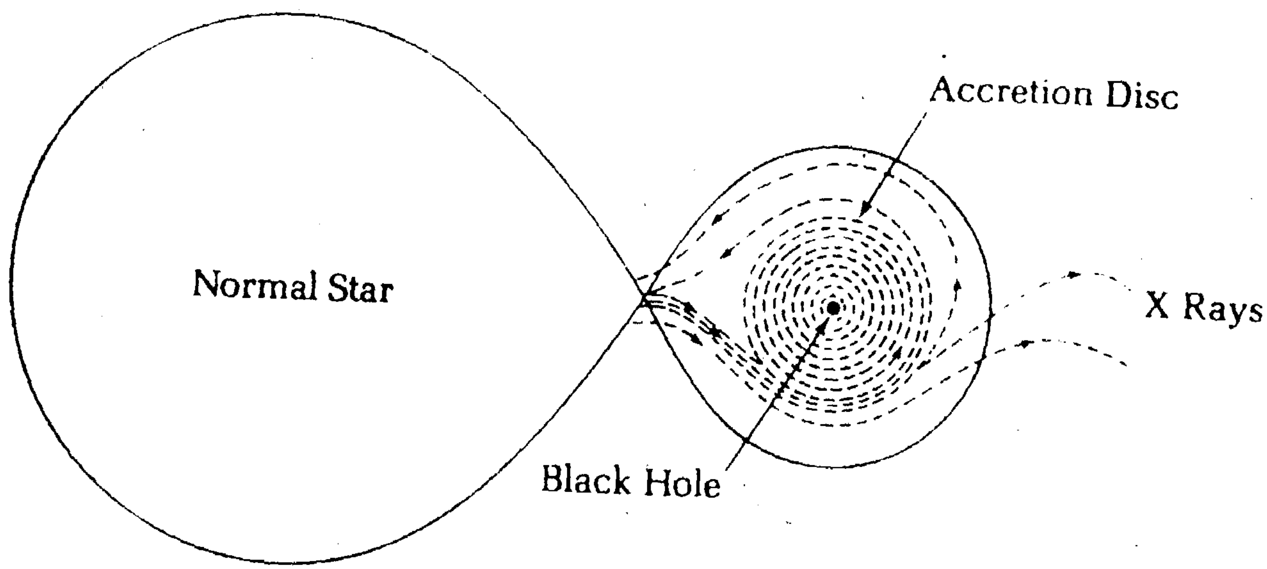
C H A P T E R - 2

SPECIAL GEOMETRICAL SYMMETRIES OF KERR-NEWMAN BLACK HOLE

The black hole as 'Experimental Model' for gravitational collapse, brings us back full-circle to the paradox that continually confronts us, and all science, the paradox of big-bang and gravitational collapse of the universe itself.

... J.A.Wheeler



Artist's impression of a double star system containing a black hole. Many believe such a system exists at the location of the X-ray source Cygnus X-1. Figure shows that the matter falling into the black hole forms a disc round it, called the 'accretion disc'.

Introduction :

In this chapter we enumerate the different types of geometrical symmetries in n-dimensional differential manifolds. Later we confine to the 4-dimensional space-time manifold of the Kerr-Newman Black Hole and study the strongest null symmetry viz., Killing equations for the null congruencew. The hypersurface on which $\pounds_{\underline{l}} F_{ab} = 0$ is determined. It is shown that $\underline{l}^a$ cannot generate the dynamical symmetry $\pounds_{\underline{l}} T'_{ab} = 0$ in the last section. The nonexistence of $\underline{l}^a$ It is found that the null vectors of the NP tetrad do not form Killing rays.

1. The Lie-derivative :

The Lie-derivative is an operator for comparing tensors in two different regions of space time manifolds, whenever there is a 'diffeomorphism' between the two regions (Pirane,1964).

Definition : The Lie derivative of a general tensor field $\Omega^{..i.}_{.j..}$ with respect to the vector field v^t is given by

$$\pounds_{\underline{v}} \Omega^{..i.}_{.j..} = \Omega^{..i.}_{.j..;k} v^k - \Omega^{..k.}_{.j..} v^i_{;k} + \dots + \\ + \Omega^{..i.}_{.k..} v^k_{;j} + \dots , \qquad \dots(1.1)$$

The Lie derivative has the following properties :

(a) $\pounds_{\underline{v}}$ does not change the form of the tensor i.e., if $\Omega^{..i.}_{.j..}$ is a tensor field of type (r,s) then $\pounds_v \Omega^{..i.}_{.j..}$ is also a tensor field of the same type (r,s).

(b) $\pounds_{\underline{v}}$ transforms tensors linearly and preserves contractions

i.e., $\pounds_{\underline{v}}(\alpha\underline{v} + \beta\underline{w}) = \alpha\pounds_{\underline{v}}\underline{v} + \beta\pounds_{\underline{v}}\underline{w}$.

where $\underline{v}$, $\underline{w}$ are vectors and α, β are numbers.

(c) $\pounds_{v}$ obeys Leibnetz's rule :

$\pounds_{\underline{v}}(\underline{v}\otimes\underline{w}) = (\pounds_{\underline{v}}\underline{V}) \otimes \underline{W} + \underline{V} \otimes (\pounds_{\underline{v}}\underline{W})$.

where the symbol $\otimes$ denotes the tensor product.

(d) $\pounds_{v} f = \underline{v} f$, for any function f .

Note : The spectacular feature of Lie derivative operator is that it is independent of the affine connection of the space-time manifold.

(In other words in the expression for Lie derivative semicolons representing covariant derivatives may be replaced by commas representing partial derivatives).

2. Geometrical Symmetry :

The set of all infinitesimal point transformation

$$x^{k'} = x^{k} + v^{k}\delta t \qquad \ldots (2.1)$$

forms a continuous group under the binary operation of composition of transformation. These transformations in due upto first order in δt the following transformation on the metric tensor, the Christoffel symbols and the curvature-tensor respectively,

$$g'_{ab}(x') = g_{ab}(x) + (\pounds_{\underline{v}} g_{ab})\delta t , \qquad \ldots (2.2)$$

$$\{b{}^a{}c\}' = \{b{}^a{}c\} + (\pounds_{\underline{v}} \{b{}^a{}c\})\,\delta t\,, \qquad \ldots (2.3)$$

$$R'{}^{\cdot\cdot\cdot d}_{abc\cdot} = R^{\cdot\cdot\cdot d}_{abc\cdot} + (\pounds_{\underline{v}} R^{\cdot\cdot\cdot d}_{abc\cdot})\,\delta t, \qquad \ldots (2.4)$$

Here a, b, c, d range over $1 \ldots n$,

<u>Definition</u> :

A tensor field $\Omega^{\cdot\cdot i\cdot}_{\cdot j\cdot\cdot}$ which is a function of the metric tensor g_{ij} is said to exhibit geometrical symmetry if it is functionally form invariant,

i.e., $\Omega^{\cdot\cdot i\cdot}_{\cdot j\cdot\cdot}(x'^k) = \Omega^{\cdot\cdot i\cdot}_{\cdot j\cdot\cdot}(x^k)$

with respect to infinitesimal point transformation

$$x'^k = x^k + v^k \delta t\ .$$

where v^a is a function of coordinates x^a .

For instance we get three symmetries from (2.2), (2.3) and (2.4)

(i) $g_{ab}(x^k) = g'_{ab}(x'^k)$ or equivalently $\pounds_v g_{ab} = 0$.

(ii) $\{b{}^a{}c\}(x^k) = \{b{}^a{}c\}'(x'^k)$ or equivalently $\pounds_v \{b{}^a{}c\} = 0$

(iii) $R_{abc}{}^d(x^k) = R'{}_{abc}{}^d(x'^k)$ or equivalently $\pounds_v R_{abc}{}^d = 0$.

The three symmetries are not independent, for instance (i) implies (ii) and (ii) implies (iii). This is the reason for calling (i) as the strongest symmetry. However (iii) need not imply (ii). The case when (iii) is satisfied and (ii), (i) are not valid will be referred as 'proper' symmetry of the curvature tensor.

3. Sixteen geometrical symmetries :

Davis (1974) gave specific names to these symmetries under the title collineations and motions. All the sixteen are listed below :

1. Weyl projective collineation (WPC) : $\pounds_v W_{abc}{}^d = 0$.
2. Projective collineation (PC) : $\pounds_v \left\{ {a \atop bc} \right\} = \delta^a_b \phi_{;c} + \delta^a_c \phi_{;b}$.
3. Special projective collineation (SPC) :

$$\pounds_v \left\{ {a \atop bc} \right\} = \delta^a_b \phi_{;c} + \delta^a_c \phi_{;b} \ , \ \phi_{;a} = 0 \ .$$

4. Ricci collineation (RC) : $\pounds_v R_{ab} = 0$.
5. Curvature collineation (CC) : $\pounds_v R_{abc}{}^d = 0$.
6. Special curvature collineation (SCE) $\left(\pounds_v \left\{ b^a{}_c \right\} \right)_{;d} = 0$.
7. Affine collineation (AC) : $\pounds_v \left\{ b^a{}_c \right\} = 0$.
8. Homothetic motion (HM) : $\pounds_v g_{ab} = 2hg_{ab}$, h = constant.
9. Motions (M) : $\pounds_v g_{ab} = 0$.
10. Special conformal collineation (S conf C) :

$$\pounds_v \left\{ b^a{}_c \right\} = \delta^a_b h_{;c} + \delta^a_c h_{;b} - g_{bc} g^{ad} h_{;d} , \ h_{,a;b} = 0 \ .$$

11. Special conformal motion (S conf M) :

$$\pounds_v g_{ab} = 2\ hg_{ab} \ , \quad h, \quad a;b = 0 \ .$$

12. Weyl conformal collineation (W conf C) : $\pounds_v C_{abc}{}^d = 0$.
13. Conformal collineation (conf C) :

$$\pounds_v \left\{ b^a{}_c \right\} = \delta^a_b h_{;c} + \delta^a_c h_{;b} - g_{bc} g^{ad} h_{;d} \ .$$

14. Conformal motion (conf M) : $\pounds_v g_{ab} = 2h g_{ab}$.

15. Null geodesic collineation (NC) :

$$\pounds_v \left\{ {}_b{}^a{}_c \right\} = g_{bc} g^{ad} \psi_{;d} \ .$$

16. Specific Null geodesic collineation (SNC) :

$$\pounds_v \left\{ {}_b{}^a{}_c \right\} = g_{bc} g^{ad} \psi_{;d} \ , \quad \psi_{;b;c} = 0 \ .$$

Here h, ψ are arbitrary and signature of the metric tensor g_{ab} is (- - - +).

4. Importance of Symmetries :

The first paper of the groups of motions in General Relativity appeared in 1925 (Eiesland). Ten conservation laws result due to the ten parameter group of motions in the special theory of relativity.

$$\pounds_{v(\alpha)} n_{ab} = 0 \ , \quad \alpha = 1,2 \ .... \ 10,$$

where n_{ab} is the Minkowski flat metric.

4 translations, 3 rotations and 3 hyperbolic rotations yield respectively the conservation of energy and linear momentum, angular momentum in the special Relativistic mechanics. Davis (1962). Homogeneous and isotropic space-times admit a 6 parameter group of motions (Ozsvath and Schucking, 1962). Spherically symmetric (and plane symmetric) space-times admit 3 parameter group of motions (Taub, 1951).

The different types of uses of the infinitesimal point transformation (2.1) can be described as follows :

(i) the preservation of distances;

(ii) the scaling of all distances by the same constant factor (self similar space-times);

(iii) the preservation of angles between two directions at a point;

(iv) the mapping of a geodesic into geodesic;

(v) the mapping of a null geodesic into a null geodesic;

(vi) the preservation of the affine parameters on geodesics;

(vii) the preservation of the null geodesic affine parameters.

The transformations exhibiting the utility (i) are motions (Yano, 1955); and (iii) and (iv) are conformal motions (Yano,1955 and Petrov 1969); (ii), (iii),(v) and (vii) are Homothetic motions (McIntosh 1976, 1980); (iv) are projective collineations (Katzin and Levine, 1972); (iv) and (vi) are Affine collineations (Katzin and Levine 1972).

5. Mathematical Technique :

We transcribe the tensor relations characterizing the geometrical symmetry of KNBH into conditions on Newman Penrose spin-coefficients and the dynamical variables characterizing KNBH. The efficiency of this technique is in generating not only the necessary conditions but also the sufficient restrictions. This explains the reason for calling the NP formalism as an "amazing" technique. As an illustration of the main mathematical technique used in this chapter we observe that :

(i) $A\underline{l}_a + Bn_a + Cm_a + C\bar{m}_a = 0$ iff $A = B = C = 0$.

since $\underline{l}^a$, m^a, n^a are linearly independent.

(ii) $A'\underline{l}_a m_b + B'\underline{l}_a n_b + C'\underline{l}_a \bar{m}_b + D'\underline{l}_a \underline{l}_b + E'm_a m_b + F'm_a \underline{l}_b \ldots = 0$

$$\Longleftrightarrow A' = B' = C' = D' = E' = F' = \ldots = 0 .$$

Note that $\underline{l}^a$, n^a, m^a, $\bar{m}^a$ is a basics of the 4-dimensional space-time and their linear independence is the cause of the inferences (i), (ii).

6. Symmetries of Kerr Newman Black Hole :

The metric characterizing KNBH is;

$$ds^2 = -\left(1 - \frac{2\,mr}{\Sigma}\right) dt^2 - \left(\frac{4\,mar\,\sin^2\theta}{\Sigma}\right) dt d\phi + \frac{\Sigma}{\Delta} dr^2 + $$
$$+ \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2\,m^2 ar\,\sin^2\theta}{\Sigma}\right) \sin^2\theta d\phi^2 \;. \quad \ldots (6.1)$$

where,

$a^2 + e^2 \leqslant m^2$

$m \equiv$ mass, $c \equiv$ charge

$a \equiv$ angular momentum per unit mass.

$\Delta \equiv r^2 - 2\,mr + a^2 + e^2$

$\Sigma = (r^2 + a^2\,(as^2\theta)^{-1}$.

The Two Trivial Geometrical Symmetries :

The metric coefficients are independent of t , ϕ. Hence

$k^a = (1,0,0,0)$ and $k^{*a} = (0,0,0,1)$; $a = 4,1,2,3$ ...(6.2)

are Killing fields i.e., $k_{a;b} + k_{b;a} = 0$.

These two vectors give the obivious geometrical symmetries

$$\pounds_k g_{ab} = 0, \quad \pounds_{k^*} g_{ab} = 0 .$$

From Wooley's result (1973)

$$\pounds_k F_{ab} = (k^c \Theta_c)^* F_{ab} ,$$

where Θ_c is the complexion vector of the non-null field.

For KNBH we have the following results due to (6.2),

(1) $\pounds_k F_{ab} = \Theta_4 {}^* F_{ab}$,

(2) $\pounds_{k^*} F_{ab} = \Theta_3 {}^* F_{ab}$.

since $k^4 \neq 0$, $k^{*3} \neq 0$.

<u>Some consequences of the symmetries</u> :

We observe that,

$$\begin{aligned} k^a k_a &= k^4 k_4 \\ &= k^4 (g_{4b} k^b) \\ &= k^4 g_{44} k^4 \\ &= g_{44} \text{ since } k^4 = 1 \end{aligned} \qquad \dots (6.3a)$$

similarly,

$$k^a k^*_a = g_{41} \qquad \dots (6.3b)$$

and

$$k^{*a} k^*_a = g_{33} . \qquad \dots (6.3c)$$

<u>Angular velocity</u> (Ω) :

For convenience we now introduce the notation

$$\Omega = \frac{d\phi}{dt} = \frac{d\phi/ds}{dt/ds} = \frac{\dot{\phi}}{\dot{t}}$$

$$= \frac{-(aE - Lz/\sin^2\theta) + a/\Delta\, P}{-a(aE\,\sin^2\theta - Lz) + \frac{r^2+a^2}{\Delta}\, P}$$

Lightman (1972) calls Ω as the "angular velocity" relative to a distant stationary observer.

4. <u>Velocity u^a</u> :

Suppose

$$u^a = \frac{k^a + \Omega \overset{*}{k}{}^a}{k^2 + \Omega \overset{*}{k}{}^2}$$

$$= \frac{(1,0,0,\ \Omega)}{\sqrt{g_{ab}(k^a+\Omega \overset{*}{k}{}^a)(k^b+\Omega \overset{*}{k}{}^b)}}$$

i.e., $u^4 = (1 + \Omega^2)^{-1/2}$,

$u^3 = (1 + \Omega^2)^{-1/2}$,

$u^1 = u^2 = 0$.

i.e., $u^a = (q,\ 0,\ 0,\ \Omega\, q)$, $\quad q = (1 + \Omega^2)^{-1/2}$.

7. <u>Stationary Observer and Killing Vector Field</u> :

The metric tensor g_{ij} is stationary Killing field if there exists a time like Killing field. An observer moving along a world line with constant r, θ and uniform angular velocity sees on unchanging space-time geometry and is thus a stationary observer. His angular velocity, measured at infinity is

$$\Omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{s}} = \frac{u^3}{u^4} ,$$

where u^a is the four velocity.

The four velocity of a stationary observer is proportional to a Killing field.

We consider a combination of k^a, k^{*a} the generators of the geometrical symmetry called motions.

$$u^a = k^a + \Omega\, k^{*a}$$

obviously $k^a + \Omega k^{*a}$ must be timelike

$$g_{44} + 2\,\Omega g_{41} + \Omega^2\, g_{33} = 0 .$$

The left hand side vanishes for

$$\Omega = \frac{-\,g_{41} \pm \sqrt{g_{41}{}^2 - g_{44}\, g_{33}}}{g_{33}}$$

Let $W = -\,g_{41}/g_{33}$; then

$$\Omega\ \text{min} = W - \sqrt{W^2 - g_{44}/g_{33}} .$$

Note that,

$$W = \frac{a(2mr - e^2)}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}$$

For an interpretation of W consider stationary observers who are non-rotating with respect to total freely falling test particles that have been dropped in radially from infinity. Since the angular momentum of such test particles vanishes, we have for these special stationary observers (socalled " Barden observers")

$$u^a k^*{}_a = 0 .$$

Hence,

$$(k^a + \Omega k^{*a})\ k^{*a} = 0 \qquad \text{i.e.,} \quad \Omega = W ,$$

we assume $a > o$. Obviously Ω Min = 0 if and only if $g_{44} = 0$

i.e., for $k^a k_a = 0$; $r = r_o(\Theta) = m + \sqrt{m^2 - e^2 - a^2 \cos^2 \Theta}$.

we assume that $m^2 > e^2 + a^2$; otherwise there exists a "naked singularity".

8. Static Observer :

An observer is said to be static (relative to the "fixed stars") if $\Omega = 0$, so that u is proportional to k^a. Static observers can exist only outside the static limit $r = r_o(\Theta)$. At the static limit, k^a becomes light like. An observer would then have to move at the speed of light in order to remain at rest with respect to the fixed stars.

We now consider the red shift which can asymptotic observer measures for light from a source at rest ($u = k^a / \| k_a \|$) outside the static limit.

Ergo surface and Killing Horizon :

A Killing horizon is a null hyper surface generated by a Killing vector. An ergosurface ("static limit") is an infinite red shift surface for static observers.

We quote Lightman's theorem :

" For a stalic Black Hole the ergosurface is a Killing horizon".

It follows that for a Schwarzschild black hole, where $r = 2m$ is both the horizon and infinite red-shift surface for static observers. (infact, for all observers). A Kerr black hole is an example of the case where the ergosurface does not coincide with the horizon. Thus we can conclude that the defining property of a Black Hole is that it should have a horizon, a surface through which matter can fall, but from which no matter or information can escape to infinity. For a Kerr black hole, it is located at r_{+} , the larger root of the equation $\Delta = 0$. The stationary limit of a rotating hole is the surface within which all observers are dragged around the hole. For Kerr, the stationary limit is at r_0, the larger root of $g_{44} = 0$. The region between the horizon and stationary limit is called the ergo sphere.

9. Special geometrical symmetries and the non-existence of $\underline{l}^a$ as null Killing vector for a black hole :

After considering the obvious geometrical symmetries generated by k^a, k^{*a} we explore the existence of special symmetries.

We have
$$\pounds_{\underline{l}}\, g_{ab} = \underline{l}_{a;b} + \underline{l}_{b;a} \qquad \ldots \quad (9.1)$$

Now, from the equation (Chapter 1, Appendix 1)

$$\underline{l}_{a;b} + \underline{l}_{b;a} = 2(\gamma+\bar{\gamma})\underline{l}_a\underline{l}_b - (\alpha+\bar{\beta}+\bar{\tau})\underline{l}_a m_b - (\alpha+\bar{\beta}+\bar{\tau}) m_a \underline{l}_b -$$
$$- (\bar{\alpha}+\beta+\tau)\underline{l}_a\bar{m}_b - (\bar{\alpha}+\beta+\tau)\bar{m}_a\underline{l}_b - (\rho+\bar{\rho})m_a\bar{m}_b + (\rho+\bar{\rho})\bar{m}_a m_b \qquad \ldots \quad (9.2)$$

It follows that,

$$\pounds_{\underline{l}} g_{ab} = 0 \quad \text{iff} \quad \rho + \bar{\rho} = \gamma + \bar{\gamma} = \alpha + \bar{\beta} + \bar{\tau} = 0$$

$$\underline{\text{OR}} \quad \text{Re}\ \rho = \text{Re}\ \gamma = \alpha + \bar{\beta} + \bar{\tau} = 0 .$$

Here 'Re' means Real part (...).

For the KNBH we have (vide chapter 1, 5.1)

$$\text{Re}\rho = \frac{-r}{r^2 + a^2\cos^2\theta}, \qquad \ldots (9.3)$$

and hence

$$\text{Re}\ \rho = 0 \Rightarrow r = 0 .$$

this restriction on the coordinate r is not tenable.

Hence $\pounds_{\underline{l}}\, g_{ab} = 0$, is not valid for KNBH .

Thus $\underline{l}^a$ is not a Killing ray and so $\underline{l}^a$ does not correspond to a geometrical symmetry for KNBH. Similarly n^a, m^a, $\bar{m}^a$ do not generate geometrical symmetries.

10. <u>Self similar distribution and their Incompatibility with Kerr-Newman Black Hole</u> :

We have the completeness relation (vide chapter 1; 3.4)

$$g_{ab} = \underline{l}_a n_b + n_a \underline{l}_b - m_a \bar{m}_b - \bar{m}_a m_b \quad \text{and}$$

self similar distributions are defined by

$$\pounds_{\underline{l}} g_{ab} = A\, g_{ab} \qquad \ldots (10.1)$$

where A is a non-zero scalar function.

The Newman-Penrose scalar version of equation for KNBH dueto (9.2) will be

5292
A

$$2(\gamma+\bar{\gamma})\underline{l}_a\underline{l}_b - (\alpha+\bar{\beta}+\bar{\tau})\underline{l}_a m_b - (\alpha+\bar{\beta}+\bar{\tau})m_a\underline{l}_b - (\bar{\alpha}+\beta+\tau)\underline{l}_a\bar{m}_b -$$
$$- (\bar{\alpha}+\beta+\tau)\bar{m}_a\underline{l}_b + (\rho+\bar{\rho})m_a\bar{m}_b + (\rho+\bar{\rho})\bar{m}_a m_b = K(\underline{l}_a n_b + n_a\underline{l}_b -$$
$$- m_a\bar{m}_b - \bar{m}_a m_b)$$

The equation holds good only when the coefficient of $\underline{l}_a n_b$ vanishes i.e., $A = 0$.

Thus, we conclude that,

$$\pounds_{\underline{l}} g_{ab} = A\, g_{ab} \text{ is not compatible for KNBH .}$$

11. Dynamical symmetries of Kerr-Newman Black Hole :

The primary dynamical tensor in KNBH is the stress tensor T^{ab} .

A dynamical symmetry is provided by

$$\pounds_{\underline{l}}\, T_{ab} = 0 \,. \qquad \ldots \qquad (11.1)$$

when the electromagnetic field tensor F_{ab} satisfies

$$\pounds_{\underline{l}} F_{ab} = 0 \qquad \ldots \qquad (11.2)$$

We observe that another dynamical symmetry exists.

(I) Electromagnetic Field Tensor F_{ab} :

For KNBH we have (vide Chapter 1; 5.4),

$$F_{ab} = - \mathrm{Re}\phi_1\underline{l}_a n_b + \mathrm{Re}\phi_1 n_a\underline{l}_b + i\,\mathrm{Im}\phi_1 m_a\bar{m}_b - i\,\mathrm{Im}\phi_1\bar{m}_a m_b$$

Let $\mathrm{Re}\phi_1 = A$ and $\mathrm{Im}\phi_1 = B$ and

Hence,

$$F_{ab} = - A(\underline{l}_a n_b - n_a\underline{l}_b) + iB\,(m_a\bar{m}_b - \bar{m}_a m_b) \qquad \ldots \qquad (11.3)$$

Now,

$$\pounds_{\underline{l}}\, F_{ab} = F_{ab;c}\underline{l}^{c} + F_{cb}\underline{l}^{c}{}_{;a} + F_{ac}\underline{l}^{c}{}_{;b} \quad \ldots \qquad (11.4)$$

We have

$$F_{ab;c}\underline{l}^{c} = -\,DA(\underline{l}_a n_b - n_a \underline{l}_b) + iDB(m_a \bar{m}_b - \bar{m}_a m_b) -$$

$$-\,A(\underline{l}_{a;c} n_b + \underline{l}_a n_{b;c} - n_{a;c}\underline{l}_b - n_a \underline{l}_{b;c})\underline{l}^{c} +$$

$$+\,iB(m_{a;c}\bar{m}_b + m_a \bar{m}_{b;c} - \bar{m}_{a;c} m_b - \bar{m}_a m_{b;c})\underline{l}^{c}\,.$$

Here $DA = A_{;c}\underline{l}^{c}$ and $DB = B_{;c}\underline{l}^{c}$.

On simplification we get

$$F_{ab;c}\underline{l}^{c} = -\,DA(\underline{l}_a n_b - n_a \underline{l}_b) + iDB(m_a \bar{m}_b - \bar{m}_a m_b) -$$

$$-\,A[\,\pi m_b \underline{l}_a + \bar{\pi}\bar{m}_b \underline{l}_a - \pi m_a \underline{l}_b - \bar{\pi}\bar{m}_a \underline{l}_b\,] +$$

$$+\,iB(\,\bar{\pi}\underline{l}_a \bar{m}_b + \pi \underline{l}_b m_a - \pi \underline{l}_a m_b - \bar{\pi}\underline{l}_b \bar{m}_a\,] \quad \ldots \; (11.5)$$

similarly after straight forward simplification we obtain;

$$F_{cb}\underline{l}^{c}{}_{;a} = A\,[\,(\gamma+\bar{\gamma})\underline{l}_a \underline{l}_b - (\alpha+\bar{\beta})m_a \underline{l}_b - (\bar{\alpha}+\beta)\bar{m}_a \underline{l}_b\,] +$$

$$+\,iB\,[\,\tau\,\underline{l}_a \bar{m}_b - \rho\, m_a \bar{m}_b - \bar{\tau}\,\underline{l}_a m_b + \bar{\rho}\,\bar{m}_a m_b\,] \ldots \; (11.6)$$

on interchanging a and b in (11.6) we have

$$F_{ac}\underline{l}^{c}{}_{;b} = -\,A[(\gamma+\bar{\gamma})\underline{l}_a \underline{l}_b - (\alpha+\bar{\beta})\underline{l}_a m_b - (\bar{\alpha}+\beta)\underline{l}_a \bar{m}_b\,] +$$

$$+\,iB\,[\,\bar{\tau}\,\underline{l}_b \bar{m}_a - \bar{\rho}\, m_a \bar{m}_b - \tau\,\bar{m}_a \underline{l}_b + \rho\,\bar{m}_a m_b\,] \; \ldots \; (11.7)$$

Combining (11.5), (11.6) and (11.7) and simplifying we infer (using 11.3)

$$\pounds_{\underline{l}} F_{ab} = - DA\,(\underline{l}_a n_b - n_a \underline{l}_b) + iDB\,(m_a \bar{m}_b - \bar{m}_a m_b) + $$
$$+ iB\Big[(\bar{\pi} + \tau)\underline{l}_a \bar{m}_b + (\pi + \bar{\tau})\underline{l}_b m_a + (-\pi - \bar{\tau})\underline{l}_a m_b + $$
$$+ (-\bar{\pi} - \tau)\underline{l}_b \bar{m}_a + (-\rho - \bar{\rho}) m_a \bar{m}_b + (\rho + \bar{\rho}) \bar{m}_a m_b\Big] \quad \ldots \ (11.8)$$

We now conclude that;

$\pounds_{\underline{l}} F_{ab} = 0$ if and only if

(a) $DA = 0$ ... (11.9a)

(b) $DB = B(\rho + \bar{\rho})$ ... (11.9b)

(c) $B\,(\bar{\pi} + \tau) = 0$ ... (11.9c)

Thus the necessary and sufficient conditions for the Lie invariance of the electromagnetic field tensor with respect to $\underline{l}^a$ are given above.

Consequences

From (11.9a)

$DA = 0$ gives

$$\left[\frac{r^2 - a^2 \cos^2\theta}{(r^2 + a^2 \cos^2\theta)^2} \left(\frac{e}{2}\right) \right]_{;c} \underline{l}^c = 0$$

or equivalently,

$$\frac{\partial}{\partial r} \left[\frac{r^2 - a^2 \cos^2\theta}{(r^2 + a^2 \cos^2\theta)^2} \right] = 0$$

Since $\underline{l}^1 = 1$, $\underline{l}^2 = \underline{l}^3 = \underline{l}^4 = 0$.

This yields the condition,

$$2r(r^2+a^2\cos^2\theta)^2 - 4r(r^2+a^2\cos^2\theta)(r^2-a^2\cos^2\theta) = 0$$

<u>OR</u> $\quad -r^2 + 3a^2 \cos^2\theta = 0 \quad \ldots \quad (11.10)$

This is a hyper surface.

From (11.9c) $B(\bar{\pi} + \tau) = 0$ we infer;

The case $B = 0$ is not tenable because

$$\mathrm{Im}\phi_1 = 0 \quad \underline{\text{or}} \quad r + a\cos\theta = 0 \quad \ldots \quad (11.11)$$

From (11.9c) and (11.11) we have

$$a^2 \cos^2\theta = 0$$

$\Rightarrow \quad r = 0$. from (11.10).

Hence $B \neq 0$.

We have to consider,

$$\bar{\pi} + \tau = 0 .$$

i.e., $a\sin\theta\,(\rho)\,(\rho - \bar{\rho}) = 0$ (vide Chapter 1, 5.1)

$$\Rightarrow \rho - \bar{\rho} = 0$$
$$\Rightarrow \mathrm{Im}\rho = 0$$
$$\Rightarrow a\cos\theta = 0$$
$$\Rightarrow \theta = \pi/2 .$$

which is a plane.

Now we shall discuss (11.9b),

$$DB = B(\rho + \bar{\rho})$$

i.e., $\left[\dfrac{ra\cos\theta}{(r^2+a^2\cos^2\theta)^2}\right]_{;c} \underline{1}^c = \dfrac{ra\cos\theta}{(r^2+a^2\cos^2\theta)^2}(\rho+\bar{\rho})$

Substituting the values of $\rho + \bar{\rho}$ and $\underline{1}^2 = \underline{1}^3 = \underline{1}^4 = 0$; $\underline{1}' = 1$.

we have,

$$\frac{\partial}{\partial r}\left[\frac{ra\cos\theta}{(r^2+a^2\cos^2\theta)^2}\right] = \frac{ra\cos\theta}{(r^2+a^2\cos^2\theta)^2}\left(\frac{-2r}{r^2+a^2\cos^2\theta}\right)$$

simplifying we have,

$$\frac{a\cos\theta\,(r^2+a^2\cos^2\theta)^2 - 2(r^2+a^2\cos^2\theta)2r(ra\cos\theta)}{(r^2+a^2\cos^2\theta)^4}$$

$$= \frac{ra\cos\theta}{(r^2+a^2\cos^2\theta)^2}\left(\frac{-2r}{r^2+a^2\cos^2\theta}\right)$$

$$\frac{a\cos\theta(r^2+a^2\cos^2\theta)^2 - 4r(r^2+a^2\cos^2\theta)(ra\cos\theta)}{(r^2+a^2\cos^2\theta)^4} = \frac{-2r^2a\cos\theta(r^2+a^2\cos^2\theta)}{(r^2+a^2\cos^2\theta)^4}$$

$$\frac{r^2+a^2\cos^2\theta\,[(a\cos\theta)(r^2+a^2\cos^2\theta)-4r^2a\cos\theta]}{} = \frac{-2r^2a\cos\theta(r^2+a^2\cos^2\theta)}{(r^2+a^2\cos^2\theta)^4}$$

$$\Rightarrow a\cos\theta\,(r^2+a^2\cos^2\theta) - 4r^2a\cos\theta = -2r^2a\cos\theta\,.$$

$$\Rightarrow a\cos\theta\,(r^2+a^2\cos^2\theta) - 2r^2a\cos\theta = 0$$

$$\Rightarrow a\cos\theta\,(r^2+a^2\cos^2\theta - 2r^2) = 0$$

$$\Rightarrow a\cos\theta\,(-r^2+a^2\cos^2\theta) = 0$$

$$\Rightarrow \theta = \pi/2\,,$$

which is a plane.

<u>Conclusion</u> :

Thus only on the hyper surface $\theta = \pi/2$ we have for KNBH $\pounds_{\underline{1}} F_{ab} = 0$.

(II) <u>Stress Tensor</u> T_{ab} :

For KNBH we have the expression for stress tensor in the form

$$T_{ab} = \frac{e^2}{4\pi(r^2+a^2\cos^2\theta)^2} \left(\underline{l}_{(a}n_{b)} + m_{(a}\bar{m}_{b)} \right) \qquad \ldots (11.12)$$

Let

$$W = \frac{e^2}{4\pi(r^2+a^2\cos^2\theta)^2} \qquad \ldots (11.13)$$

then,

$$T_{ab} = \frac{1}{2} W \left(\underline{l}_a n_b + n_a \underline{l}_b + m_a \bar{m}_b + \bar{m}_a m_b\right) \qquad \ldots (11.14)$$

Now

$$\pounds_{\underline{l}} T_{ab} = T_{ab;c}\underline{l}^c + T_{cb}\underline{l}^c{}_{;a} + T_{ac}\underline{l}^c{}_{;b} \qquad \ldots (11.15)$$

we have on simplication (from 11.14)

$$T_{ab;c}\underline{l}^c = \frac{1}{2} \Big\{ DW(\underline{l}_a n_b + n_a \underline{l}_b + m_a \bar{m}_b + \bar{m}_a m_b) +$$

$$+ W[2\pi\, m_b \underline{l}_a + 2\bar{\pi}\, \bar{m}_b \underline{l}_a + 2\pi\, m_a \underline{l}_b + 2\bar{\pi}\, \bar{m}_a l_b \,]\Big\} \ldots (11.16)$$

Similarly after straight forward simplification we obtain ;

$$T_{cb}\underline{l}^c{}_{;a} = \frac{1}{2} W[(\gamma+\bar{\gamma})\underline{l}_a \underline{l}_b - (\alpha+\bar{\beta}) m_a \underline{l}_b - (\bar{\alpha}+\beta)\bar{m}_a \underline{l}_b +$$

$$+ \tau\, \underline{l}_a \bar{m}_b - \rho\, m_a \bar{m}_b + \tau\, \underline{l}_a m_b - \bar{\rho}\, \bar{m}_a m_b \,] \qquad \ldots (11.17)$$

On interchanging a and b in (11.17) we have :

$$T_{ac}\underline{l}^c{}_{;b} = \frac{1}{2} W\, [(\gamma+\bar{\gamma})\underline{l}_a \underline{l}_b - (\alpha+\bar{\beta})\underline{l}_a m_b - (\bar{\alpha}+\beta)\bar{m}_b \underline{l}_a +$$

$$+ \bar{\tau}\, m_a \underline{l}_b - \bar{\rho}\, m_a \bar{m}_b + \tau\, \bar{m}_a \underline{l}_b - \rho\, \bar{m}_a m_b \,] \qquad \ldots (11.18)$$

Combining (11.16), (11.17), (11.18) and simplifying we infer (using, 11.15);

$$\pounds_{\underline{l}} T_{ab} = \tfrac{1}{2}(DW)\underline{l}_a n_b + \tfrac{1}{2}(DW) n_a \underline{l}_b + \tfrac{1}{2}(DW - W\rho - W\bar{\rho}) m_a \bar{m}_b +$$

$$+ \tfrac{1}{2}(DW - W\rho - W\bar{\rho}) \bar{m}_a m_b + \tfrac{1}{2}[W(2\pi + \bar{\tau} - (\alpha + \bar{\beta})]\underline{l}_a m_b +$$

$$+ \tfrac{1}{2}[W(2\bar{\pi} + \tau - (\bar{\alpha} + \beta)]\underline{l}_a \bar{m}_b + \tfrac{1}{2}[W(2\pi + \bar{\tau} - (\alpha + \bar{\beta})] m_a \underline{l}_b +$$

$$+ \tfrac{1}{2}[W(2\bar{\pi} + \tau - (\bar{\alpha} + \beta)]\bar{m}_a \underline{l}_b + [W(\gamma + \bar{\gamma})]\underline{l}_a \underline{l}_b . \qquad \ldots (11.19)$$

We now conclude that the following statements are equivalent

(I) $\pounds_{\underline{l}} T_{ab} = 0$

(II) (a) $DW = 0$ ...(11.20a)

(b) $DW - W(\rho + \bar{\rho}) = 0$ ...(11.20b)

(c) $[2\pi + \bar{\tau} - (\alpha + \bar{\beta})] = 0$ ...(11.20c)

(d) $\gamma + \bar{\gamma} = 0$ . ...(11.20d)

Thus the necessary and sufficient conditions for the Lie invariance of the stress tensor T_{ab} with respect to $\underline{l}^a$ are as given above.

Consequences :

We have from (11.20b);

$$DW - W(\rho + \bar{\rho}) = 0 .$$

Then from (11.20a) consequently (11.20b) will be

$$W\,(\rho+\bar{\rho}) = 0\ .$$

$$\Rightarrow \rho + \bar{\rho} = 0\ .$$

which is not tenable as discussed in Section 10(Chapter 2).

From (11.20c),

$$[2\pi + \bar{\tau} - (\alpha + \bar{\beta})] = 0\ .$$

$\Rightarrow (2\pi + \bar{\tau} - \pi) = 0$, since $\alpha + \bar{\beta} = \pi$ for a Black Hole.

$$\Rightarrow (\pi + \bar{\tau}) = 0$$

$=\ \rho + \bar{\rho} = 0$, since $\pi + \bar{\tau} = \rho + \bar{\rho}$ for a Black Hole.

which is untenable as discussed in Section 10 (vide Chapter 2).

Conclusion :

The null vector $\underline{l}^a$ cannot generate the dynamical symmetry $\pounds_{\underline{l}}\, T_{ab} = 0$.