}\left[\Gamma(\eta+\alpha+\frac{1}{2}+it)\right]^{n}} \right.$$

$$\left. + \frac{\left[\Gamma(\eta+\frac{1}{2}-it)\right]^{n}\left[\Gamma(\eta+\frac{1}{2}+it)\right]^{m}}{\left[\Gamma(\eta+\alpha+\frac{1}{2}-it)\right]^{n}\left[\Gamma(\eta+\alpha+\frac{1}{2}+it)\right]^{m}} \right\}\psi(\tfrac{1}{2}+it).$$

Since $\psi_1(\frac{1}{2}+it)$ satisfies the functional relation

$$\psi_1(\tfrac{1}{2}+it) = \psi_1(\tfrac{1}{2}-it),$$

the theorem easily follows in view of Theorem 1. The latter part of the theorem can be proved by proceeding on similar lines.

2.21 COROLLARIES

(i) With $\gamma = \frac{1}{2}$, $\beta = 1$, and $\alpha_j = 1$ $(j = 1,2, \ldots,p)$, $\beta_j = 1$ $(j = 1,2, \ldots,q)$ the above theorem reduces to that of Sharma [61, p. 198] .

(ii) When $m = n$, we have that :

If $f(x)$ is $R_T\left[(a_p, \alpha_p); (b_q,\beta_q)\right]$ then

$(I^{+}_{\eta,\alpha})^m (K^{-}_{\eta,\alpha})^m f(x)$ is $R_T\left[(a_p, \alpha_p);(b_q,\beta_q)\right]$,

provided that $R(\alpha) > 0$, $R(\eta) > -\frac{1}{2}$.

(iii) If $p = 1$, $q = 2$, $\alpha_1 = 1$, $\beta_1 = \beta_2 = 1$, and $a_1 = k - m - \frac{1}{2} - \nu/2$, $b_1 = \nu/2$, $b_2 = \nu/2 + 2m$, the above theorem yields a result given by Bhise [7, p. 202].

(iv) By putting $p = 0$, $q = 2$, $\beta_1 = \beta_2 = 1$, and $b_1 = \nu/2$, $b_2 = \mu/2$, we get a result due to Srivastava [65, p. 53, 60] .

2.22 ILLUSTRATION

Now, we illustrate the use of the above theorem with $m = n = 1$ in transforming one $R_T\left[(a_p, \alpha_p); (b_q,\beta_q)\right]$ - function in to another $R_T\left[(a_p,\alpha_p); (b_q,\beta_q)\right]$ - function.

Consider

$$2\gamma\, H^{l+q,k}_{p+m,q+n}\left[\beta x^{2\gamma}\left|\begin{matrix}(c_m+\frac{2\gamma-1}{4\gamma}\gamma_m,\ \gamma_m),(1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\(b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q),(d_n+\frac{2\gamma-1}{4\gamma}\delta_n,\delta_n)\end{matrix}\right.\right]$$

Then, by virtue of the definition of Fox's H-function we have

$$2\gamma\, H^{l+q,k}_{p+m,q+n}\left[\beta x^{2\gamma}\left|\begin{matrix}(c_m+\frac{2\gamma-1}{4\gamma}\gamma_m,\gamma_m),(1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\(b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q),\ (dn+\frac{2\gamma-1}{4\gamma}\delta_n,\ \delta_n)\end{matrix}\right.\right]$$

$$=\frac{2\gamma}{2\pi i}\int_{c-i\infty}^{c+i\infty}\beta^{-s}\ \frac{\prod_{j=1}^{q}\Gamma(b_j+\frac{2\gamma-1}{4\gamma}\beta_j+s\beta_j)}{\prod_{j=1}^{p}\Gamma(1-a_j-\frac{2\gamma+1}{4\gamma}\alpha_j+s.\alpha_j)}\ x^{-2\gamma s}\ x$$

$$x\ \frac{\prod_{j=1}^{l}\Gamma(d_j+\frac{2\gamma-1}{4\gamma}\delta_j+s.\delta_j)\ \prod_{j=1}^{k}\Gamma(1-e_j-\frac{2\gamma-1}{4\gamma}\gamma_j-s.\gamma_j)}{\prod_{j=l+1}^{n}\Gamma(1-d_j-\frac{2\gamma-1}{4\gamma}\delta_j-s.\delta_j)\ \prod_{j=k+1}^{m}\Gamma(c_j+\frac{2\gamma-1}{4\gamma}\gamma_j+s\gamma_j)}\ ds$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\beta^{-s/2\gamma}\ \frac{\prod_{j=1}^{q}\Gamma(b_j+\frac{2\gamma-1}{4\gamma}\beta_j+\frac{s}{2\gamma}\beta_j)}{\prod_{j=1}^{p}\Gamma(1-a_j-\frac{2\gamma+1}{4\gamma}\alpha_j+\frac{s}{2\gamma}\alpha_j)}\ x^{-s}\ x$$

$$\times \frac{\prod_{j=1}^{\lambda}\Gamma\left(dj + \frac{2\gamma-1}{4\gamma}\delta_j + \frac{s}{2\gamma}\delta_j\right)\prod_{j=1}^{k}\Gamma\left(1-c_j - \frac{2\gamma-1}{4\gamma}\gamma_j - \frac{s}{2\gamma}\gamma_j\right)}{\prod_{j=\lambda+1}^{n}\Gamma\left(1-d_j - \frac{2\gamma-1}{4\gamma}\delta_j - \frac{s}{2\gamma}\delta_j\right)\prod_{j=k+1}^{m}\Gamma\left(c_j + \frac{2\gamma-1}{4\gamma}\gamma_j + \frac{s}{2\gamma}\gamma_j\right)} ds$$

On replacing s by $s/2\gamma$.

Hence, we have

$$2\gamma H^{\lambda+q,k}_{p+m,q+n}\left[\beta x^{2\gamma}\left|\begin{matrix}(c_m + \frac{2\gamma-1}{4\gamma}\gamma_m, \gamma_m),\ (1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p\\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q),\ (dn + \frac{2\gamma-1}{4\gamma}\delta_n, \delta_n)\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\beta^{-\frac{s}{2\gamma}}\frac{\prod_{j=1}^{q}\Gamma\left(b_j + \frac{2\gamma-1}{4\gamma}\beta_j + \frac{s}{2\gamma}\beta_j\right)}{\prod_{j=1}^{p}\Gamma\left(1-a_j - \frac{2\gamma+1}{4\gamma}\alpha_j + \frac{s}{2\gamma}\alpha_j\right)}\psi(s)x^{-s}ds,$$

where

$$\psi(s) = \frac{\prod_{j=1}^{l}\Gamma\left(d_j + \frac{2\gamma-1}{4\gamma}\delta_j + \frac{s}{2\gamma}\delta_j\right)\prod_{j=1}^{k}\Gamma\left(j-c_j - \frac{2\gamma+1}{4\gamma}\gamma_j - \frac{s}{2\gamma}\gamma_j\right)}{\prod_{j=\lambda+1}^{n}\Gamma\left(1-d_j - \frac{2\gamma-1}{4\gamma}\delta_j - \frac{s}{2\gamma}\delta_j\right)\prod_{j=k+1}^{m}\Gamma\left(c_j + \frac{2\gamma-1}{4\gamma}\gamma_j + \frac{s}{2\gamma}\gamma_j\right)}$$

which satisfies the functional relation (2.3), if $k = \lambda$. $m = n$, $\delta_j = \gamma_j$ $(j = 1,2, \ldots, m)$, and $c_j + d_j + \gamma_j = 1 (j=1,2,\ldots, m)$.

Therefore, by virtue of Theorem 1, we see that the function

$$f(x) = 2\gamma\, H^{\lambda+q,\lambda}_{p+m,q+m}\left[\beta x^{2\gamma}\,\middle|\,\begin{matrix}(c_m + \frac{2\gamma-1}{4\gamma}\gamma_m, \gamma_m), \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q),\end{matrix}\right.$$

$$\left.\begin{matrix}(1 - a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p) \\ (1 - c_m - \frac{2\gamma+1}{4\gamma}\gamma_m, \gamma_m)\end{matrix}\right]$$

is $R_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$, provided that $p - q < 2\ (2\lambda - m)$.

Now

$$I^{+}_{\eta,\alpha}\ f(x)$$

$$= \frac{1}{\Gamma(\alpha)}\ z^{-\eta-\alpha} \int_0^z (z-t)^{\alpha-1}\ t^{\eta}\ f(t)dt$$

$$= \frac{2\gamma}{\Gamma(\alpha)}\ z^{-\eta-\alpha} \int_0^z\ (z-t)^{\alpha-1}\ t^{\eta}\ H^{\lambda+q,\lambda}_{p+m,q+m}\left[\beta t^{2\gamma}\,\middle|\,\begin{matrix}(e_m + \frac{2\gamma-1}{4\gamma}\gamma_m, \gamma_m), \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q),\end{matrix}\right.$$

$$\left.\begin{matrix}(1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p) \\ (1 - c_m - \frac{2\gamma+1}{4\gamma}\gamma_m, \gamma_m)\end{matrix}\right]\ dt$$

$$= \frac{2\gamma}{\Gamma(\alpha)}\ z^{-\eta-\alpha}\ \beta^{-\eta/2\gamma} \int_0^z (z-t)^{\alpha-1} \times$$

$$\times\, H^{\lambda+q,\lambda}_{p+m,q+m}\left[\beta t^{2\gamma}\ \middle|\ \begin{matrix}(c_m + \frac{2\gamma-1}{4\gamma}\gamma_m + \frac{\eta}{2\gamma}\gamma_m, \gamma_m), \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q + \frac{\eta}{2\gamma}\beta_q, \beta_q),\end{matrix}\right.$$

$$\left.\begin{matrix}(1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p + \frac{\eta}{2\gamma}\alpha_p, \alpha_p) \\ (1-c_m - \frac{2\gamma+1}{4\gamma}\gamma_m + \frac{\eta}{2\gamma}\gamma_m, \gamma_m)\end{matrix}\right] dt,$$

on using the identity (1.45).

Evaluating this integral with the help of (1.75) and then using the identity (1.45), we obtain

(2.17) $I^{+}_{\eta,\alpha} f(x)$

$$= 2\gamma\, H^{\lambda+q,\lambda+1}_{p+m+1,\, q+m+1}\left[\beta z^{2\gamma}\middle|\begin{matrix}(-\eta,2\gamma),\ (c_m + \frac{2\gamma-1}{4\gamma}\gamma_m, \gamma_m), \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q), (1-c_m - \frac{2\gamma+1}{4\gamma}\gamma_m, \gamma_m),\end{matrix}\right.$$

$$\left.\begin{matrix}(1 - a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p) \\ (-\eta-\alpha,\ 2\gamma\ )\end{matrix}\right]$$

Now using the operator, $K^{-}_{\eta,\alpha}$ on (2.17), we get

$(I^{+}_{\eta,\alpha})\ (K^{-}_{\eta,\alpha})\ f(x)$

$$= \frac{2\gamma}{\Gamma(\alpha)}\, y^{\eta} \int_{y}^{\infty} (t-y)^{\alpha-1}\, t^{-\eta-\alpha}\, H^{\lambda+q,\lambda+1}_{p+m+1,q+m+1}\left[\beta t^{2\gamma}\middle|\begin{matrix}(-\eta,2\gamma), \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q),\end{matrix}\right.$$

$$\left.\begin{matrix} (c_m + \frac{2\gamma-1}{4\gamma}\gamma_m, \gamma_m),\ (1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p,\ \alpha_p) \\ (1-c_m - \frac{2\gamma+1}{4\gamma}\gamma_m,\ \gamma_m),\ (-\eta-\alpha,\ 2\gamma) \end{matrix}\right] dt$$

$$= \frac{2\gamma}{\Gamma(\alpha)} y^{\eta} \beta^{\frac{\eta+\alpha}{2\gamma}} \int_y^{\infty} (t-y)^{\alpha-1} H^{\ell+q,\ \ell+1}_{p+m+1,\ q+m+1}\left[\beta t^{2\gamma}\right|$$

$$(-2\eta-\alpha, 2\gamma),\ (c_m + \frac{2\gamma-1}{4\gamma}\gamma_m - \frac{\eta+\alpha}{2\gamma}\gamma_m,\ \gamma_m),$$

$$(b_q + \frac{2\gamma-1}{4\gamma}\beta_q - \frac{\eta+\alpha}{2\gamma}\beta_q, \beta_q),(1-c_m - \frac{2\gamma+1}{4\gamma}\gamma_m - \frac{\eta+\alpha}{2\gamma}\gamma_m, \gamma_m),$$

$$\left.\begin{matrix} (1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p - \frac{\eta+\alpha}{2\gamma}\alpha_p,\ \alpha_p) \\ (-2\eta-2\alpha,\ 2\gamma) \end{matrix}\right] dt$$

On using the identity (1.45).

Evaluating the integral with the help of (1.76) and then using the identity (1.45) we obtain

$$(I^{+}_{\eta,\alpha})(K^{-}_{\eta,\alpha})f(x)$$

$$= 2\gamma\ H^{\ell+q+1,\ \ell+1}_{p+m+2,\ q+m+2}\left[\beta y^{2\gamma} \left| \begin{matrix} (-\eta,2\gamma),(c_m + \frac{2\gamma-1}{4\gamma}\gamma_m, \gamma_m), \\ (\eta,2\gamma),\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q,\ \beta_q), \end{matrix}\right.\right.$$

$$\left.\begin{array}{l}(1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p,\ \alpha_p),\ (\eta+\alpha,\ 2\gamma) \\ \\ (1-c_m - \frac{2\gamma+1}{4\gamma}\gamma_m, \gamma_m),\ (-\eta-\alpha,\ 2\gamma)\end{array}\right] ,$$

which is $R_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$ by virtue of Theorem 2, provided that

$p - q < 2(2\lambda - m),\ R(\alpha) > 0,$,

$$R(b_j/\beta_j + \frac{2\eta+2\gamma-1}{4\gamma}) > -1 \quad (j = 1,2,\ldots,q),$$

$$R\left(\frac{1-c_j}{\gamma_j} + \frac{2\eta-2\gamma-1}{4\gamma}\right) > -1 \quad (j = 1,2,\ \ldots,\lambda),$$

$$R(\eta) > -\tfrac{1}{2} + (\gamma-\tfrac{1}{2})\,R(\alpha),$$

$$R\left(\frac{1-c_j}{\gamma_j} + \frac{2\eta-2\gamma+1}{4\gamma}\right) > \left(1 - \frac{1}{2\gamma}\right)R(\alpha).$$

2.3 Rules for connecting different classes of self-reciprocal functions

In this section, we shall consider the operation of changing the pair of functions f and its $H_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$- transform to another pair of functions, one being $H_T\left[(c_p,\ \gamma_p);\ (d_q,\ \delta_q)\right]$- transform of the other.

2.31 <u>THEOREM 3</u> IF

(2.18) $$F = \left\{ H_T \left[(a_p,\alpha_p);\ (b_q,\beta_q) \right] \right\} f \text{ that is}$$

$$F(x) = 2\gamma\beta^{1/2\gamma} \int_0^\infty H^{q,p}_{2p,2q} \left[\beta^2 (xy)^{2\gamma} \middle| \begin{matrix} (a_p + \frac{2\gamma-1}{4\gamma}\alpha_p, \alpha_p), \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q), \end{matrix} \begin{matrix} (1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p) \\ (1-b_q - \frac{2\gamma+1}{4\gamma}\beta_q, \beta_q) \end{matrix} \right] f(y)dy,$$

then

(2.19) $$\int_0^\infty K(xy)F(y)dy = \left\{ H_T \left[(c_p,\gamma_p);(d_q,\delta_q) \right] \right\} \times \left\{ \int_0^\infty K(xy)f(y)dy \right\},$$

Provided that

(2.20) $$\beta^{\frac{it}{\gamma}} \frac{\prod_{j=1}^{q} \Gamma(d_j + \tfrac{1}{2}\delta_j - i.\frac{t}{2\gamma}\delta_j) \prod_{j=1}^{p} \Gamma(1-c_j - \tfrac{1}{2}\gamma_j + i\frac{t}{2\gamma}\gamma_j)}{\prod_{j=1}^{q} \Gamma(b_j + \tfrac{1}{2}\beta_j + i.\frac{t}{2\gamma}\beta_j) \prod_{j=1}^{p} \Gamma(1-a_j - \tfrac{1}{2}\alpha_j - i.\frac{t}{2\gamma}\alpha_j)} M_t(K)$$

$$= \beta^{\frac{-it}{\gamma}} \frac{\prod_{j=1}^{q}\Gamma(d_j+\tfrac{1}{2}\delta_j+i\,\frac{t}{2\gamma}\,\delta_j)\ \prod_{j=1}^{p}\Gamma(1-c_j-\tfrac{1}{2}\gamma_j-i.\,\frac{t}{2\gamma}\,\gamma_j)}{\prod_{j=1}^{q}\Gamma(b_j+\tfrac{1}{2}\beta_j-i.\frac{t}{2\gamma}\beta_j)\ \prod_{j=1}^{p}\Gamma(1-a_j-\tfrac{1}{2}\alpha_j+i.\frac{t}{2\gamma}\,\alpha_j)} M_{-t}(k),$$

the integrals on both sides of (2.19) are absolutely convergent; Mellin transforms of $|L.H.S.|$, $|R.H.S.|$ and $\left|H_T\left[(c_p, \gamma_p);\ (d_q, \delta_q)\right]\right.$ - transform of $\left.K(xy)\right|$ exist; $H_T\left[(c_p,\gamma_p);\ (d_q, \delta_q)\right]$ transform of $k(xy)$ exists; and

$$\min_{1\le j\le q} R\,(b_j/\beta_j) > -\tfrac{1}{2}\,, \qquad \min_{1\le j\le q} R(d_j/\delta_j) > -\tfrac{1}{2}\,,$$

$$\max_{1\le j\le p} R\left(\frac{a_j-1}{\alpha_j}\right) < -\tfrac{1}{2}\,, \qquad \max_{1\le j\le p} R\left(\frac{c_j-1}{\gamma_j}\right) < -\tfrac{1}{2}\,.$$

PROOF Taking Mellin transform of both the sides of (2.19), we get

$$(2.21)\quad \int_0^{\infty} x^{-\frac{1}{2}+it}\,dx \int_0^{\infty} K(xy)F(y)dy = 2\gamma\beta^{1/2\gamma} \int_0^{\infty} x^{-\frac{1}{2}+it}dx\ x$$

$$x \int_0^{\infty} H^{q,p}_{2p,2q}\left[\beta^2 (xy)^{2\gamma}\left|\begin{matrix}(c_p+\frac{2\gamma-1}{4\gamma}\,\gamma_p,\gamma_p),(1-c_p-\frac{2\gamma+1}{4\gamma}\,\gamma_p,\gamma_p)\\ (d_q+\frac{2\gamma-1}{4\gamma}\delta_q,\delta_q),(1-d_q-\frac{2\gamma+1}{4\gamma}\delta_q,\delta_q)\end{matrix}\right.\right] dy\ x$$

$$x \int_0^{\infty} K(zy)\ f(z)\ dz.$$

On L.H.S. we replace xy by x after changing the order of integration, which is valid, provided that the integration on L.H.S. of (2.19) is absolutely convergent and Mellin transform of $\left| \int_0^\infty K(xy)F(y)dy \right|$ exists, Hence

$$\text{L.H.S.} = \int_0^\infty F(y)y^{-\frac{1}{2}-it}\,dy \int_0^\infty K(x)x^{-\frac{1}{2}+it}\,dx$$

$$(2.22) = M_{-t}(F)M_t(K).$$

Now

$$M_{-t}(F)$$

$$= \int_0^\infty y^{-\frac{1}{2}-it} F(y)dy$$

$$= 2\gamma\beta^{1/2\gamma} \int_0^\infty y^{-\frac{1}{2}-it}\,dy \int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma}\left|\begin{matrix}(a_p+\frac{2\gamma-1}{4\gamma}\alpha_p,\alpha_p), & (1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\ (b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q), & (1-b_q-\frac{2\gamma+1}{4\gamma}\beta_q,\beta_q)\end{matrix}\right.\right] f(x)dx$$

$$= \beta^{1/2\gamma} \int_0^\infty x^{-\frac{1}{2}+it} f(x)dx \int_0^\infty v^{1/4-\frac{it}{2\gamma}-1} \times$$

$$\times H^{q,p}_{2p,2q}\left[\beta^2 v \left| \begin{matrix} (a_p + \frac{2\gamma-1}{4\gamma}\alpha_p, \alpha_p), (1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p) \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q), (1-b_q - \frac{2\gamma+1}{4\gamma}\beta_q, \beta_q) \end{matrix} \right.\right] dv,$$

On changing the order of integration and replacing $(xy)^{2\gamma}$ by v.

Now, evaluating the v-integral, by using (1.80), we get

$$M_{-t}(F) = \beta^{it/\gamma} \frac{\prod_{j=1}^{q}\Gamma(b_j + \frac{1}{2}\beta_j - i\,\frac{t}{2\gamma}\beta_j)\prod_{j=1}^{p}\Gamma(1-a_j - \frac{1}{2}\alpha_j + i\,\frac{t}{2\gamma}\alpha_j)}{\prod_{j=1}^{q}\Gamma(b_j + \frac{1}{2}\beta_j + i.\frac{t}{2\gamma}\beta_j)\prod_{j=1}^{p}\Gamma(1-a_j - \frac{1}{2}\alpha_j - i.\frac{t}{2\gamma}\alpha_j)} \times$$

$$\times M_t(f).$$

Therefore, putting it in (2.22), we get

$$(2.23)\quad \text{L.H.S.} = \beta^{it/\gamma} \frac{\prod_{j=1}^{q}\Gamma(b_j + \frac{1}{2}\beta_j - i.\frac{t}{2\gamma}\beta_j)}{\prod_{j=1}^{q}\Gamma(b_j + \frac{1}{2}\beta_j + i.\frac{t}{2\gamma}\beta_j)} \times$$

$$\times \frac{\prod_{j=1}^{p}\Gamma(1-a_j - \frac{1}{2}\alpha_j + i.\frac{t}{2\gamma}\alpha_j)}{\prod_{j=1}^{p}\Gamma(1-a_j - \frac{1}{2}\alpha_j - i.\frac{t}{2\gamma}\alpha_j)} M_t(K)M_t(f)$$

Also from (2.21), we have

$$\text{R.H.S.} = 2\gamma\beta^{1/2\gamma} \int_0^\infty x^{-\frac{1}{2}+it}\,dx \int_0^\infty f(z)\,dz \int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma} \,\middle|\, \begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p,\ \gamma_p),\ (1-c_p-\frac{2\gamma+1}{4\gamma}\gamma_p,\ \gamma_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q,\delta_q),\ (1-d_q-\frac{2\gamma+1}{4\gamma}\delta_q,\delta_q) \end{matrix}\right] k(yz)\,dy$$

$$= 2\gamma\beta^{1/2\gamma} \int_0^\infty f(z)\,dz \int_0^\infty x^{-\frac{1}{2}+it}\,dx \int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma} \,\middle|\, \begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p,\gamma_p),\ (1-c_p-\frac{2\gamma+1}{4\gamma}\gamma_p,\ \gamma_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q,\delta_q),\ (1-d_q-\frac{2\gamma+1}{4\gamma}\delta_q,\delta_q) \end{matrix}\right] k(yz)\,dy$$

$$= 2\gamma\beta^{1/2\gamma} \int_0^\infty f(z)\,dz \int_0^\infty k(zy)\,dy \int_0^\infty x^{-\frac{1}{2}+it} H^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma} \,\middle|\, \begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p,\gamma_p),\ (1-c_p-\frac{2\gamma+1}{4\gamma}\gamma_p,\gamma_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q,\delta_q),\ (1-d_q-\frac{2\gamma+1}{4\gamma}\delta_q,\delta_q) \end{matrix}\right] dx.$$

$$= \beta^{1/2\gamma} \int_0^{\infty} f(z)dz \int_0^{\infty} y^{-\frac{1}{2}-it}\, k(zy)dy \int_0^{\infty} v^{1/4\gamma + i.\frac{t}{2\gamma} - 1} \; x$$

$$X\, H^{q,p}_{2p,2q}\left[\beta^2_v \left| \begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p, \gamma_p),\; (1-c_p - \frac{2\gamma+1}{4\gamma}\gamma_p, \gamma_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q, \delta_q),\; (1-d_q - \frac{2\gamma+1}{4\gamma}\delta_q, \delta_q) \end{matrix} \right.\right] dv$$

On changing the order of integration and replacing $(xy)^{2\gamma}$ by v.

Now, evaluating the v - integral, by using (1.80) and putting $yz = u$, we obtain

$$(2.24)\quad R.H.S. = \beta^{-it/\gamma} \frac{\prod_{j=1}^{q} \Gamma(d_j + \frac{1}{2}\delta_j + i.\frac{t}{2\gamma}\delta_j)}{\prod_{j=1}^{q} \Gamma(d_j + \frac{1}{2}\delta_j - i.\frac{t}{2\gamma}\delta_j)}$$

$$\frac{\prod_{j=1}^{p} \Gamma(1-c_j - \frac{1}{2}\gamma_j - i.\frac{t}{2\gamma}\gamma_j)}{\prod_{j=1}^{p} \Gamma(1-c_j - \frac{1}{2}\gamma_j + i.\frac{t}{2\gamma}\gamma_j)} \; X\, M_{-t}(k).\, M_t(f).$$

The changes in the order of integration are permissible only if $\int_0^{\infty} k(zy)f(z)dz$ is absolutely convergent; $H_T\left[(c_p, \gamma_p);\right.$ $\left.(d_q, \delta_q)\right]$ - transform of $k(zy)$ and $\left| \int_0^{\infty} k(zy)f(z)dz \right|$

exist ; Mellin transform of $\left| H_T \left[(c_p, \gamma_p); (d_q, \delta_q) \right] - \right.$ – transform of $\left. k(zy) \right|$ exists;

$$\min_{1 \leq j \leq q} R\left(\frac{d_j}{\delta_j}\right) > -\tfrac{1}{2}, \text{ and } \max_{1 \leq j \leq p} R\left(\frac{c_j - 1}{\gamma_j}\right) < -\tfrac{1}{2}.$$

Hence, equating the value of L.H.S. and R.H.S. from (2.23) and (2.24), the relation (2.20) can easily be obtained, which completes the proof.

2.311 CORO LLARIES

(i) Putting $\gamma = \frac{1}{2}$, $\beta = 1$ and $\alpha_j = \gamma_j = 1$ $(j = 1, 2, \ldots, p)$, $\beta_j = \delta_j = 1$ $(j = 1, 2, \ldots, q)$ the above theorem reduces to that of Sharma [61, p. 205].

(ii) When $p = 1$, $q = 2$, $\alpha_1 = \gamma_1$, $\beta_1 = \beta_2 = 1$, $\delta_1 = \delta_2 = 1$, $\gamma = \frac{1}{2}$, $\beta = 1$, and $a_1 = k - m - \frac{1}{2} - \nu/2$, $b_1 = \nu/2$, $b_2 = \nu/2 + 2m$, $c_1 = k' - m' - \nu'/2 - \frac{1}{2}$, $d_1 = \nu'/2$, $d_2 = \nu'/2 + 2m'$, the above theorem reduces to a result, due to Bhise [7, p. 204].

(iii) With $p = 0$, $q = 2$, $\gamma = \frac{1}{2}$, $\beta = 1$, $\beta_1 = \beta_2 = 1$, $\delta_1 = \delta_2 = 1$, and $b_1 = \nu$, $b_2 = \mu$, $d_1 = \xi$, $d_2 = \eta$, we get a known result, given by

Srivastava [65, p. 62].

2.32 THEOREM 4 If

$$F = \left\{ H_T \left[(a_p, \alpha_p);\ (b_q, \beta_q) \right] \right\} f$$

that is

$$F(x) = 2\gamma\beta^{1/2\gamma} \int_0^{\infty} H^{q,p}_{2p,2q} \left[\beta^2 (xy)^{2\gamma} \left| \begin{matrix} (a_p + \frac{2\gamma-1}{4\gamma}\alpha_p, \alpha_p), & (1 - a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p) \\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q), & (1 - b_q - \frac{2\gamma+1}{4\gamma}\beta_q, \beta_q) \end{matrix} \right. \right] f(y)dy,$$

then

$$(2.25) \quad \int_0^{\infty} (1/y)\ K(x/y)\ F(y)dy = \left\{ H_T \left[(c_p, \gamma_p); (d_q, \delta_q) \right] \right\} \times$$

$$\times \left\{ \int_0^{\infty} (1/y) K(x/y) f(y) dy \right\},$$

provided that

$$\frac{\prod_{j=1}^{q} \Gamma(d_j + \tfrac{1}{2}\delta_j - i.\frac{t}{2\gamma}\delta_j) \prod_{j=1}^{p} \Gamma(1 - c_j - \tfrac{1}{2}\gamma_j + i.\frac{t}{2\gamma}\gamma_j)}{\prod_{j=1}^{q} \Gamma(b_j + \tfrac{1}{2}\beta_j - i.\frac{t}{2\gamma}\beta_j) \prod_{j=1}^{p} \Gamma(1 - a_j - \tfrac{1}{2}\alpha_j + i.\frac{t}{2\gamma}\alpha_j)} M_t(K)$$

$$= \frac{\prod_{j=1}^{q} \Gamma(d_j + \tfrac{1}{2}\delta_j + i.\tfrac{t}{2\gamma}\,\delta_j) \prod_{j=1}^{p} \Gamma(1-c_j - \tfrac{1}{2}\gamma_j - i.\tfrac{t}{2\gamma}\gamma_j}{\prod_{j=1}^{q} \Gamma(b_j + \tfrac{1}{2}\beta_j + i.\tfrac{t}{2\gamma}\,\beta_j) \prod_{j=1}^{p} \Gamma(1-a_j - \tfrac{1}{2}\alpha_j - i.\tfrac{t}{2\gamma}\alpha_j)} M_{-t}(K),$$

the integrals on both the sides of (2.25) are absolutely convergent; Mellin transforms of $|L.H.S.|$, $|R.H.S.|$ and $\left|H_T \left[(c_p, \gamma_p); (d_q, \delta_q)\right] \text{ - transform of } k(x/y)\right|$ exist; $H_T \; (c_p, \gamma_p); (d_q, \delta_q)]$ - transform of $k(x/y)$ exists;

$$\min_{1 \le j \le q} R(b_j/\beta_j) > -\tfrac{1}{2}, \quad \min_{1 \le j \le q} R(d_j/\delta_j) > -\tfrac{1}{2},$$

$$\max_{1 \le j \le p} R\left(\frac{a_j-1}{\alpha_j}\right) < -\tfrac{1}{2}, \quad \max_{1 \le j \le p} R\left(\frac{c_j-1}{\gamma_j}\right) < -\tfrac{1}{2}.$$

PROOF Taking Mellin transform of both the sides of (2.25), we get

$$(2.27) \qquad \int_0^\infty x^{-\frac{1}{2}+it}\,dx \int_0^\infty (1/y)\; K(x/y)\; F(y)dy$$

$$= 2\gamma_\beta^{1/2\gamma} \int_0^\infty x^{-\frac{1}{2}+it}\,dx \int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2\,(xy)^{2\gamma} \left| \begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p, \gamma_p), (1-c_p - \frac{2\gamma+1}{4\gamma}\gamma_p, \gamma_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q, \delta_q), (1-d_q - \frac{2\gamma+1}{4\gamma}\delta_q, \delta_q) \end{matrix} \right.\right] dy \int_0^\infty \frac{1}{z} k(y/z) f(z)dz.$$

On L.H.S., we replace x/y by u after changing the order of integration which is valid, provided that the integral on L.H.S. of (2.25) is absolutely convergent and Mellin transform of $\left| \int_0^\infty 1/y\ K(x/y)\ F(y)dy \right|$ - exists, we have

$$\text{L.H.S.} = \int_0^\infty x^{-\frac{1}{2}+it}\,dx \int_0^\infty (1/y)K(x/y)\,F(y)dy$$

$$= \int_0^\infty y^{-\frac{1}{2}+it}\,F(y)dy \int_0^\infty u^{-\frac{1}{2}+it}\,K(u)du$$

(2.28) $\quad M_t(F).\ M_t(K).$

Now

$$M_t(F) = 2\gamma\beta^{1/2\gamma}\int_0^\infty y^{-\frac{1}{2}+it}\,dy \int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma}\left|\begin{matrix}(a_p+\frac{2\gamma-1}{4\gamma}\alpha_p,\alpha_p),(1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\ (b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q),\ (1-b_q-\frac{2\gamma+1}{4\gamma}\beta_q,\beta_q)\end{matrix}\right.\right] f(x)dx$$

$$= \beta^{1/2\gamma}\int_0^\infty x^{-\frac{1}{2}-it}\,f(x)dx \int_0^\infty v^{\frac{1}{4\gamma}+i.\frac{t}{2\gamma}-1} \times$$

$$\times\ H^{q,p}_{2p,2q}\left[\beta^2 v\left|\begin{matrix}(a_p+\frac{2\gamma-1}{4\gamma}\alpha_p,\alpha_p),(1-a_p-\frac{2\gamma+1}{4\gamma}\alpha_p,\alpha_p)\\ (b_q+\frac{2\gamma-1}{4\gamma}\beta_q,\beta_q),(1-b_q-\frac{2\gamma+1}{4\gamma}\beta_q,\beta_q)\end{matrix}\right.\right] dv\ ,$$

on changing the order of integration and replacing $(xy)^{2\gamma}$ by v .

Now, evaluating the v-integral by using (1.80) we have

$$M_t(F) = \beta^{-it/\gamma} \frac{\prod_{j=1}^{q} \Gamma(b_j+\tfrac{1}{2}\beta_j+i.\frac{t}{2\gamma}\beta_j) \prod_{j=1}^{p} \Gamma(1-a_j-\tfrac{1}{2}\alpha_j-i.\frac{t}{2\gamma}\alpha_j)}{\prod_{j=1}^{q} \Gamma(b_j+\tfrac{1}{2}\beta_j-i.\frac{t}{2\gamma}\beta_j) \prod_{j=1}^{p} \Gamma(1-a_j-\tfrac{1}{2}\alpha_j+i.\frac{t}{2\gamma}\alpha_j)} \times$$

$$\times M_{-t}(f).$$

Therefore, putting it in (2.28), we get

$$(2.29)\quad L.H.S. = \beta^{-it/\gamma} \frac{\prod_{j=1}^{q} \Gamma(b_j+\tfrac{1}{2}\beta_j+i.t/2\gamma\cdot\beta_j)}{\prod_{j=1}^{q} \Gamma(b_j+\tfrac{1}{2}\beta_j-\frac{it}{2\gamma}\beta_j)} \times$$

$$\times \frac{\prod_{j=1}^{p} \Gamma(1-a_j-\tfrac{1}{2}\alpha_j-\frac{it}{2\gamma}\alpha_j)}{\prod_{j=1}^{p} \Gamma(1-a_j-\tfrac{1}{2}\alpha_j+\frac{it}{2\gamma}\alpha_j)} \times M_t(K)M_{-t}(f).$$

Also from (2.27), we have

$$R.H.S.= 2\gamma\beta^{1/2\gamma} \int_0^\infty x^{-\frac{1}{2}+it}\,dx \int_0^\infty \frac{1}{z}f(z)dz \int_0^\infty H_{2p,2q}^{q,p}\left[\beta^2(xy)^{2\gamma}\right|$$

$$\left.\begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p,\gamma_p), (1-c_p - \frac{2\gamma+1}{4\gamma}\gamma_p,\gamma_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q,\delta_q), (1-d_q - \frac{2\gamma+1}{4\gamma}\delta_q,\delta_q) \end{matrix}\right] k(y/z)dy$$

$$= 2\gamma\beta^{1/2\gamma} \int_0^\infty \frac{1}{z} f(z)dz \int_0^\infty x^{-\frac{1}{2}+it}\, dx \int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma} \left|\begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p,\gamma_p) (1-c_p - \frac{2\gamma+1}{4\gamma}\gamma_p,\gamma_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q,\delta_q), (1-d_q - \frac{2\gamma+1}{4\gamma}\delta_q,\delta_q) \end{matrix}\right.\right] k(y/z)dy$$

$$= 2\gamma\beta^{1/2\gamma} \int_0^\infty \frac{1}{z} f(z)dz \int_0^\infty k(y/z)dy \int_0^\infty x^{-\frac{1}{2}+it} \times$$

$$\times H^{q,p}_{2p,2q}\left[\beta^2(xy)^{2\gamma} \left|\begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p,\gamma_p), (1-c_p - \frac{2\gamma+1}{4\gamma}\gamma_p,\gamma_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q,\delta_q), (1-d_q - \frac{2\gamma+1}{4\gamma}\delta_q,\delta_q) \end{matrix}\right.\right] dx$$

$$= \beta^{1/2\gamma} \int_0^\infty \frac{1}{z} f(z)dz \int_0^\infty y^{-\frac{1}{2}-it} k(y/z)dy \int_0^\infty v^{\frac{1}{4\gamma}+\frac{it}{2\gamma}-1} \times$$

$$\times H^{q,p}_{2p,2q}\left[\beta^2 v \left|\begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p,\gamma_p), (1-c_p - \frac{2\gamma+1}{4\gamma}\gamma_p,\gamma_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q,\delta_q), (1-d_q - \frac{2\gamma+1}{4\gamma}\delta_q,\delta_q) \end{matrix}\right.\right] dv,$$

On changing the order of integration and replacing $(xy)^{2\gamma}$ by v.

Now, evaluating v-integral with the help of (1.80) and then putting $y/z = u$, we get

$$(2.30)\quad \text{R.H.S.} = \beta^{-it/\gamma} \frac{\prod_{j=1}^{q} \Gamma(d_j + \frac{1}{2}\delta_j + \frac{it}{2\gamma}\delta_j)}{\prod_{j=1}^{q} \Gamma(d_j + \frac{1}{2}\delta_j - \frac{it}{2\gamma}\delta_j)} \times$$

$$\times \frac{\prod_{j=1}^{p} \Gamma(1-c_j - \frac{1}{2}\gamma_j - \frac{it}{2\gamma}\gamma_j)}{\prod_{j=1}^{p} \Gamma(1-c_j - \frac{1}{2}\gamma_j + \frac{it}{2\gamma}\gamma_j)} M_{-t}(k)\, M_{-t}(f).$$

The changes in the orders of integration are admissible provided that $\int_0^\infty k(z/y)f(z)\,dz$ is absolutely convergent

$H_T\left[(c_p,\gamma_p);\ (d_q,\delta_q)\right]$ - transform of $k(z/y)$ and

$\left|\ H_T\left[(c_p,\gamma_p);\ (d_q,\delta_q)\right]\right.$ - transform of $k(z/y)\Big|$ exist;

$$\min_{1\le j\le q} R(d_j/\delta_j) > -\tfrac{1}{2} \text{ and } \max_{1\le j\le p} R\left(\frac{c_j-1}{\gamma_j}\right) < -\tfrac{1}{2}.$$

Hence, equating the values of L.H.S. and R.H.S. from (2.29) and (2.30), we get the relation (2.26) and hence the proof.

2.321 <u>COROLLARIES</u>

(i) With $\gamma = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $\alpha_j = \gamma_j = 1$ $(j = 1,2, \ldots,p)$,

$\beta_j = \delta_j = 1$ $(j = 1,2,\ldots,q)$ the above theorem reduces to a theorem due to Sharma [61, p. 209].

(ii) With $p = 1$, $q = 2$, $\gamma = \frac{1}{2}$, $\beta = 1$, $\alpha_1 = \gamma_1 = 1$, $\beta_1 = \beta_2 = 1$, $\delta_1 = \delta_2 = 1$ and $a_1 = k - m - \nu/2 - \frac{1}{2}$, $b_1 = \nu/2$, $b_2 = \nu/2 + 2m$, $c_1 = k' - m' - \nu'/2 - \frac{1}{2}$, $d_1 = \nu'/2$, and $d_2 = \nu'/2 + 2m'$, we get a result due to Bhise [7, p. 206].

(iii) When $p = 0$, $q = 2$, $\beta = 1$, $\gamma = \frac{1}{2}$ and $\beta_1 = \beta_2 = 1$, $\delta_1 = \delta_2 = 1$, $b_1 = \nu$, $b_2 = \mu$, $d_1 = \xi$, $d_2 = \eta$, the above theorem yields a known result, given by Srivastava [65, p. 62].

2.4 Representation of the Kernels

In this section, we now obtain the representations for the Kernels, used in the Theorem 3 and Theorem 4.

2.41 FOR THEOREM 3

In view of (2.17), we can easily write K(x) and k(x) as

$$(2.31)\quad \begin{matrix} K(x) \\ k(x) \end{matrix} = M_x^{-1}\left[\beta^{-it/\gamma}\left\{\frac{\prod_{j=1}^{q}\Gamma\left(d_j + \frac{1}{2}\delta_j + \frac{it}{2\gamma}\delta_j\right)}{\prod_{j=1}^{q}\Gamma\left(b_j + \frac{1}{2}\beta_j - \frac{it}{2\gamma}\beta_j\right)}\right. \times\right.$$

$$\times \frac{\prod_{j=1}^{p} \Gamma(1 - c_j - \tfrac{1}{2}\gamma_j - \frac{it}{2\gamma}\gamma_j)}{\prod_{j=1}^{p} \Gamma(1 - a_j - \tfrac{1}{2}\alpha_j + \frac{it}{2\gamma}\alpha_j)} \left\} \begin{matrix} \phi(t) \\ \phi(-t) \end{matrix} \right] ,$$

where ϕ (t) is an arbitrary function such that right hand side of (2.31) exists. It can easily be verified by putting the values of $M_t(k)$ and $M_{-t}(k)$ in (2.20) from (2.31). Let $\phi(t) = M_t h(y)$, where h (y) is an arbitrary function. Therefore, we have from (2.31)

$$(2.32) \quad \begin{matrix} K(x) \\ k(x) \end{matrix} = 1/2\pi \int_{-\infty}^{\infty} \beta^{-it/\gamma} \left[\frac{\prod_{j=1}^{q} \Gamma(d_j + \tfrac{1}{2}\delta_j + \frac{it}{2\gamma}\delta_j)}{\prod_{j=1}^{q} \Gamma(b_j + \tfrac{1}{2}\beta_j - \frac{it}{2\gamma}\beta_j)} \times \right.$$

$$\left. \times \frac{\prod_{j=1}^{p} \Gamma(1 - c_j - \tfrac{1}{2}\gamma_j - \frac{it}{2\gamma}\gamma_j)}{\prod_{j=1}^{p} \Gamma(1 - a_j - \tfrac{1}{2}\alpha_j + \frac{it}{2\gamma}\alpha_j)} \right] \begin{matrix} M_t h(y) \\ M_{-t} h(y) \end{matrix} \quad x^{-\frac{1}{2} - it}\, dt.$$

Now, replacing y by (1/y) in the first integral for $K(x)$, but not disturbing the second integral for $k(x)$, changing the order of integration in (2.32) and then using the definition of Fox's H-function, we obtain the representation of the Kernels of Theorem 3, as

$$\frac{K(x)}{k(x)} = \int_0^{\infty} y^{-1} \frac{h(y^{-1})}{h(y)} \lambda_1(xy)dy,$$

where

$$\lambda_1(x) = 2\gamma\beta^{1/2\gamma} H^{q,p}_{2p,2q}\left[\beta^2 x^{2\gamma} \middle| \begin{matrix} (c_p + \frac{2\gamma-1}{4\gamma}\gamma_p, \gamma_p), & (1 - a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p) \\ (d_q + \frac{2\gamma-1}{4\gamma}\delta_q, \delta_q), & (1 - b_q - \frac{2\gamma+1}{4\gamma}\beta_q, \beta_q) \end{matrix}\right]$$

provided that $q \geq 0$, $p \geq 0$,

$$-\min_{1 \leq j \leq q} R(d_j/\delta_j) < \tfrac{1}{2} < \min_{1 \leq j \leq p} R\left(\frac{1-c_j}{\gamma_j}\right).$$

2.42 FOR THEOREM 4

Here, we find two types of representations for the Kernels of Theorem 4.

2.421 Obviously, from (2.26), we can represent K(x) and k(x) by

$$(2.33) \qquad \frac{K(x)}{k(x)} = M_x^{-1}\left[\left\{\frac{\prod_{j=1}^{q}\Gamma(d_j + \frac{1}{2}\delta_j + \frac{it}{2\gamma}\delta_j)}{\prod_{j=1}^{q}\Gamma(b_j + \frac{1}{2}\beta_j + \frac{it}{2\gamma}\beta_j)} \times\right.\right.$$

$$\times \left. \frac{\prod_{j=1}^{p} \Gamma\left(1 - c_j - \tfrac{1}{2}\gamma_j - \frac{it}{2\gamma}\gamma_j\right)}{\prod_{j=1}^{p} \Gamma\left(1 - a_j - \tfrac{1}{2}\alpha_j - \frac{it}{2\gamma}\alpha_j\right)} \right\} \begin{matrix} \phi(t) \\ \phi(-t) \end{matrix} \Bigg] ,$$

where $\phi(t)$ is an arbitrary function such that the right hand side of (2.33) exists. It can easily be verified by putting the values of M_t (K) and M_{-t} (k) in (2.26) from (2.33).

Let $\phi(t) = M_t h(y)$, where h(y) is an arbitrary function. Therefore, we have from (2.33).

$$(2.34) \quad \begin{matrix} K(x) \\ k(x) \end{matrix} = 1/2\pi \int_{-\infty}^{\infty} \left[\frac{\prod_{j=1}^{q} \Gamma\left(d_j + \tfrac{1}{2}\delta_j + \frac{it}{2\gamma}\delta_j\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \tfrac{1}{2}\beta_j + \frac{it}{2\gamma}\beta_j\right)} \times \right.$$

$$\left. \times \frac{\prod_{j=1}^{p} \Gamma\left(1 - c_j - \tfrac{1}{2}\gamma_j - \frac{it}{2\gamma}\gamma_j\right)}{\prod_{j=1}^{p} \Gamma\left(1 - a_j - \tfrac{1}{2}\alpha_j - \frac{it}{2\gamma}\alpha_j\right)} \right] \begin{matrix} M_t h(y) \\ M_{-t} h(y) \end{matrix} x^{-\frac{1}{2} - it} \, dt.$$

Replacing y by (1/y) in the first integral for k(x), but not disturbing the second integral for K(x) , changing the order of integration in (2.34) and then using the definition of Fox's H-function, we obtain the representation for the Kernels of Theorem 4, as

$$\begin{matrix} K(x) \\ k(x) \end{matrix} = \int_{0}^{\infty} \begin{matrix} y^{-1} h(y^{-1}) \\ h(y) \end{matrix} \quad \lambda_2(xy)\,dy,$$

$$\chi(x) = 2\gamma H^{q,p}_{p+q,\ q+p}\left[x^{2\gamma} \left| \begin{matrix} (c_p, \gamma_p), (b_q, \beta_q) \\ (d_q, \delta_q), (a_p, \alpha_p) \end{matrix} \right. \right]$$

provided that $p \geqslant 0 \quad q \geqslant 0$ and

$$- \min_{1 \leq j \leq q} (d_j/\delta_j) < \tfrac{1}{2} < \min_{1 \leq j \leq p} R\left(\frac{1-c_j}{\gamma_j} \right).$$

2.422. Other representations of the Kernels of Theorem 4, may be obtained by using the operators of fractional integration. Here, we mention some of them as under :

With the help of the results (1.19) and (1.20) and using the convolution theorem [66, Th. 44, p.60] , we have, by virtue of (2.26),

$$\begin{matrix} K(x) \\ k(x) \end{matrix} = \left[\left(\prod_{j=1}^{q} (K^{d_j, b_j - d_j, \beta_{j_*}}) \right) \left(\prod_{j=1}^{p} \left(I^{-c_j, a_j - c_j, A_{j_*}} \begin{matrix} h(x) \\ x^{-1} h(x^{-1}) \end{matrix} \right) \right) \right]$$

and

$$\begin{matrix} K(x) \\ k(x) \end{matrix} = \left[\left(\prod_{j=1}^{q} (I^{b_j + \beta_j - 1, d_j - b_j, B_j}) \right)_* \left(\prod_{j=1}^{p} \left(K^{1 - a_j - \alpha_j, a_j - c_j, A_j,} \begin{matrix} h(x) \\ {}_* x^{-1} h(x^{-1}) \end{matrix} \right) \right) \right]$$

where

$\alpha_j = \gamma_j,\ i = 1, 2, \ldots, p$

$\beta_j = \delta_j,\ j = 1, 2, \ldots, q$

$A_j = 1/\alpha_j,\ j = 1, 2, \ldots, p$

$B_j = 1/\beta_j,\ j = 1, 2, \ldots, q$

and

$$\prod_{j=1}^{p} (I^{\eta_j,\alpha_j,A_j}) \text{ and } \prod_{j=1}^{q} (K^{\eta_j,\alpha_j,A_j})$$

stands for operators

$$\left[(I^{\eta_1,\alpha_1,A_1}) , \ldots\ldots\ldots , (I^{\eta_p,\alpha_p,A_p}) \right],$$

and

$$\left[(K^{\eta_1,\alpha_1,A_1}), \ldots\ldots\ldots\ldots , (K^{\eta_q,\alpha_q,A_q}) \right]$$

respectively.

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