CHAPTER - I

INTRODUCTION

1.1 The H-function :

The H-function is applicable in a number of problems arising in physical sciences, engineering and statistics. The importance of this function lies in the fact that nearly all the special functions occurring in applied mathematics and statistics are its special cases. Besides, the functions considered by Boersma [6], Mittag-Leffler, generalized Bessel function due to Wright [42], the generalization of the hyper-geometric functions studied by Fox [14], Wright [43,44] and G-function of Meijer [27] are all special cases of the H-function, hence a study of this function will cover wider range than the G-function and gives deeper, more general and more useful results directly applicable in various problems of physical and biological sciences.

On account of the presence of the coefficients of s in the definition of the H-function, the results of the H-function are obtainable in a more co mpact form and without much difficulty. This is not the case with G-function.

Definition of the H-function :

Mellin-Barnes type integrals have been studied by Pincherle in 1888, Barnes [3] and Mellin [28]. Dixon and Ferrar [10] have given the asymptotic expansion of general Mellin-Barnes type integrals [Also, see Erdelyi.et al. [12,p.49] in this connection].

Functions close to an H-function occur in the study of the solutions of certain functional equations considered by Bochner [5] and Chandrasekharah and Narasimhan [8].

In an attempt to unify and extend the existing results on symmetrical Fourier kernels, Fox [15] has defined the H-function in terms of a general Mellin-Barnes type integral. He has also investigated the most general Fourier kernel associated with the H-function and obtained the asymptotic expansions of the kernel for large values of the arguments, by following his earlier method [14].

It is not out of place to mention that symmetrical Fourier kernels are useful in characterization of probability density functions and in obtaining the solutions of certain dual integral equations. The relevant details are available in the works of Fox [16,17], Saxena [35,36], Mathai and Saxena [23] and Saxena and Kushwaha [37,38]. The asymptotic expansions and analytic continuations of this function have been studied in detail by Braaksma [7].

An H-function is defined in terms of a Mellin-Barnes type integral as follows:

$$(1.1.1)\quad H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix} \right]$$

$$= H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_L \chi(s) z^s ds ,$$

where $i = (-1)^{1/2}$, $z \neq 0$

and

$$(1.1.2)\quad z^s = \exp[\, s\operatorname{Log}|z| + i \arg z \,],$$

in which $\operatorname{Log}|z|$ represents the natural logarithm of z and $\arg z$ is not necessarily the principle value. An empty product is interpreted as unity.

Here

$$(1.1.3)\quad \chi(s) = \frac{\prod_{1}^{m}\Gamma(b_j-B_js)\ \prod_{1}^{n}\Gamma(1-a_j+A_js)}{\prod_{m+1}^{q}\Gamma(1-b_j+B_js)\ \prod_{n+1}^{p}\Gamma(a_j-A_js)} ,$$

where m, n, p and q are non-negative integers such that $0 \leq n \leq p$, $1 \leq m \leq q$; $A_j (j = 1,\ldots,p)$, $B_j (j = 1, \ldots, q)$ are positive numbers; $a_j (j = 1, \ldots, p)$, $b_j (j = 1, \ldots, q)$ are complex numbers such that

(1.1.4) $A_j(b_h + \nu) \neq B_h(a_j - \lambda - 1)$

for $\nu, \lambda = 0, 1, 2, \ldots;\ h = 1, \ldots, m;\ j = 1, \ldots, n.$

L is the contour separating the points

(1.1.5) $s = \left(\frac{b_j + \nu}{B_j}\right),\ (j = 1, \ldots, m;\ \nu = 0, 1, \ldots)$

which are the poles of $\Gamma(b_j - B_j s)$ $(j = 1, \ldots, m)$,

from the points

(1.1.6) $s = \left(\frac{a_j - \nu - 1}{A_j}\right),\ (j = 1, \ldots, n;\ \nu = 0, 1, \ldots)$,

which are the poles of $\Gamma(1 - a_j + A_j s)$, $(j = 1, \ldots, n)$.

The contour L exist on account of (1.1.4). These assumptions will be retained throughout. In contracted form the H-function in (1.1.1) will be denoted by one of the following notations:

$$H(z), \quad H^{m,n}_{p,q}(z), \quad H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}(a_p, A_p)\\(bq, B_q)\end{matrix}\right].$$

The H-function is analytic function of z and makes sense if the following existence conditions are satisfied.

(1.1.7) CASE 1. For all $z \neq 0$ with $\mu > 0$.

(1.1.8) CASE 2. for $0 < |z| < \beta^{-1}$ with $\mu = 0$.

Here

(1.1.9) $\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j$ and

(1.1.10) $\beta = \prod_{j=1}^{p} A_j^{A_j} \prod_{j=1}^{q} B_j^{-B_j}$

It does not depend on the choice of L. Due to the occurrence of the factor z^s in the integrand of (1.1.1) it is, in general, multiple-valued but one valued on the Riemann surface of logz.

1.2 Some known results involving the H-function :

In this section we state for immediate reference, some of th the known results involving the H-function, which we use in the present work.

(i) $$H^{m,\,n+1}_{p+1,q+1}\left[z\,\middle|\,\begin{matrix}(\alpha,\delta),\,(a_p,A_p)\\(b_q,B_q),\,(\alpha+r,\delta)\end{matrix}\right]$$

$$= (-1)^r\, H^{m+1,n}_{p+1,q+1}\left[z\,\middle|\,\begin{matrix}(a_p,A_p),\,(\alpha,\delta)\\(\alpha+r,\delta),(b_q,B_q)\end{matrix}\right].$$

[1,p.191]

(ii) Asymptotic expansions:

The behaviour of the H-function for small and large values of the argument has been discussed by Braaksma [7] in detail. In this section we enumerate some of his results which are useful in applied problems. In order to present the results, the following notations as defined below will be used.

$$\alpha = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j ;$$

$$\beta = \prod_{j=1}^{p} (A_j)^{A_j} \prod_{j=1}^{q} (B_j)^{-B_j} ; \quad \gamma = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + p/2 - q/2 ;$$

$$\lambda = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j - \sum_{j=1}^{p} A_j , \quad \text{and}$$

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j .$$

According to Braaksma [7,p.278]

$$H_{p,q}^{m,n}(x) = O(|x|^{c}) \quad \text{for small } x ,$$

where $\mu \geq 0$ and $c = \min R(b_j/B_j)$ $(j = 1, \ldots, m)$;

and $$H_{p,q}^{m,n}(x) = O(|x|^{d}) \quad \text{for large } x ,$$

where $\mu \geq 0$, $\alpha > 0$, $|\arg x| < \frac{\alpha\pi}{2}$ and

$$d = \max R\left(\frac{a_j - 1}{A_j}\right) \quad (j = 1, \ldots, n).$$

For n=0, the H-function vanishes exponentially for large x in certain cases. We have

$$H_{p,q}^{m,o}[x] \sim O\left\{\exp(-\mu x^{1/\mu} \beta^{1/\mu}) x^{1/\mu(\gamma+1/2)}\right\}$$

provided that $\lambda > 0$, $|\arg x| < \pi\lambda/2$ and $\mu > 0$.

(iii) Special cases :

The H-function covers a vast number of analytic functions as special cases. These analytic functions appear in various problems arising in theoretical and applied brnaches of mathematics, statistics and engineering sciences.

In the first place, when

$A_j = B_h = 1$ ($j = 1, \ldots, p$; $h = 1, \ldots, q$), (1.1.1) reduces to Meijer's G-function,

$$(1.2.1) \quad H^{m,n}_{p,q}\left[x \,\middle|\, {(a_p,1) \atop (b_q,1)} \right] = G^{m,n}_{p,q}\left[x \,\middle|\, {a_p \atop b_q} \right].$$

A detailed account of Meijer's G-function and its applications can be found in the monograph by Mathai and Saxena [24]. We list here the few interesting cases of the H-function which are useful in the present work. We have

$$(1.2.2) \quad H^{1,0}_{0,1}\left[z \,\middle|\, {- \atop (b,B)} \right] = B^{-1} z^{b/B} \exp(-z^{1/B}) :$$

$$(1.2.3) \quad H^{1,1}_{1,1}\left[z \,\middle|\, {(1-\nu,1) \atop (0,1)} \right] = \Gamma(\nu)(1+z)^{-\nu} {}_1F_0(\nu;-z);$$

$$(1.2.4) \quad H^{1,p}_{p,q+1}\left[z \,\middle|\, {(1-a_p,A_p) \atop (0,1),(1-b_q,B_q)} \right]$$

$$= \sum_{r=0}^{\infty} \frac{\prod_{1}^{p} \Gamma(a_j+A_j r)\,(-z)^r}{\prod_{1}^{q} \Gamma(b_j+B_j r)\; r!}$$

$$= {}_p\Psi_q \left[{(a_p,\, A_p) \; ; \atop (b_q,\, B_q)} - z \right],$$

which is called Maitland's generalized hypergeometric function. The series in (1.2.4) has been studied in detail by Wright [43].

(iv) The H-function of two variables :

The H-function of two variables defined by Munot and Kalla [31] is given by

$$(1.2.5)\qquad H[x,y] \equiv$$

$$H\left[\begin{matrix}\begin{bmatrix} m_1, 0 \\ p_1-m_1, q_1\end{bmatrix} \\ \begin{bmatrix} m_2, n_2 \\ p_2-m_2, q_2-n_2\end{bmatrix} \\ \begin{bmatrix} m_3, n_3 \\ p_3-m_3, q_3-n_3\end{bmatrix}\end{matrix}\;\middle|\;\begin{matrix}\{(a_{p_1}, A_{p_1})\} \;;\; \{(b_{q_1}, B_{q_1})\} \\ \{(c_{p_2}, c_{p_2})\} \;;\; \{(d_{q_2}, D_{q_2})\} \\ \{(e_{p_3}, E_{p_3})\} ;\; \{(f_{q_3}, F_{q_3})\}\end{matrix}\;\middle|\;\begin{matrix} x \\ \\ y\end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{c_1}\int_{c_2} F(\xi+\eta)\phi(\xi)\psi(\eta)\,d\xi\, d\eta$$

where

$$F(\xi+\eta) = \frac{\prod_{j=1}^{m_1}\Gamma(a_j + A_j\,\overline{\xi+\eta})}{\prod_{j=m_1+1}^{p_1}\Gamma(1-a_j-A_j\,\overline{\xi+\eta})\prod_{j=1}^{q_1}\Gamma(b_j+B_j\,\overline{\xi+\eta})},$$

$$\phi(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma(1-c_j + C_j\xi)\prod_{j=1}^{n_2}\Gamma(d_j - D_j\xi)}{\prod_{j=m_2+1}^{p_2}\Gamma(c_j-C_j\xi)\prod_{j=n_2+1}^{q_2}\Gamma(1-d_j+D_j\xi)}\, x^{\xi}$$

and

$$\psi(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(1-e_j + E_j\eta) \prod_{j=1}^{n_3} \Gamma(f_j - F_j\eta)}{\prod_{j=m_3+1}^{p_3} \Gamma(e_j - E_j\eta) \prod_{j=n_3+1}^{q_3} \Gamma(1-f_j+F_j\eta)} y^{\eta} .$$

The details of the above function are available in [39].

(v) Here we state some of the known integrals [39,18,9 and 26] which we have used in the present investigations.

$$(1.2.6) \quad \int_0^1 t_1^{\gamma_1}(1-t_1)^{\beta} P_k^{(\alpha,\beta)}(1-2t_1) H_{p,q}^{m,n}\left[\alpha t_1^h \,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] dt_1$$

$$= \frac{(-1)^k \Gamma(\beta+k+1)}{k!} H_{p+2,\ q+2}^{m,\ \ n+2} \left[\alpha \,\middle|\, \begin{matrix}(-\gamma_1,h),\ (\alpha-\gamma_1,h),\ (a_p,\ A_p)\\ (b_q,B_q),(\alpha-\gamma_1+k,h),(-1-\beta-\gamma_1-k,h)\end{matrix}\right]$$

where

$Re(\beta) > -1,\ h > 0,\ \lambda > 0,\ |\arg\alpha| < \frac{1}{2}\lambda\pi$,

$\mu \leq 0$, where

$$\lambda = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j$$

$$\mu = \sum_{j=1}^{p} A_j - \sum_{j=1}^{q} B_j$$

$Re(\gamma_1+1) + h \min_{1 \leq j \leq m} [Re(b_j/B_j)] > 0$, $P_n^{(\alpha,\beta)}(x)$ being the Jacobi polynomial [34, p.254].

$$(1.2.7)\quad \int_0^1 t_1^{\gamma_1}(1-t_1^2)^{\nu/2} P_k^{\nu}(t_1) H_{p,q}^{m,n}\left[\alpha t_1^h \,\Big|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] dt_1$$

$$= \frac{(-1)^{\nu}\Gamma(1+\nu+k)}{2^{\nu+1}\Gamma(1-\nu+k)} H_{p+2,q+2}^{m,n+2}$$

$$\left[\alpha \,\Bigg|\, \begin{matrix}\left(\frac{1-\gamma_1}{2},\frac{h}{2}\right),\left(\frac{-\gamma_1}{2},\frac{h}{2}\right),(a_p,A_p)\\ (b_q,B_q),\left(\frac{-\gamma_1+k-\nu}{2},\frac{h}{2}\right),\left(\frac{-1-\gamma_1-\nu-k}{2},\frac{h}{2}\right)\end{matrix}\right]$$

provided $h>0$, $\lambda>0$, $|\arg\alpha|<\frac{1}{2}\lambda\pi$, $\mu\leq 0$,

$Re(\gamma_1)>-1$, $Re(\gamma_1)+h\min\limits_{1\leq j\leq m}[Re(b_j/B_j)]+1>0$, and

ν is a positive integer.

$\{P_r^{\nu}(x)$ is the associated Legendre function [12,p.148]$\}$

$$(1.2.8)\quad \int_0^1 \cdots \int_0^1 \prod_{j=1}^{r}\{t_j^{\gamma_j-1}(1-t_j)^{-1/2} T_{n_j}(2t_j-1)\}\times$$

$$H_{p,q}^{m,n}\left[c(t_1\ldots t_r)^h \,\Big|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]dt_1\ldots dt_r$$

$$=\sqrt{\pi}\; H_{p+2r,q+2r}^{m,n+2r}\left[\alpha \,\Big|\, \begin{matrix}(1-\gamma_r,h) & (1/2-\gamma_r,h) & ,(a_p,A_p)\\ (b_a;B_q), & (1/2-\gamma_r-n_r,h) & (1/2-\gamma_r+n_r,h)\end{matrix}\right]$$

where $h>0$, $\mu\leq 0$, $\lambda>0$ $|\arg\alpha|<\frac{\lambda\pi}{2}$,

$Re(\gamma_i)+h\min\limits_{1\leq j\leq m}Re(b_j/B_j)>-1$, $(j=1,2,\ldots,m)(i=1,\ldots,r)$.

$$(1.2.9)\quad \int_0^{\infty} x^{\gamma}e^{-x}L_k^{(\sigma)}(x)\, H_{p,q}^{m,n}\left[zx^{\delta}\,\Big|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]dx$$

$$= \frac{(-1)^k (2\pi)^{(1-\delta)/2} \delta^{\gamma+k+1/2}}{k!} \times$$

$$H^{m,n+2\delta}_{p+2\delta,q+\delta}\left[z\delta^{\delta} \middle| \begin{matrix} (\Delta(\delta,-\gamma),1),(\Delta(\delta,\sigma-\gamma),1),(a_p,A_p) \\ (b_q\ B_q).\ (\Delta(\delta,\sigma-\gamma+k),1) \end{matrix} \right]$$

where δ is a positive integer, $\mu \leq 0$ and $\lambda > 0$,

$|\arg z| < \lambda\pi/2$, $\mathrm{Re}(\gamma+\delta b_j/B_j) > -1 (j=1,\ldots,m)$.

Taking $\delta = 1$ it reduces to

$$\int_0^\infty x^{\gamma} e^{-x} L_k^{(\sigma)}(x)\, H^{m,n}_{p,q}\left[zx \middle| \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix} \right] dx$$

$$= \frac{(-1)^k}{k!} H^{m,n+2}_{p+2,q+1}\left[z \middle| \begin{matrix} (-\gamma,1),(\sigma-\gamma,1),(a_p,A_p) \\ (b_q,B_q),\ (\sigma-\gamma+k,1) \end{matrix} \right]$$

(1.2.10) $$\int_0^\infty e^{-t} t^{\gamma-1}\, Y_n(1,a;t)\, H^{m,n}_{p,q}\left[bt \middle| \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix} \right] dt$$

$$= H^{m,n+2}_{p+2,q+1}\left[b \middle| \begin{matrix} (1+n-\gamma,1),(2-\gamma-n-a,1),(a_p,A_p) \\ (b_q,B_q),\ (2-\gamma-a,1) \end{matrix} \right]$$

where $\mathrm{Re}(\gamma+\min b_j/B_j) > 0$ $(j=1,\ldots,m)$; $|\arg b| < \frac{\lambda\pi}{2}$

$\lambda > 0$ and $\mu \leq 0$.

1.3 Laplace Transform :

Define $\bar{f}(p)$ by the equation

$$(1.3.1)\quad \bar{f}(p) = \int_0^\infty f(t)e^{-pt}dt$$

this equation can be written in the form

$$(1.3.2)\quad f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{f}(p)e^{pt}dp ,$$

where γ is a positive constant of such a nature that integral $\int_0^\infty e^{-\gamma t} f(t)dt$ does converge.

The function $\bar{f}(p)$ defined by equation (1.3.1) is called Laplace Transform of the function $f(t)$ [40]. We use the notation

$$(1.3.3)\quad \bar{f}(p) = \mathcal{L}[f(t);p]$$

for function of single variable t.

In similar manner we shall write the relation between the functions $f(t)$ and $\bar{f}(p)$ in the inverse form

$$f(t) = \mathcal{L}^{-1}[\bar{f}(p);\, t].$$

In case of functions of several variables $t_1,\ldots,t_n$ we use the notation

$$\bar{f}(t_1,\ldots,p,\ldots,t_n) = \mathcal{L}[f(t_1,\ldots,t_r,\ldots,t_n);\, t_r \to p]$$

to denote the integral

$$\int_{o}^{\infty} f(t_1,\ldots,t_r,\ldots,t_n)e^{-pt_r}dt_r \quad .$$

We shall also use the inverse notation

$$f(t_1,\ldots,t_r,\ldots,t_n) = \pounds^{-1}\left[\ \bar{f}(t_1,\ldots,p,\ \ldots,t_n);\ p \rightarrow t_r\ \right] .$$

The above heuristic argument suggests that the inversion theorem for the Laplace transform is of the form (1.3.2).

An integral of the form (1.3.1) in which the function f(t) is integrable over the integral (o,a) for any positive a, is called Laplace integral. The integral does not exist for every function f(t). The sufficient conditions for the Laplace integral of a function f(t) to exist are as follows:

[I] If (a) f(t) is integrable over any finite interval [a,b], $o < a < b$, (b) there exists a real number c such that for any $b > o$, $\int_{b}^{\lambda} e^{-ct}f(t)dt$ tends to a finite limit as $\lambda \rightarrow \infty$, (c) for arbitrary $a > o$, $\int_{\epsilon}^{a} |f(t)|\, dt$ tends to a finite limit as $\epsilon \rightarrow 0^+$, then $\pounds[f(t);p]$ exists for $Re(p) \geq c$.

[II] If (a) f(t) is integrable over any finite interval (a,b), $o < a < b$, (b) for arbitrary positive a the integral $\int_{\epsilon}^{a} |f(t)|\, dt$ tends to a finite limit as $\epsilon \rightarrow o^+$, (c) $f(t) = O(e^{ct})$ as $t \rightarrow \infty$, then the integral defining $\pounds[f(t);p]$ converges absolutely for $Re p > o$.

It is easy to show that if ν is a constant such that $Re(\nu) > -1$, then the Laplace transform of t^{ν} exist and is given by

(1.3.4) $$\mathcal{L}[\,t^{\nu};p\,] = p^{-\nu-1}\,\Gamma(\nu+1)\,.$$

The convolution of two functions:

The convolution $(f*g)$ of two functions f and g is defined by the equation

(1.3.5) $$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau\,.$$

Convolution Theorem :

The Laplace transform of the covolution $f*g$ in terms of the Laplace transforms $\bar{f}(p)$, $\bar{g}(p)$ of the functions $f(t)$, $g(t)$ is given by

(1.3.6) $$\mathcal{L}[f*g;p] = \bar{f}(p)\bar{g}(p).$$

This may also be written in the inverse form

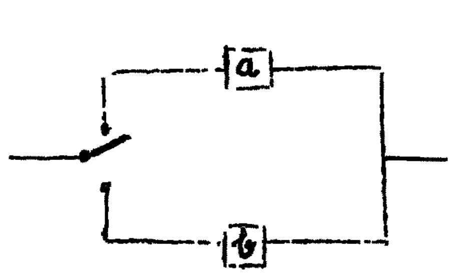
(1.3.7) $$\mathcal{L}^{-1}[\,\bar{f}(p)\bar{g}(p);\,t] = (f*g)(t).$$

1.4 Probability Distributions :

As is well known, every probability density function must satisfy two basic requirements: firstly that it must be non-negative everywhere and secondly that its total integral over the range of variation must equal unity. Hence, in all the subsequent work we shall suppose, without mentioning so explicitely every time, that the parameters involved are always so chosen that the function which we name as the probability density function satisfies the above two requirements.

The term probability density function will sometimes be abbreviated as pdf.

The sum $X_1 + X_2$ of two independent random variables is an important concept in probability theory. To illustrate its utility let us suppose that there is a system S consisting of two components a and b. The arrangement is such that while a is in operation first, b is not in operation. But the moment a fails b is put in to operation. If now X_1 and X_2 are the random times of failure of the components a and b after they are put into operation $X_1 + X_2$ represents the random time of failure of the system S (S fails when both a and b have failed). Assuming that the behaviour of a and b is independent of each other then X_1 and X_2 are independent. Suitable probability density functions may be chosen as models for X_1 and X_2. The system S may be any mechanical or electrical system or a human relay system and is diagramatically represented as follows:



The idea is easily extended to the sum of n random variables $X_1 + \dots + X_n$. We have, however, not considered the time variations of the random variables in this work but have restricted ourselves to the consideration of the situation at any fixed point of time.

We have considered, the general family of probability distributions introduced by Mathai and Saxena [22]; viz:

$$(1.4.1) \quad p(x) = \frac{d a^{c/d} \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\gamma - c/d)}{\Gamma(c/d)\, \Gamma(\gamma)\, \Gamma(\alpha - c/d)\, \Gamma(\beta - c/d)} x^{c-1} \cdot {}_2F_1(\alpha, \beta; \gamma; -ax^d), \quad x > 0$$

$$= 0 \quad \text{otherwise}$$

in which $c > 0$, $\alpha - c/d > 0$ and $\beta - c/d > 0$ and ${}_2F_1(\cdot)$ is the usual hypergeometric function.

In their paper, Mathai and Saxena have pointed out that most of the wellknown probability distributions and the probability distributions given by Patil [33], Stacy [41], Mathai [21] and others are special cases of the general distribution discussed by them. Therefore, the probability distribution of the sum of two independent random variables discussed here will yield particular case of interest in all these distributions.

1.5 Here we write some of the known polynomials used in our present work.

BESSEL POLYNOMIAL :

(1.5.1) $$Y_n(x,a,b) = \sum_{r=o}^{n} \frac{(-n)_r(a+n-1)_r}{r!}\left(-\frac{x}{b}\right)^r$$

$$= {}_2F_0\,(-n,\ a+n-1;\ -\ ;\ -\ x/b).$$

Orthogonality Property :

(1.5.2) $$\int_o^1 x^{1-a}e^{-x}Y_n(1,a,x)\ Y_m(1,b,x)dx$$

$$= \begin{cases} 0 & \text{if } m \neq n \\ n!\ (2-a-n) & \text{if } m=n \end{cases}$$

TCHEBICHEF POLYNOMIALS :

(1.5.3) $$T_n(x) = \cos(n \cos^{-1}x)$$

$$U_n(x) = \frac{[\sin(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)}$$

Orthogonality Property :

(1.5.4) $$\int_{-1}^{1} (1-x^2)^{-1/2}\, T_m(x)T_n(x)dx = \frac{\pi}{2}\, \delta m,n$$

JACOBI POLYNOMIAL :

(1.5.5) $$P_n^{(\alpha;\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}]$$

Orthogonality Property :

$$(1.5.6)\quad \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x)\; P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{2^{\alpha+\beta+1}\;\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\alpha+\beta+2n+1\;(n)!\;\Gamma(\alpha+\beta+n+1)}\,\delta m,n,$$

where $Re(\alpha) > -1$, $Re(\beta) > -1$.

LAGUEREE POLYNOMIAL :

$$(1.5.7)\quad L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\;\frac{d^n}{dx^n}\,(e^{-x}x^{n+\alpha})$$

$$L_n^{o}(x) = L_n(x)\;.$$

Orthogonality Property :

$$(1.5.8)\quad \int_{0}^{\infty} e^{-x}x^{\alpha}L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)dx$$

$$= \frac{\Gamma(\alpha+n+1)}{n!}\,\delta m;n,\; Re(\alpha+n+1) > 0\;.$$

1.6 Motivation of the work done :

Moharir [29] has considered the general family of probability distributions introduced by Mathai and Saxena [22]. He has investigated the probability distribution of the sum of two independent stochastic variables utilizing similar types of probability density functions by using some known integrals. It is found that same probability density functions can be derived by using the technique of convolution theorem in Laplace transforms. This idea motivated us to undertake the present investigation. In order to achieve our aim, firstly we have established the results given by Moharir [29] by using Laplace transforms technique and secondly utilizing the techniques given by Mathai [20], Mathai and Saxena [25] and Moharir and Saxena [30], we have investigated some multivariate probability density functions of implicit type.