CHAPTER - III

ON MULTIVARIATE PROBABILITY DENSITY FUNCTIONS

3.1 We introduce here some new theoretical joint multivariate probability density functions for the set of r independent stochastic variables $(X_1, \ldots, X_r)$. In each case the joint k^{th} moment has been calculated. The special features of these new pdfs are (i) each of them is a function of $r(r=1,2, \ldots)$ and hence actually represents a family of pdfs and (ii) each of them is an implicit function whereby separation into individual densities is not possible.

It is well known that if $X_1, \ldots, X_r$ is a set of r random variables their joint cumulative distribution function is defined as [32,ch.8]

(3.1.1) $$F(t_1, \ldots, t_r) = P\{X_1 \leq t_1, \ldots, X_r \leq t_r\}$$.

and that their joint pdf is given by

(3.1.2) $$f(t_1, \ldots, t_r) = \partial^r F(t_1, \ldots, t_r)/\partial t_1 \ldots \partial t_r$$.

The function $f(\cdot)$ so defined is everywhere non-negative and its integral over the entire range of variation of $t_1 \ldots t_r$ equals unity i.e.

$$(3.1.3) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1,\ldots,t_r)dt_1\ldots dt_r = 1 .$$

Also, if $k = k_1 + \ldots + k_r$, the joint k^{th} moment of $(x_1,\ldots,x_r)$ is defined as

$$(3.1.4) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} t_1^{k_1}\ldots t_r^{k_r} f(t_1,\ldots,t_r)dt_1\ldots dt_r .$$

The characteristic function (c.f.) of random variable has two-fold importance:
(i) it is an additional mathematical tool for studying the mathematical behaviour of the raondom variable and (ii) since the c.f. and its inversion are unique the random variable might be specified either by its pdf or its c.f.

The characteristic function of the set of random variables $(X_1,\ldots,X_r)$ which is governed by the pdf $f(t_1,\ldots,t_r)$ is defined as [13,p.69]

$$(3.1.5) \quad \phi(w_1,\ldots,w_r) = E\left\{ e^{i(w_1X_1 + \ldots + w_rX_r)} \right\}$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i(w_1t_1+\ldots+w_rt_r)} f(t_1,\ldots t_r)dt_1\ldots dt_r$$

in which E . represents the expectation and $i = \sqrt{-1}$.

We now state without proof the inversion theorem of c.f.

Inversion Theorem

If $\phi(w_1,\ldots,w_r)$ is the characteristic function of the set of random variables $(X_1,\ldots,X_r)$ is the characteristic function of the set of random variables $(X_1,\ldots,X_r)$ with pdf $f(t_1,\ldots,t_r)$, as defined by (3.1.5), then

$$(3.1.6)\quad \frac{1}{(2\pi)^r}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i(t_1w_1+\ldots+t_rw_r)}\phi(w_1,\ldots w_r)\,dw_1\ldots dw_r$$

$$= f(t_1,\ldots,t_r)\ .$$

The theorem can be proved by using the following two properties of the Diroc delta function $\delta(t)$:

$$\text{(i)}\quad \int_{-\infty}^{\infty} f(y)\delta(x-y)\,dy = \int_{-\infty}^{\infty} f(y)\delta(y-x)\,dy = f(x)$$

and

$$\text{(ii)}\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(x-a)}\,dw = \delta(x-a)\ .$$

3.2 Let us denote the integral (1.2.6) by I_1 and replace α by αt_2^h , multiply both the sides by $t_2^{\gamma_2}(1-t_2)^{\beta}p_k^{(\alpha,\beta)}(1-2t_2)$ and integrate with respect to t_2 from 0 to 1 to get I_2. Repeating the same process (r-1) times we get

$$(3.2.1)\quad I_r = \int_0^1\cdots\int_0^1 \prod_{j=1}^{r}[\, t_j^{\gamma_j}(1-t_j)^{\beta}P_k^{(\alpha,\beta)}(1-2t_j)\,]$$

$$\times H_{p,q}^{m,n}\left[\alpha(t_r\ldots t_1)\ \middle|\ {(a_p,A_p) \atop (b_q,B_q)}\right]dt_1\ldots dt_r\ .$$

$$= \left[\frac{(-1)^k \Gamma(\beta+k+1)}{k!}\right]^r H^{m,\,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}(-\gamma_r,h)\ ,\ (\alpha-\gamma_r,h)\ ,\ (a_p,A_p)\\ (b_q,B_q),\ (\alpha-\gamma_r+k,h)\ ,\ (-1-\beta-\gamma_r-k,h)\end{matrix}\right]$$

where $Re(\beta) > -1$, $h > 0$, $\lambda > 0$, $|\arg\alpha| < \frac{1}{2}\lambda\pi$; $\mu \leq 0$,

$Re(\gamma_i + 1) + h \min_{1 \leq j \leq m} [Re\ b_j/B_j] > 0\ (i = 1, \ldots, r)$,

$P_k^{(\alpha,\beta)}$ being the Jacobi polynomial

Now taking limit $\gamma_j \to 0$, the pdf can be formulated as

$$(3.1.2)\quad f(t_1,\ldots,t_r) = \frac{\prod_{j=1}^{r}\left[(1-t_j)^{\beta} P_k^{(\alpha,\beta)}(1-2t_j)\right]}{\left[\frac{(-1)^k \Gamma(\beta+k+1)}{k!}\right]^r}$$

$$\times \frac{H^{m,n}_{p,q}\left[\alpha(t_r\ldots t_1)^h \,\middle|\, \begin{matrix}(a_p,A_p)\\ (b_q,B_q)\end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}\{(0,h)\}_r,\ \{(\alpha,h)\}_r,\ (a_p,A_p)\\ (b_q,B_q),\{(\alpha+k,h)\}_r,\{(-1-\beta-k,h)\}_r\end{matrix}\right]}$$

Taking the pdf as in (3.2.2) we have after using the result (3.2.1)

$$(3.2.3)\quad \int_0^1 \cdots \int_0^1 \prod_{j=1}^{r} t_j^{\gamma_j} f(t_1,\ldots,t_r)dt_1\ldots dt_r$$

$$= \frac{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}(-\gamma_r,h)\ ,\ (\alpha-\gamma_r,h)\ ,\ (a_p,A_p)\\ (b_q,B_q),\ (\alpha-\gamma_r+k,h)\ ,\ (-1-\beta-\gamma_r-k,h)\end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}\{(0,h)\}_r,\ \{(\alpha,h)\}_r,\ (a_p,A_p)\\ (b_q,B_q),\{(\alpha+k,h)\}_r,\{(-1-\beta-k,h)\}_r\end{matrix}\right]}$$

This gives after substituting $\gamma_j = k_j (j=1,\ldots,r)$ the joint k^{th} moment defined in (3.1.4) as

$$(3.2.4)\quad \frac{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}(-k_r,h), & (\alpha-k_r,h), & (a_p,A_p)\\ (b_q,B_q), & (\alpha-k_r+k,h), & (-1-\beta-k_r-k,h)\end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}\{(o,h)\}_r, \{(\alpha,h)\}_r, (a_p,A_p)\\ (b_q,B_q), \{(\alpha+k,\mu)\}_r, \{(-1-\beta-k,h)\}_r\end{matrix}\right]}.$$

The conditions under which this result is valid are the same as in (3.2.1) with γ_j replaced by k_j.

Example 1 : In the pdf given by (3.2.2) if we take $m=1$, $n=0$, $p=1$, $q=1$, $A_1=1$, $B_1=1$, the H-function in numerator becomes

$$H^{1,0}_{1,1}\left[\alpha(t_r\ldots t_1)^h \,\middle|\, \begin{matrix}(a_1,1)\\(b_1,1)\end{matrix}\right] = G^{1,0}_{1,1}\left[\alpha(t_r\ldots t_1)^h \,\middle|\, \begin{matrix}a_1\\b_1\end{matrix}\right]$$

On using the identity

$$G^{1,0}_{1,1}\left(z \,\middle|\, \begin{matrix}a_1\\b_1\end{matrix}\right) = \frac{z^{b_1}(1-z)^{-1-b_1+a_1}}{\Gamma(a_1-b_1)},$$

we get

$$G^{1,0}_{1,1}\left(\alpha(t_r\ldots t_1)^h \,\middle|\, \begin{matrix}a_1\\b_1\end{matrix}\right) = \frac{\alpha^{b_1}(t_r\ldots t_1)^{hb_1}[1-\alpha(t_r\ldots t_1)^h]^{-1-b_1+a_1}}{\Gamma(a_1-b_1)}.$$

Therefore the pdf can be formulated as

$$(3.2.6)\quad f(t_1,\ldots,t_r) = \frac{\prod_{j=1}^{r}\left[(1-t_j)^{\beta} P_k^{(\alpha,\beta)}(1-2t_j)\right]}{\Gamma(a_1-b_1)\left[\frac{(-1)^k\,(\beta+k+1)}{k!}\right]^r}$$

$$\times \frac{\alpha^{b_1}(t_r\ldots t_1)^{hb_1}[1-\alpha(t_r\ldots t_1)^h]^{-1-b_1+a_1}}{H^{1,2r}_{1+2r,1+2r}\left[\alpha \left| \begin{matrix} \{(o,h)\}_r, \{(\alpha,h)\}_r, (a_1,1) \\ (b_1,1), \{(\alpha+k,h)\}_r, \{(-1-\beta-k,h)\}_r \end{matrix} \right.\right]}$$

for $o \leq t_j \leq 1$

$= 0$ otherwise.

If $K = k_1 + \ldots + k_r$, the joint k^{th} moment will be given by

$$(3.2.7)\quad \frac{H^{1,2r}_{1+2r,1+2r}\left[\alpha \left| \begin{matrix} (-k_r,h) & , (\alpha-k_r,h) & , (a_1,1) \\ (b_1,1), & (\alpha-k_r+k,h) & (-1-\beta-k_r-k,h) \end{matrix} \right.\right]}{H^{1,2r}_{1+2r,1+2r}\left[\alpha \left| \begin{matrix} \{(o,h)\}_r, \{(\alpha,h)\}_r, (a_1,1) \\ (b_1,1)\ \{(\alpha+k,h)\}_r \{(-1-\beta-k,h)\}_r \end{matrix} \right.\right]}$$

provided $(k_i+1) + h \operatorname{Re} b_1 > 0$ $(i = 1, \ldots r)$.

3.3 Let us denote the integral (1.2.7) by I_1 and replace α by αt_2^h , multiply both the sides by $t_2^{\gamma_2} (1-t_2^2)^{\nu/2} P_k^{\nu}(t_2)$ and integrate with respect to t_2 from 0 to 1 to get I_2. Repeating the same process (r-1) times we get

$$(3.3.1)\quad I_r = \int_o^1 \ldots \int_o^1 \prod_{j=1}^{r} [\, t_j^{\gamma_j}(1-t_j^2)^{\nu/2} P_k^{\nu}(t_j)\,]\, H^{m,n}_{p,q}\left[\alpha(t_r\ldots t_1)^h \left| \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix} \right. dt_1\ldots dt_r\right].$$

$$= \left[\frac{(-1)^{\nu}\Gamma(1+\nu+k)}{2^{\nu+1}\Gamma(1-\nu+k)}\right]^r H^{m,n+2r}_{p+2r,q+2r}$$

$$\left[\begin{matrix} (\frac{1-\gamma_r}{2}, \frac{h}{2}) , & (\frac{-\gamma_r}{2}, \frac{h}{2}) & , (a_p,A_p) \\ (b_q,B_q), & (\frac{-\gamma_r+k-\nu}{2}, \frac{h}{2}) , & (\frac{-1-\gamma_r-k-\nu}{2}, \frac{h}{2}) \end{matrix}\right]$$

provided $h > 0$, $\lambda > 0$, $|\arg\alpha| < \frac{1}{2}\lambda\pi$, $\mu \leqslant 0$, $\text{Re}(\gamma_i) > -1$, $\text{Re}(\gamma_i) + h \min_{1 \leqslant j \leqslant m} [\text{Re}(bj/Bj)] + 1 > 0$ $(i = 1, \ldots, r)$ and ν is positive integer. Taking limit as $\gamma_j \to 0$, the pdf formulated is

$$(3.3.2)\quad f(t_1,\ldots,t_r) = \frac{\prod_{j=1}^{r} [\,(1-t_j^2)^{\nu/2}\; P_k^{\nu}(t_j)\,]}{\left[\dfrac{(-1)^{\nu}\,\Gamma(1+\nu+k)}{2^{\nu+1}\Gamma(1-\nu+k)}\right]^r}$$

$$\times \frac{H^{m,n}_{p,q}\left[\alpha(t_r\ldots t_1)^h \,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}\{(1/2,h/2)\}_r, \{(o,h/2)\}_r, (a_p,A_p)\\ (b_q,B_q), \{(\frac{k-\nu}{2}, \frac{h}{2})\}_r, \{(-\frac{1-\nu-k}{2}, \frac{h}{2})\}_r\end{matrix}\right]}$$

Now taking pdf (3.3.2) and using the result (3.3.1), we get

$$(3.3.3)\quad \int_0^1 \cdots \int_0^1 \prod_{j=1}^{r} t_j^{\gamma_j}\, f(t_1,\ldots,t_r)dt_1\ldots dt_r$$

$$= \frac{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}(\frac{1-\gamma_r}{2}, \frac{h}{2}), (-\frac{\gamma_r}{2}, \frac{h}{2}), (a_p,A_p)\\ (b_q,B_q), (-\frac{\gamma_r+k-\nu}{2}, \frac{h}{2}), (-\frac{1-\gamma_r-\nu-k}{2}, \frac{h}{2})\end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}\{(1/2,h/2)\}_r, \{(o,h/2)\}_r, (a_p,A_p)\\ (b_q,B_q), \{(\frac{k-\nu}{2}, \frac{h}{2})\}_r, \{(-\frac{1-\nu-k}{2}, \frac{h}{2})\}_r\end{matrix}\right]};$$

which gives after substituting $\gamma_j = k_j (j=1,\ldots,r)$ the K^{th} moment as

$$(3.3.4)\quad \frac{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \left| \begin{matrix} (\frac{1-k_r}{2},\frac{h}{2}), (\frac{-k_r}{2},\frac{h}{2}), (a_p,A_p) \\ (b_q,B_q), (\frac{-k_r+k-\nu}{2},\frac{h}{2}), (\frac{-1-k_r-\nu-k}{2},\frac{h}{2}) \end{matrix}\right.\right]}{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \left| \begin{matrix} \{(1/2,h/2)\}_r, \{(o,h/2)\}_r, (a_p,A_p) \\ (b_q,B_q), \{(\frac{k-\nu}{2},\frac{h}{2})\}_r, \{(\frac{-1-\nu-k}{2},\frac{h}{2})\}_r \end{matrix}\right.\right]}.$$

The conditions of validity of this result are same as in (3.3.1) with γ_j replaced by k_j .

Example : In the pdf given by (3.3.2), let $m=1$, $n=0$, $p=1$, $q = 1$, $A_1 = 1$, $B_1 = 1$, the H-function in numerator becomes

$$H^{1,0}_{1,1}\left[\alpha(t_r\ldots t_1)^h \left| \begin{matrix} (a_1,1) \\ (b_1,1) \end{matrix}\right.\right]$$

$$= G^{1,0}_{1,1}\left[\alpha(t_r\ldots t_1)^h \left| \begin{matrix} a_1 \\ b_1 \end{matrix}\right.\right]$$

$$= \frac{\alpha^{b_1}(t_r\ldots t_1)^{hb_1}[1-\alpha(t_r\ldots t_1)^h]^{-1-b_1+a_1}}{\Gamma(a_1-b_1)}$$

Therefore the pdf can be formulated as

$$(3.3.5)\quad f(t_1,\ldots,t_r) = \frac{\prod_{j=1}^{r}\left[(1-t_j^2)^{\nu/2} P_k^{\nu}(t_j)\right]}{\Gamma(a_1-b_1)\left[\frac{(-1)^{\nu}\Gamma(1+\nu+k)}{2^{\nu+1}\Gamma(1-\nu+k)}\right]^r}$$

$$\times \frac{\alpha^{b_1}(t_r\ldots t_1)^{hb_1}[1-\alpha(t_r\ldots t_1)^h]^{-1-b_1+a_1}}{H^{1,2r}_{1+2r,1+2r}\left[\alpha \left| \begin{matrix} \{(1/2,h/2)\}_r, \{(o,h/2)\}_r, (a_1,1) \\ (b_1,1), \{(\frac{k-\nu}{2},\frac{h}{2})\}_r, \{(\frac{-1-\nu-k}{2},\frac{h}{2})\}_r \end{matrix}\right.\right]},$$

for $o \leqslant t_j \leqslant 1$

$= 0$ otherwise.

If $\kappa = k_1 \ldots + k_r$, the joint κ^{th} moment will be given by

$$(3.3.6)\quad \frac{H^{1,2r}_{1+2r,1+2r}\left[\alpha \,\middle|\, \begin{matrix} (\frac{1-k_r}{2},\frac{h}{2}), & (\frac{-k_r}{2},\frac{h}{2}), & (a_1,1) \\ (b_1,1), & (\frac{-k_r+k-\nu}{2},\frac{h}{2}), & (\frac{-1-k_r-\nu-k}{2},\frac{h}{2}) \end{matrix}\right]}{H^{1,2r}_{1+2r,1+2r}\left[\alpha \,\middle|\, \begin{matrix} \{(1/2,\, h/2)\}_r, \{(0,h/2)\}_r, (a_1,1) \\ (b_1,1), \{(\frac{k-\nu}{2},\frac{h}{2})\}_r \{(-\frac{1-\nu-k}{2},\frac{h}{2})\}_r \end{matrix}\right]}$$

provided $(k_i+1) + h\,Re(b_1) > 0$ $(i = 1, \ldots, r)$.

3.4 By considering the integral (1.2.8) we are in a position to formulate the desired probability density function.

From (1.2.8) after taking limit as $\gamma_j \to 1$ $(j = 1,\ldots,r)$, the pdf formulated is

$$(3.4.1)\quad f(t_1,\ldots,t_r)$$

$$= \frac{\prod_{j=1}^{r} [(1-t_j)^{-1/2}\, Tn_j(2t_j-1)]\, H^{m,n}_{p,q}\left[\alpha(t_1\ldots t_r)^h \,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]}{\sqrt{\pi}\; H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \,\middle|\, \begin{matrix}\{(0,h)\}_r, \{(-1/2,h)\}_r, (a_p,A_p) \\ (b_q,B_q), (-1/2-n_r,h), (-1/2+n_r,h)\end{matrix}\right]}$$

Now taking the pdf (3.4.1) and using (1.2.8) we get

$$(3.4.2)\quad \int_0^1 \cdots \int_0^1 \prod_{j=1}^{r} [\, t_j^{\gamma_j-1}]\, f(t_1,\ldots,t_r)dt_1\ldots dt_r$$

$$= \frac{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \left| \begin{matrix} (1-\gamma_r,h), (1/2-\gamma_r,h), (a_p,A_p) \\ (b_q,B_q), (1/2-\gamma_r-n_r,h), (1/2-\gamma_r+n_r,h) \end{matrix}\right.\right]}{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \left| \begin{matrix} \{(o,h)\}_r, \{(-1/2,h)\}_r, (a_p,A_p) \\ (b_q,B_q), (-1/2-n_r,h), (-1/2+n_r,h) \end{matrix}\right.\right]}$$

This gives after substituting $k_j = \gamma_j - 1$, the joint k^{th} moment as

$$(3.4.3) = \frac{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \left| \begin{matrix} (-k_r,h), (-1/2-k_r,h), (a_p,A_p) \\ (b_q,B_q), (-1/2-k_r-n_r,h), (-1/2-k_r+n_r,h) \end{matrix}\right.\right]}{H^{m,n+2r}_{p+2r,q+2r}\left[\alpha \left| \begin{matrix} \{(o,h)\}_r, \{(-1/2,h)\}_r, (a_p,A_p) \\ (b_q,B_q), (-1/2-n_r,h), (-1/2+n_r,h) \end{matrix}\right.\right]}$$

The conditions of validity of this result are same as (1.2.8) with γ_j replaced by $k_j + 1$ $(j=1,\ldots,r)$.

Example : In (3.4.1) if we take $m=1$, $n=0$, $p=1$, $q=1$, $A_1=1$, $B_1=1$, the H-function in numerator becomes

$$H^{1,0}_{1,1}\left[\alpha(t_1\ldots t_r)^h \left| \begin{matrix} (a_1,1) \\ (b_1,1) \end{matrix}\right.\right] = G^{1,0}_{1,1}\left[\alpha\,(t_1\ldots t_r)^h \left| \begin{matrix} a_1 \\ b_1 \end{matrix}\right.\right]$$

$$= \frac{\alpha^{b_1}(t_r\ldots t_1)^{hb_1}\left[1-\alpha(t_r\ldots t_1)^h\right]^{-1-b_1+a_1}}{\Gamma(a_1 - b_1)}$$

Therefore the pdf can be formulated as

(3.4.4) $f(t_1,\dots,t_r) =$

$$= \frac{\prod_{j=1}^{r}[(1-t_j)^{-1/2} T_{nj}(2t_j-1)]\alpha^{b_1}(t_r\dots t_1)^{hb_1}[1-\alpha(t_r\dots t_1)^h]^{-1-b_1+a_1}}{\sqrt{\pi}\; H^{1,2r}_{1+2r,1+2r}\left[\alpha \,\middle|\, \begin{matrix}\{(o,h)\}_r,\{(-1/2,h)\}_r,\ (a_1,1)\\ (b_1,1),\ (-1/2-n_r,h),\ (-1/2+n_r,h)\end{matrix}\right]}$$

for $o \leq t_j \leq 1$

= O otherwise.

If $K = K_1 + \dots + K_r$, the joint K^{th} moment will be given by

$$(3.4.5)\quad \frac{H^{1,2r}_{1+2r,1+2r}\left[\alpha \,\middle|\, \begin{matrix}(-k_r,h),\ (-1/2-k_r,h),\ (a_1,1)\\ (b_1,1),\ (-1/2-k_r-n_r,h),\ (-1/2-k_r+n_r,h)\end{matrix}\right]}{H^{1,2r}_{1+2r,1+2r}\left[\alpha \,\middle|\, \begin{matrix}\{(o,h)\}_r,\{(-1/2,h)\}_r,\ (a_1,1)\\ (b_1,1),\ (-1/2-n_r,h),\ (-1/2+n_r,h)\end{matrix}\right]}$$

with $\mathrm{Re}\,[(k_j + 1)+ hb_1] > -1,\ (i = 1,\dots r).$

3.5 Let us denote the integral (1.2.9) by I_1 and replace α by αt_2, multiply both the sides by $t_2^{\gamma_2} e^{-t_2} L_k^{(\sigma)}(t_2)$ and integrate with respect to t_2 from o to ∞ to get I_2. Proceeding in the same manner we get

$$(3.5.1)\quad I_r = \int_o^{\infty}\dots\int_o^{\infty} \prod_{j=1}^{r}[t_j^{\gamma_j} e^{-t_j} L_k^{(\sigma)}(t_j)] H^{m,n}_{p,q}\left[\alpha\, t_r\dots t_1 \,\middle|\, \begin{matrix}(a_p,A_p)\\ (b_q,B_q)\end{matrix}\right] dt_1\dots dt_r$$

$$= \left[\frac{(-1)^k}{k!}\right]^r H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}(-\gamma_r,1),\ (\sigma-\gamma_r,1),\ (a_p,A_p)\\ (b_q,B_q),\ (\sigma-\gamma_r+k,1)\end{matrix}\right]$$

where $\mu \leq 0$, and $\lambda > 0$, $|\arg\alpha| < \frac{\lambda\pi}{2}$,

$Re(\gamma_j + \frac{b_h}{B_h}) > -1$, $(h=1,\ldots,m)$, $(j=1,\ldots,r)$.

Hence from (3.5.1) on taking the limit as $\gamma_j \to 0$ the required pdf is

$$(3.5.2)\quad f(t_1,\ldots,t_r) = \frac{\prod_{j=1}^{r}[\, e^{-t_j} L_k^{(\sigma)}(t_j)\,]}{[\frac{(-1)^k}{k!}]^r}$$

$$\times \frac{H^{m,n}_{p,q}\left[\alpha t_r \ldots t_1 \,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}\{(o,1)\}_r, \{(\sigma,1)\}_r\ (a_p,A_p)\\(b_q,B_q), \{(\sigma+k,1)\}_r\end{matrix}\right]}$$

for $t_j \geq o$ $(i=1,\ldots r)$

= otherwise.

Now taking pdf (3.5.2) and using the result (3.5.1) we have

$$(3.5.3)\quad \int_0^\infty \ldots \int_0^\infty \prod_{j=1}^{r} t_j^{\gamma_j} f(t_1,\ldots,t_r)dt_1\ldots dt_r$$

$$\frac{H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}(-\gamma_r,1),\ (\sigma-\gamma_r,1),\ (a_p,A_p)\\(b_q,B_q),\ (\sigma-\gamma_r+k,1)\end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}\{(o,1)\}_r, \{(\sigma,1)\}_r, (a_p,A_p)\\(b_q,B_q), \{(\sigma+k,1)\}_r\end{matrix}\right]}$$

This gives after substituting $\gamma_j = k_j$ the joint K^{th} moment in the form

$$(3.5.4)\quad \frac{H^{m,n+2r}_{p+2r,q+r}\left[\alpha\,\middle|\,\begin{matrix}(-k_r,1)\,,\;(\sigma-k_r,1)\,,\;(a_p,A_p)\\(b_q,B_q),\;(\sigma-k_r+k,1)\end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+r}\left[\alpha\,\middle|\,\begin{matrix}\{(o,1)\}_r,\{(\sigma,1)\}_r,\;(a_p,A_p)\\(b_q,B_q),\;\{(\sigma+k,1)\}_r\end{matrix}\right]}$$

The conditions under which this result is valid are same as given in (3.5.1) with γ_j replaced by k_j .

Example 1 : In the pdf given by (3.5.2) let

$m=1,\ n=0,\ p=0,\ B_1 = 1,\ r = 2.$

The H=function in the numarator reduces to

$$H^{1,0}_{0,1}\left[\alpha t_1 t_2 \,|\,(b_1,1)\right] = (\alpha t_1 t_2)^{b_1}\, e^{-\alpha t_1 t_2}$$

So that the bivariate pdf can be formulated as

$$(3.5.5)\quad f(t_1,t_2) = \frac{(k!)^2 e^{-(t_1+t_2+\alpha t_1 t_2)}\,\alpha^{b_1} t_1^{b_1} t_2^{b_1} L_k^{(\sigma)}(t_1) L_k^{(\sigma)}(t_2)}{G^{1,4}_{4,3}\left[\alpha\,\middle|\,\begin{matrix}\{o\}_2,\;\{\sigma\}_2\\b_1,\;\{\sigma+k\}_2\end{matrix}\right]}\qquad \text{for } t_1,t_2 > 0$$

$= 0$ otherwise.

The k^{th} moment is given as

$$(3.5.6)\quad \frac{G^{1,4}_{4,3}\left[\alpha\,\middle|\,\begin{matrix}-k_1,\;-k_2,\;\sigma-k_1,\;\sigma-k_2\\b_1;\;\sigma-k_1+k,\;\sigma-k_2+k\end{matrix}\right]}{G^{1,4}_{4,3}\left[\alpha\,\middle|\,\begin{matrix}\{o\}_2,\;\{\sigma\}_2\\b_1,\;\{\sigma+k\}_2\end{matrix}\right]}\,.$$

Example 2 : In the pdf given by (3.5.2), take

$$m = 1,\ n = 0,\ p = 0,\ q = 1,\ r = 1$$

The H-function in the numerator reduces to

$$H^{1,0}_{0,1}\left[\alpha t_1 \,\middle|\, {}^{-}_{(b_1,1)}\right] = (\alpha t_1)^{b_1} e^{-\alpha t_1} .$$

So that the pdf can be formulated as

$$(3.5.7)\quad f(t_1) = \frac{(-1)^k k!\, e^{-(1+\alpha)t_1} (\alpha t_1)^{b_1} \alpha^{b_1} L_k^{(\sigma)}(t_1)}{G^{1,2}_{2,2}\left[\alpha \,\middle|\, {}^{o,\ \sigma}_{b_1,\sigma+k}\right]} \quad \text{for } t_1 > 0$$

$$= 0 \text{ otherwise.}$$

The moment is given by

$$(3.5.8)\quad \frac{G^{1,2}_{2,2}\left[\alpha \,\middle|\, {}^{-k_1,\ \sigma-k_1}_{b_1,\ \sigma-k_1+k}\right]}{G^{1,2}_{2,2}\left[\alpha \,\middle|\, {}^{o,\ \sigma}_{b_1,\sigma+k}\right]} .$$

3.6 Let us denote the integral (1.2.10) by I_1, multiply both the sides by $e^{-t_2} t_2^{\gamma_2 - 1} Y_n(1,a;t_2)$, replace α by αt_2 and integrate with respect to t_2 from 0 to ∞ to get I_2. Repeating the same process (r-1) times we get

$$(3.6.1)\quad I_r = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} \left[e^{-t_j} t_j^{\gamma_j - 1} Y_n(1,a;t_j) \right.$$

$$\left. \times H^{m,n}_{p,q}\left[\alpha t_r \cdots t_1 \,\middle|\, {}^{(a_p,A_p)}_{(b_q,B_q)}\right] \right] dt_1 \cdots dt_r .$$

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$$= H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}(1+n-\gamma_r,1), & (2-\gamma_r-n-a,1), & (a_p,A_p) \\ (b_q,B_q), & (2-\gamma_r-a,1) & \end{matrix}\right]$$

where. $Re(\gamma_i+\min b_j/B_j) > 0$ $(j=1,\ldots,m)(i=1,\ldots,r)$

$$|\arg\alpha| < \frac{\lambda\pi}{2}, \quad \lambda > 0, \quad \mu \leqslant 0 .$$

Taking limit as $\gamma_j \to 1$, the pdf formulated is

$$(3.6.2) \quad f(t_1,\ldots,t_r)$$

$$= \frac{\prod_{j=1}^{r} [e^{-t_j} Y_n(1,a;t_j)] \, H^{m,n}_{p,q} \left[\alpha t_r \cdots t_1 \,\middle|\, \begin{matrix}(a_p,A_p) \\ (b_q,B_q)\end{matrix} \right]}{H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}\{(n,1)\}_r, & \{(1-n-a,1)\}_r, & (a_p,A_p) \\ (b_q,B_q), & \{(1-a,1)\}_r & \end{matrix}\right]}$$

for $t_j \geqslant 0$

$= 0$ otherwise.

Now taking pdf (3.6.2) and using (3.6.1) we have

$$(3.6.3) \quad \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} t_j^{\gamma_j - 1} f(t_1,\ldots,t_r)dt_1\ldots dt_r$$

$$= \frac{H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}(1+n-\gamma_r,1), & (2-\gamma_r-n-a,1), & (a_p,A_p) \\ (b_q,B_q), & (2-\gamma_r-a,1) & \end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}\{(n,1)\}_r, & \{(1-n-a,1)\}_r, & (a_p,A_p) \\ (b_q,B_q), & \{(1-a,1)\}_r & \end{matrix}\right]}$$

This gives after substituting $\gamma_j = k_j + 1$ the joint k^{th} moment as

$$(3.6.4)\quad \frac{H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}(n-k_r,1)\ ,\ (1-k_r-n-a,1)\ ,\ (a_p,A_p)\\ (b_q,B_q),\ \ (1-k_r-a,1)\end{matrix}\right]}{H^{m,n+2r}_{p+2r,q+r}\left[\alpha \,\middle|\, \begin{matrix}\{(n,1)\}_r,\ \{(1-n-a,1)\}_r,\ (a_p,A_p)\\ (b_q,B_q),\ \{(1-a,1)\}_r\end{matrix}\right]}$$

The conditions under which (3.6.4) is valid are same as that of (3.6.1) with γ_j replaced by $k_j + 1$.

Example 1 : In the pdf given by (3.6.2) let $r = 2$, $m = 1$, $n = 0$, $p = 0$, $q = 1$, $B_1 = 1$, then the H-function in numerator becomes

$$H^{1,0}_{0,1}\left[\alpha\, t_1 t_2 \middle| (b_1,1)\right] = (\alpha t_1 t_2)^{b_1} e^{-\alpha t_1 t_2},$$

so that the bivariate pdf is given by

$$(3.6.5)\quad f(t_1,t_2) = \frac{\alpha^{b_1}(t_1 t_2)^{b_1} e^{-(t_1+t_2+\alpha t_1 t_2)}\, Y_n(1,a,t_1)\, Y_n(1,a;t_2)}{G^{1,4}_{4,3}\left[\alpha \,\middle|\, \begin{matrix}\{n\}_2\ ,\ \{1-n-a\}_2\\ b_1,\ \ \{1-a\}_2\end{matrix}\right]} \quad \text{for } t_1,\ t_2 > 0$$

$$= 0 \quad \text{otherwise.}$$

From (3.6.4) the k^{th} moment is given as

$$(3.6.6)\quad \frac{G^{1,4}_{4,3}\left[\alpha \,\middle|\, \begin{matrix}n-k_1,\ n-k_2,\ 1-k_1-n-a,\ 1-k_2-n-a\\ b_1,\ 1-k_1-a,\ \ 1-k_2-a\end{matrix}\right]}{G^{1,4}_{4,3}\left[\alpha \,\middle|\, \begin{matrix}\{n\}_2,\ \{1-n-a\}_2\\ b_1,\ \{1-a\}_2\end{matrix}\right]}.$$

Example 2 : Let $r = 1$, $m = 1$, $n = 0$, $p = 0$, $q = 1$, $B_1 = 1$.

From (3.6.4)

$$(3.6.7)\quad f(t_1) = \frac{e^{-t_1(1+\alpha)}(\alpha t_1)^{b_1} Y_n(1,,a;t_1)}{G^{1,2}_{2,2}\left[\alpha \middle| \begin{matrix} n, & 1-n-a \\ b_1, & 1-a \end{matrix}\right]} \quad \text{for } t_1 > 0$$

$$= 0 \text{ otherwise.}$$

And the moment is given by

$$(3.6.8)\quad \frac{G^{1,2}_{2,2}\left[\alpha \middle| \begin{matrix} n-k_1, & 1-k_1-n-a \\ b_1, & 1-k_1-a \end{matrix}\right]}{G^{1,2}_{2,2}\left[\alpha \middle| \begin{matrix} n, & 1-n-a \\ b_1, & 1-a \end{matrix}\right]}.$$

3.7 By direct evaluation we get

$$(3.7.1)\quad I_1 = \int_0^\infty t_1^{\gamma_1-1} e^{-t_1} H^{m,n}_{p,q}\left[\alpha t_1^h \middle| \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix}\right] dt_1$$

$$= H^{m,n+1}_{p+1,q}\left[\alpha \middle| \begin{matrix} (1-\gamma_1,h), (a_p,A_p) \\ (b_q, B_q) \end{matrix}\right]$$

provided $\gamma_1 + \min \operatorname{Re}(b_i/B_i) > 0$, $(i=1,\ldots,m)$ and

$\mu \equiv \sum_{j=1}^{p} A_j - \sum_{j=1}^{q} B_j \leq 0$, h is positive number.

Now replace α by αt_2^h in (3.7.1), multiply both the sides by $t_2^{\gamma_2-1} e^{-t_2}$ and integrate with respect to t_2 from 0 to ∞ to get I_2. Proceeding in this way we get

$$(3.7.2)\quad I_r = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} \{ t_j^{\gamma_j - 1} e^{-t_j} \} H_{p,q}^{m,n} \left[\alpha (t_r \cdots t_1)^h \,\middle|\, \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] dt_1 \cdots dt_r$$

$$= H_{p+r,q}^{m,n+r} \left[\alpha \,\middle|\, \begin{matrix} (1-\gamma_r, h) \;, (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right]$$

provided $\gamma_j + \min \operatorname{Re}(b_i/B_i) > 0$, $(i=1, \ldots, m)(j=1, \ldots, r)$

$\mu \leqslant 0$, h positive number.

Hence from (3.7.2) we formulate the pdf as

$$(3.7.3)\quad f(t_1, \ldots, t_r) = \frac{\prod_{j=1}^{r} [t_j^{\gamma_j - 1} e^{-t_j}] \, H_{p,q}^{m,n} \left[\alpha (t_r \cdots t_1)^h \,\middle|\, \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right]}{H_{p+r,q}^{m,n+r} \left[\alpha \,\middle|\, \begin{matrix} (1-\gamma_r, h) \;, (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right]}$$

for $t_j > 0$, $(j=1, \ldots, r)$

$= 0$ otherwise.

The characteristic function for pdf (3.7.3), by using (3.7.2) is given by

$$(3.7.4)\quad \phi(w_1, \ldots, w_r) = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} e^{i w_j t_j} f(t_1, \ldots, t_r) dt_1 \cdots dt_r$$

$$= \frac{H_{p+r,q}^{m,n+r} \left[\dfrac{\alpha}{\prod_{j=1}^{r} (1 - i w_j)^h} \,\middle|\, \begin{matrix} (1-\gamma_2, h) \;, (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right]}{\prod_{j=1}^{r} [1 - i w_j]^{\gamma_j} H_{p+r,q}^{m,n+r} \left[\alpha \,\middle|\, \begin{matrix} (1-\gamma_r, h), (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right]}$$

Example 1 : In the pdf given by (3.7.3), let

$$r = 2,\ m = 1,\ n = 0,\ p = 0,\ q = 1,\ B_1 = 1 .$$

The H-function in numerator becomes

$$H^{1,0}_{0,1}\left[\ \alpha(t_1t_2)^h \Big|_{(b_1,1)}^{-}\ \right] = \alpha^{b_1}(t_1t_2)^{hb_1} e^{-\alpha(t_1t_2)^h}$$

So that we have the birariate pdf :

$$(3.7.5)\quad f(t_1,t_2) = \frac{\alpha^{b_1} t_1^{\gamma_1+hb_1-1}\ t_2^{\gamma_2+hb_1-1}\ e^{-(t_1+t_2+\alpha t_1^h t_2^h)}}{H^{1,2}_{2,1}\left[\alpha \Big|^{(1-\gamma_1,h),\ (1-\gamma_2,h)}_{(b_1,1\ )}\right]} \quad \text{for } t_1,\ t_2 > 0$$

$$= 0 \ \text{ otherwise.}$$

And the characteristic function is

$$(3.7.6)\quad \phi(w_1,w_2) = \frac{H^{1,2}_{2,1}\left[\frac{\alpha}{(1-iw_1)^h(1-iw_2)^h} \Big|^{(1-\gamma_1,h),(1-\gamma_2,h)}_{(b_1,1)}\right]}{[1-iw_1]^{\gamma_1}[1-iw_2]^{\gamma_2}\ H^{1,2}_{2,1}\left[\ \alpha \ ^{(1-\gamma_1,h),(1-\gamma_2,h)}_{(b_1,1)}\right]}$$

Example 2 : Let $r = 1,\ m = 1,\ n = 0,\ p = 0,\ q = 1,\ B_1 = 1$.

The H-function in numerator becomes

$$H^{1,0}_{0,1}\left[\ \alpha t_1^h \Big|_{(b_1,1)}^{-}\right] = \alpha^{b_1} t_1^{hb_1} e^{-\alpha t_1^h} ,$$

so that we have the pdf

$$(3.7.7)\quad f(t_1) = \frac{t^{\gamma_1-1} e^{-t_1} \alpha^{b_1} t_1^{hb_1} e^{-\alpha t_1^h}}{H^{1,1}_{1,1}\left[\alpha \,\middle|\, \begin{matrix}(1-\gamma_1,h)\\(b_1,1)\end{matrix}\right]} .$$

And the characteristic function is

$$(3.7.8)\quad \phi(w_1) = \frac{H^{1,1}_{1,1}\left[\frac{\alpha}{(1-iw_1)^h} \,\middle|\, \begin{matrix}(1-\gamma_1,h)\\(b_1,1)\end{matrix}\right]}{[1-iw_1]^h\, H^{1,1}_{1,1}\left[\alpha \,\middle|\, \begin{matrix}(1-\gamma_1,h)\\(b_1,1)\end{matrix}\right]}$$

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