CHAPTER - II

ON THE PROBABILITY DENSITIES INVOLVING THE HYPERGEOMETRIC FUNCTION AND THE H - FUNCTION

2.1 In this chapter, we have considered, the general family of probability distributions introduced by Mathai and Saxena [22]. The object of this chapter is to investigate the probability distribution of the sum of two independent stochastic variables utilizing similar types of probability density functions. To obtain this probability density function we have used the technique of convolution theorem in Laplace transforms.

We also present here a new probability density function of the sum of two independent random variables in that, whereas the rectangular or uniform pdf has been assumed for the random variable X_2 new pdf have been introduced for the random variable X_1 and the pdf of $X_1 + X_2$ has been obtained.

2.2 The Distribution of the Sum of Two Independent Variables :

We consider here the probability distribution of the sum of two independent stochastic variables having the pdfs belonging to the same class of density functions as p(x) in (1.4.1). Let X_1 and X_2 be two independent stochastic variables with pdfs

$$(2.2.1)\quad p_j(u_j) = k_j u_j^{c_j-1}\, {}_2F_1(\alpha_j,\ \beta_j;\ \gamma_j;\ -a_j u_j^{d_j}) \qquad \text{for } u_j > 0$$

$$= 0 \quad \text{otherwise}$$

with $c_j > 0$, $\alpha_j - c_j/d_j > 0$, $\beta_j - c_j/d_j > 0$,

$\gamma_j \neq 0, -1, -2, \ldots$ and

$$(2.2.2)\quad k_j = \frac{d_j a_j^{c_j/d_j}\,\Gamma(\alpha_j)\,\Gamma(\beta_j)\,\Gamma(\gamma_j - c_j/d_j)}{\Gamma(c_j/d_j)\,\Gamma(\gamma_j)\,\Gamma(\alpha_j - c_j/d_j)\,\Gamma(\beta_j - c_j/d_j)}$$

with $j = 1, 2$.

Then the pdf of the sum $x_1 + x_2$ is given by the convolution

$$(2.2.3)\quad q(x) = \int_0^x p_1(x-u_2)\,p_2(u_2)\,du_2, \qquad x > 0$$

$$= 0 \quad \text{otherwise.}$$

Taking Laplace transforms of both the sides of (2.2.3) and using (1.3.6) we get

$$(2.2.4)\quad \mathcal{L}\{q(x)\} = \mathcal{L}\{p_1(x)\}\ \mathcal{L}\{p_2(x)\}.$$

Writing $p_1(x)$ and $p_2(x)$ from (2.2.1) in terms of hypergeometric series, taking Laplace transforms and using (1.3.4), equation (2.2.4) becomes

$$(2.2.5)\quad \mathcal{L}\{q(x)\} = k_1 k_2 \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_\nu (\beta_1)_\nu}{\nu!\,(\gamma_1)_\nu}\,\Gamma(\nu d_1 + c_1)(-a_1)^\nu$$

$$\times \sum_{n=0}^{\infty} \frac{(\alpha_2)_n (\beta_2)_n}{n!(\gamma_2)_n} \Gamma(nd_2+c_2)(-a_2)^n \times$$

$$\times \frac{1}{s^{\nu d_1+nd_2+c_1+c_2}}$$

By taking inverse Laplace transform for (2.2.5) we get that

$$(2.2.6) \quad q(x) = k_1 k_2 x^{c_1+c_2-1} \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_\nu (\beta_1)_\nu}{(\gamma_1)_\nu} \frac{(-a_1 x^{d_1})^\nu}{\nu!} \Gamma(\nu d_1+c_1)$$

$$\times \sum_{n=0}^{\infty} \frac{(\alpha_2)_n (\beta_2)_n}{(\gamma_2)_n} \frac{(-a_2 x^{d_2})}{n!} \frac{\Gamma(c_2+nd_2)}{\Gamma(\nu d_1+nd_2+(c_1+c_2))}$$

Now using Gauss multiplication theorem [34,p.26], (2.2.6) takes the form

$$(2.2.7) \quad q(x) = k_1 k_2 x^{c_1+c_2-1} \frac{\Gamma(\gamma_2)\Gamma(c_1)}{\Gamma(\alpha_2)\Gamma(\beta_2)} \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_\nu (\beta_1)_\nu}{\nu!(\gamma_1)_\nu} (-a_1 x^{d_1})^\nu$$

$$\times \prod_{j=1}^{d_1} \left(\frac{c_1+j-1}{d_1}\right)_\nu \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_2+n)\Gamma(\beta_2+n)\Gamma(c_2+d_2n)}{\Gamma(\gamma_2+n)\Gamma(c_1+c_2+d_2n)}$$

$$\times \frac{\prod_{j=1}^{d_1}\Gamma\left(\frac{c_1+c_2+j-1}{d_1} + \frac{d_2}{d_1}n\right) (-a_2 x^{d_2})^n}{\prod_{j=1}^{d_1}\Gamma\left(\frac{c_1+c_2+\nu d_1+j-1}{d_1} + \frac{d_2}{d_1}n\right) n!} .$$

Summing the inner series in (2.2.7) with the help of (1.2.4) we have

$$(2.2.8)\quad q(x) = k_1 k_2 x^{c_1+c_2-1} \frac{\Gamma(\gamma_2)}{(\alpha_2)} \frac{\Gamma(c_1)}{(\beta_2)} \sum_{\nu=o}^{\infty} \frac{(\alpha_1)_\nu (\beta_1)_\nu}{(\gamma_1)_\nu \nu!} (-a_1 x^{d_1})^\nu$$

$$\times \prod_{j=1}^{d_1} \left(\frac{c_1+j-1}{d_1} \right)_\nu H^{1,\, d_1+3}_{d_1+3,\, d_1+3}$$

$$\left[a_2 x^{d_2} \middle| \begin{matrix} (1-\alpha_2, 1), (1-\beta_2, 1), (1-c_2, d_2), \{\Delta(d_1, 1-c_1-c_2) \frac{d_2}{d_1}\} \\ (0,1), (1-\gamma_2, 1), (1-c_1-c_2, d_2), \{\Delta(d_1, 1-c_1-c_2-\nu d_1), \frac{d_2}{d_1}\} \end{matrix} \right]$$

for $x > 0$

$q(x) = 0$ otherwise,

with $c_1, c_2 > 0$ and $c_1 + c_2 > 1$. k_1 and k_2 are as given in (2.2.2).

2.3 In this section we consider another general class of probability distributions and study the probability law of the sum of two independent stochastic variables following this probability law.

Let the random (or stochastic) variables X_1 and X_2 be governed by the probability density function introduced by Mathai and Saxena [23] viz.

$$(2.3.1)\quad f_i(x) = \gamma \Theta^{\sigma_i/\gamma} \phi(\sigma_i) x^{\sigma_i - 1}$$

$$\times H^{m,n}_{p,q} \left[\Theta x^r \middle| \begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \right] \text{ for } x > o$$

$$= 0 \text{ otherwise}$$

for $i = 1,2$ respectively where $\theta, \gamma, \sigma_1, \sigma_2 > 0$.

$\phi(\sigma_i) = \frac{1}{\chi(\sigma_i/\beta)}$, $\beta > 0$ and $\chi(\xi)$ is given by

$$(2.3.2) \quad \chi(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_j+\beta_j\xi) \prod_{j=1}^{n} \Gamma(1-a_j-\alpha_j\xi)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j\xi) \prod_{j=n+1}^{p} \Gamma(a_j+\alpha_j\xi)}$$

Since X_1 and X_2 are mutually independent the probability density for $X_1 + X_2$ will be given by the convolution

$$(2.3.3) \quad h(x) = \int_0^x f_1(x-v) f_2(v)\,dv, \quad x > 0$$
$$= 0 \quad \text{otherwise}.$$

Taking Laplace transforms of both the sides in (2.3.3) and using convolution theorem we get that

$$(2.3.4) \quad \mathcal{L}\{h(x)\} = \mathcal{L}\{f_1(x)\} \ \mathcal{L}\{f_2(x)\}.$$

After substituting for $f_1(x)$ and $f_2(x)$ in (2.3.4) from (2.3.1) and replacing the two $H^{m,n}_{p,q}[.]$ functions by contour integrals as given by (1.1.1), taking inverse Laplace transform and by writing the resulting double contour integral in symbolic form, (2.3.4) becomes

$$(2.3.5) \quad h(x) = r^2 \theta^{(\sigma_1+\sigma_2)/r} \phi(\sigma_1)\phi(\sigma_2) x^{\sigma_1+\sigma_2-1}$$

$$\times H \left[\begin{bmatrix} 0, & 0 \\ 0, & 1 \end{bmatrix} \begin{bmatrix} n+1, m \\ p-n, q-m \end{bmatrix} \begin{bmatrix} n+1, m \\ p-n, q-m \end{bmatrix} \left| \begin{matrix} - \; ; (\sigma_1+\sigma_2, r) \\ (1-\sigma_1, r), (a_p, \alpha_p) \; ; (b_q, \beta_q) \\ (1-\sigma_2, r), (a_p, \alpha_p) \; ; (b_q, \beta_q) \end{matrix} \right| \begin{matrix} \Theta x^r \\ \Theta x^r \end{matrix} \right]$$

for $x > 0$

$= 0$ otherwise

in which $H[.]$ represents Fox's H-function of two variables defined by Munot and Kalla [31] as given in (1.2.5).

The result (2.3.5) will be true subject to the conditions

(2.3.6) $\Theta, r, \sigma_1, \sigma_2 > 0$; $\sigma_2 + r \min Re\,(b_j/\beta_j) > 0$

for $j = 1, \ldots, m$; $\alpha \equiv \sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j \leq 0$;

$|\arg(\Theta x^r)| \leq \frac{1}{2} \alpha' \pi$ where

$$\alpha' \equiv \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=n+1}^{p} \alpha_j - \sum_{j=m+1}^{q} \beta_j > 0 .$$

2.4 Case where X_2 is characterized by the Rectangular or Uniform pdf :

Following the idea given by Dwass [11]. Let us consider two independent random variables X_1 and X_2 which have their pdfs as follows :

The pdf of X_1 = $f(x)$ for $o \leq x \leq a$

= 0 otherwise,

the pdf of X_2 = $g(x)$ for $o \leq x \leq a$

= 0 otherwise.

$f(x)$ and $g(x)$ are assumed non-negative and such that

$$\int_0^a f(x)dx = \int_0^a g(x)dx = 1 .$$

Then it can be seen that the pdf of $X_1 + X_2$ is given by

$$(2.4.1) \quad h(x) = \int_0^x f(v)g(x-v)dv, \quad o \leq x \leq a$$

$$= \int_{x-a}^a f(v)g(x-v)dv , \quad a \leq x \leq 2a$$

$$= 0 \text{ otherwise.}$$

Consider now the case in which the random variable X_2 is characterized by the rectangular or uniform pdf given by

$$g(x) = 1/a , \quad o \leq x \leq a \quad (a > o)$$

$$= 0 \text{ otherwise}$$

and X_1 is characterized by the pdf $f(x)$ as defined below. The pdf $h(x)$ for $X_1 + X_2$, obtained by using (2.4.1) has been noted below. The parameters involved are supposed to be so chosen that the pdf is always non-negative. From the result in [4] it can be readily verified that

$$\int_0^a f(x)\,dx = 1 \quad \text{and} \quad \int_0^{2a} h(x)\,dx = 1 .$$

Take

$$f(x) = a^{-\rho} x^{\rho-1} H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]$$

$$\times 1 / H^{m,n+1}_{p+1,q+1}\left[a \,\middle|\, \begin{matrix}(1-\rho,1),(a_p,A_p)\\(b_q,B_q),(-\rho,1)\end{matrix}\right], \quad 0 \leq x \leq a$$

$$= 0 \quad \text{otherwise} .$$

Then

$$h(x) = Ax^{\rho} H^{m,n+1}_{p+1,q+1}\left[x \,\middle|\, \begin{matrix}(1-\rho,1),\ (a_p,A_p)\\(b_q,B_q),\ (-\rho,1)\end{matrix}\right], \quad 0 \leq x \leq a$$

$$= A\left\{a^{\rho} H^{m,n+1}_{p+1,q+1}\left[a \,\middle|\, \begin{matrix}(1-\rho,1),\ (a_p,A_p)\\(b_q,B_q),\ (-\rho,1)\end{matrix}\right]\right.$$

$$\left. -(x-a)^{\rho} H^{m,n+1}_{p+1,q+1}\left[(x-a) \,\middle|\, \begin{matrix}(1-\rho,1),\ (a_p,A_p)\\(b_q,B_q),\ (-\rho,1)\end{matrix}\right]\right\},$$

$$0 \leq x \leq 2a$$

$$= 0 \quad \text{otherwise}$$

where

$$A = a^{-\rho-1} / H^{m,n+1}_{p+1,q+1}\left[a \,\middle|\, \begin{matrix}(1-\rho,1),\ (a_p,A_p)\\(b_q,B_q),\ (-\rho,1)\end{matrix}\right] .$$

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