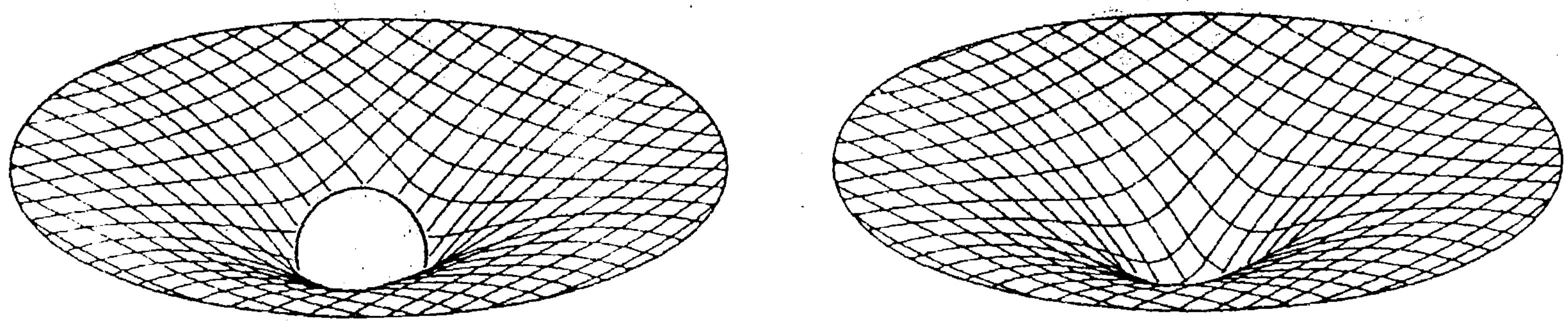
C H A P T E R - 3

SPECIAL FEATURES OF KERR-NEWMAN BLACK HOLE

No single object or concept epitomizes more completely the present stage of the revolution than the Black Hole. The data currently most likely to tell us whether black holes really do exist in our universe are the x-ray data from the UHURU satellite.

- C.Dewit and B.S.Dewit



(a) (b)

(a) The warping of spacetime produced by gravitating matter shown schematically. (b) The warping produced by a black hole. Even if the black hole is invisible, the warping of spacetime produced by it is measurable in principle. Through such measurements, black holes could be detected.

Introduction

In this chapter we enumerate some special features of the Type A field using the NP-formalism. We investigate the complexion of an electromagnetic field of Type A which is non-vanishing for KNBH. The necessary and sufficient conditions for the vanishing of the Nijenhuis tensor of KNBH is determined in Section 2. In third section existence of Zilch tensor field is shown. Special transports in KNBH (i) Jaumann Transport (ii) Fermi-Walker transport are explored in last section. We recall the non-singular electromagnetic field tensor

$$F_{ab} = -(\phi_1 + \bar{\phi}_1)\, l_{[a}n_{b]} + (\phi_1 - \bar{\phi}_1)\, m_{[a}\bar{m}_{b]}$$

The stress-energy momentum tensor for the Type A field is

$$T_{ab} = \phi_{11}\, (l_{(a}n_{b)} + m_{(a}\bar{m}_{b)})\, .$$

We list here the relations satisfied by the type A fields for utilization in the succeeding sections.

The Type A field is characterized in the Newman-Penrose formalism by (vide Chapter 1; 5.3)

$$\phi_0 = \phi_2 = 0$$

$$\phi_1 = 0$$

consequently the electromagnetic field tensor becomes (vide Chapter 1; 5.4)

$$F_{ab} = 2\, \mathrm{Re}\phi_1 l_{[a}n_{b]} + 2i\mathrm{Im}\; m_{[a}\bar{m}_{b]}$$

So we can say that KNBH is of the Type A. Later using this we confine special features of the Kerr Newman Black Hole.

1. Complexion of an electromagnetic field of type A

The special feature of the Type A field as distinguished from the Type B and the Type C field is the existence of a complexion vector (vide Appendix Chapter 3).

The complexion vector of an electromagnetic field is defined by

$$\textcircled{H}_a = \frac{\sqrt{-g}\ \epsilon_{abcd}\ R^b{}_k R^{ck;d}}{R_{pq}R^{pq}} \qquad \ldots (1.1)$$

which has to satisfy

$$\textcircled{H}_{a;b} = \textcircled{H}_{b;a} = 0 \qquad \ldots (1.2)$$

It is shown by Singh (1965) that the differential relations (1.2) are identically satisfied by cylinderically symmetric electromagnetic fields in the absence of charges.

The equation (1.2) is the geometrized form of the relations governing the behaviour of the electromagnetic field in a finite region of space-time.

For the non-singular field of the Type A we obtain:

$$\textcircled{H}_a = \frac{\phi_{11}^2\sqrt{-g}}{4R_{pq}R^{pq}} - \epsilon_{abcd}\ g^{dk}\ [\ \underline{l}^c{}_{;k}n^b + \underline{l}^b n^c{}_{;k} - m^b\bar{m}^c{}_{;k} - - \bar{m}^b m^c{}_{;k} + \underline{l}^b n^c n^r \underline{l}_{r;k} + n^b \underline{l}^c \underline{l}^r n_{r;k} + \underline{l}^b m^c n^r \bar{m}_{r;k} +$$

$$+ m^b \underline{l}^c \bar{m}^r n_{r;k} + \underline{l}^b \bar{m}^c n^r m_{r;k} + \bar{m}^b \underline{l}^c m^r n_{r;k} +$$

$$+ m^b n^c \bar{m}^r \underline{l}_{r;k} + n^b m^c \underline{l}^r \bar{m}_{r;k} + \bar{m}^b n^c m^r \underline{l}_{r;k} +$$

$$+ n^b \bar{m}^c \underline{l}^r m_{r;k} + m^b \bar{m}^c \bar{m}^r m_{r;k} + \bar{m}^b m^c m^r \bar{m}_{r;k} \quad] \qquad \ldots (1.3)$$

using the intrinsic derivatives of the null-tetrad field we obtain

$$\textcircled{H}_a = \sqrt{-g}\, \epsilon_{abcd} \, [\, \bar{\tau}\, m^{[b} n^{c]} \underline{l}^d - \bar{\rho}\, m^{[b} n^{c]} \bar{m}^d -$$

$$- \mu\, \underline{l}^{[b} m^{c]} \bar{m}^d + \pi\, \underline{l}^{[b} m^{c]} n^d] + \text{C.C.}$$

Wheeler (1962) interprets; the electromagnetic field of the Type A will be an essentially electric field iff the complex null tetrad is expansion free and twist free. We shall prove this result.

<u>Proof</u> : From the equation (1.4), we readily infer that

$$\textcircled{H}_a = 0 \text{ iff }, \quad \rho = 0 \qquad \ldots (1.5a)$$

$$\mu = 0 \qquad \ldots (1.5b)$$

$$\tau = \pi = 0 \qquad \ldots (1.5c)$$

The conditions (1.5a), (1.5b), (1.5c) give respectively, $\underline{l}^a$, n^a, m^a are expansion free and twist free. This completes the proof.

For a Black Hole $\rho = 0$ is not tenable. Accordingly the complexion field of a black hole does not vanish.

2. The Nijenhuis tensor field :

Introduction :

The relevance of the Nijenhuis tensor $N_{ab\cdot}^{\cdot\cdot c}$ for the General Theory of Relatively has been examined by Radhakrishna and Khade (1973 and 1976). They have constructed the Nijenhuis tensor viz.,

$$N_{ab}^{\cdot\cdot c} = 2 F_{k}^{\cdot c} F_{[a;b]}^{k} + 2 F_{[a}^{k} F_{b];k}^{c} , \qquad \ldots (2.1)$$

where F_{ab} is the skew-symmetric electromagnetic field tensor. In an attempt to find the physical meaning of the third rank tensor $N_{ab}{}^{c}$, Zafar Ahsan and Hussain (1980) have shown that at a point in Minkowske space-time this tensor represents "the variation of electric and magnetic fields strengths in different direction, determined by the electric and magnetic fields".

Relativistic Electrodynamics and the Newman-Penrose formalism :

The three-Maxwells scalars of an electromagnetif field :

There are three complex scalars called Maxwells scalars in electrodynamics [vide Chapter 1, (3.11a),(3.11b),(3.11c)]. The Maxwell Scalar ϕ_0 is constructed by taking the inner product of F_{ab} with $\underline{l}^a$ and m^a .

$$\phi_0 = 2 F_{ab} \underline{l}^b m^a , \qquad \ldots (2.2a)$$

This scalar is identified as the strength of the electromagnetic dipole field. The coulomb part of the field is designated by the Maxwell scalar ϕ_1 and is defined by

$$\phi_1 = F_{ab} \, (l^a n^b + \bar{m}^a m^b) \qquad \ldots (2.2b)$$

The scalar

$$\phi_2 = 2 \, F_{ab} \, \bar{m}^a n^b \qquad \ldots (2.2c)$$

measures the outgoing electromagnetic radiation.

Electromagnetic field tensor in the Null formalism :

In the General Theory of Relativity an electromagnetic field tensor is determined by a skew symmetric tensor field F_{ab} satisfying the Maxwell equations.

In the complex null tetrad formalism the electromagnetic field tensor F_{ab} is in the form (vide Chapter 1, 3.10) as

$$F_{ab} = - 2 \, \mathrm{Re}\phi_1 l_{[a} n_{b]} + 2 \, i \, \mathrm{Im}\phi_1 m_{[a} \bar{m}_{b]} + \phi_2 l_{[a} m_{b]} + \\ + \bar{\phi}_2 \, l_{[a} \bar{m}_{b]} + \bar{\phi}_0 \, m_{[a} n_{b]} + \phi_0 \, \bar{m}_{[a} n_{b]} \qquad \ldots (2.3)$$

where $\mathrm{Re}\phi$ and $\mathrm{Im}\phi$ denote the real and imaginary parts of ϕ respectively. The dual electromagnetic field tensor ${}^*F_{ab}$ is expressed as

$${}^*F_{ab} = - 2 \, i \, \mathrm{Re}\phi_1 \, m_{[a} \bar{m}_{b]} - 2 \, \mathrm{Im} \, \phi_1 l_{[a} n_{b]} - i \, \phi_2 \, l_{[a} m_{b]} + \\ + i \, \phi_2 \, l_{[a} \bar{m}_{b]} + i \, \bar{\phi}_0 \, m_{[a} n_{b]} - i \, \phi_0 \, \bar{m}_{[a} n_{b]} \qquad \ldots (2.4)$$

The two electromagnetic field scalars

$$F_{ab} \, F^{ab} \quad \text{and} \quad F_{ab} \, {}^*F^{ab}$$

are respectively

$$\phi_0 \phi_2 + \bar{\phi}_0 \bar{\phi}_2 - \phi_1^2 - \bar{\phi}_1^2 \quad \text{and} \quad i\,(\phi_1^2 - \bar{\phi}_1^2 + \bar{\phi}_0\bar{\phi}_2 - \phi_0\phi_2) \ .$$

The electromagnetic invariant K is defined as,

$$K = \frac{1}{2}\,(F_{ab}F^{ab} + i\,F_{ab}{}^{*}F^{ab}) \qquad \ldots (2.5)$$

In terms of the Maxwell scalars, K reduces to

$$K = 2\,(\phi_0\phi_2 - \phi_1^2\,) \qquad \ldots (2.6)$$

The invariant K can be expressed as

$$K = N_{ab}N^{ab} \ ,$$

where

$$N_{ab} = -\,\phi_1(l_{[a}n_{b]} + \bar{m}_{[a}m_{b]}) + \phi_2 l_{[a}m_{b]} + \phi_0\,\bar{m}_{[a}n_{b]} \qquad \ldots (2.7)$$

This can identified as the anti-self dual part of F_{ab}. The quantity $M_{ab} = \bar{N}_{ab}$ is called the self dual part of F_{ab}.

Here,

$$\mathrm{Re}\,K = \phi_0\phi_2 + \bar{\phi}_0\bar{\phi}_2 - \phi_1^2 - \bar{\phi}_1^2$$

$$i\,\mathrm{Im}\,K = \phi_0\phi_2 - \bar{\phi}_0\bar{\phi}_2 - \phi_1^2 + \bar{\phi}_1^2 \qquad \ldots (2.8)$$

For a Kerr Newman Black Hole $\phi_0 = \phi_2 = 0$. Thus we have (vide Chapter 1, 5.3),

$$\mathrm{Re}\,K = -\,\phi_1^2 - \bar{\phi}_1{}^2$$

$$i\,\mathrm{Im}\,K = -\,\phi_1^2 + \bar{\phi}_1^2 \qquad \ldots (2.9)$$

The three current scalars :

The four current vector field J^a is expressed as

$$J^a = I_2 \underline{l}^a + I_0 n^a - I_1 m^a - \bar{I}_1 \bar{m}^a \qquad \ldots (2.10)$$

where,

$$J_a J^a = 2 (I_0 I_2 - I_1 \bar{I}_1) \qquad \ldots (2.11)$$

The three complex current scalars I_A, $(A = 0,1,2)$ are easily identified from (2.10) by transvecting with the null legs as

$$I_0 = J^a \underline{l}_a$$

$$I_2 = J^a n_a$$

$$I_1 = J^a m_a$$

$\bar{I}_1$ can be obtained by taking the complex conjugate of I_1 .
For Kerr Newman Black Hole,

$$J^a = - \bar{I}_1 m^a - I_1 \bar{m}^a$$

and $$I_0 = I_2 = 0 . \qquad \ldots (2.12)$$

The Four Maxwell's equations

$$F^{ab}{}_{;b} = J^a ,$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \qquad \ldots (2.13)$$

In the complex spin coefficient formalism these eight tensor equations can be written as four scalar equations, viz.,

$$\begin{bmatrix} -\delta & D & 0 \\ -\Delta & \delta & 0 \\ 0 & -\bar{\delta} & D \\ 0 & -\Delta & \delta \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \pi - 2\alpha & 2\rho - K & I_0 \\ \mu - 2\gamma & 2\tau - \sigma & I_1 \\ -\lambda & 2\pi \quad \rho - 2\epsilon & \bar{I}_1 \\ -\nu & 2\mu \quad \tau - 2\beta & I_2 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ 1 \end{bmatrix}$$

... (2.14a)

$$D\, I_2 + \Delta I_0 - \bar{\delta}\, I_1 = (\gamma + \bar{\gamma} - \mu - \bar{\mu}) I_0 + (\bar{\beta} - \bar{\pi} - \alpha - \bar{\tau}) I_1 +$$

$$+ (\beta + \bar{\pi} - \bar{\alpha} - \tau)\, \bar{I}_1 + (\rho + \bar{\rho} - \epsilon - \bar{\epsilon})\, I_2 \qquad \text{... (2.14b)}$$

We have used here the intrinsic derivative operators viz.,

$$D\phi = \phi_{,a} l^a \;, \qquad \delta\phi = \phi_{,a} m^a \;,$$

$$\bar{\delta}\phi = \phi_{,a} \bar{m}^a \;, \qquad \Delta\phi = \phi_{,a} n^a \;,$$

'<u>NP-Concominants' of the electro-magnetic stress tensor</u> :

The electromagnetic stress energy momentum tensor T_{ab} is given by

$$T_{ab} = 1/4\ (F_{cd}F^{cd}) g_{ab} - F_{ac}F_b{}^c \;. \qquad \text{... (2.15)}$$

In terms of the complex null tetrad Z^{α}_a and the maxwells scalars the stress energy momentum tensor assumes the form (Debney and Zund 1971)

$$T_{ab} = 1/2\ [\ \phi_{22} l_a l_b + \phi_{00} n_a n_b + \phi_{02} \bar{m}_a \bar{m}_b + \phi_{20} m_a m_b\ ] +$$

$$+ \phi_{11} [l_a n_{b)} + m_{(a} \bar{m}_{b)}\ ] - \phi_{21} l_{(a} m_{b)} - \phi_{12} l_{(a} \bar{m}_{b)} +$$

$$+ \phi_{10} m_{(a} n_{b)} + \phi_{01} \bar{m}_{(a} n_{b)} \qquad \text{... (2.16)}$$

Here ϕ_{AB}, (A,B = 0,1,2) are the Ricci scalars.
There exists an interesting relation between the Ricci scalars and the Maxwell's scalars, when electromagnetism is the only source of the stress tensor namely

$$\phi_{AB} = \phi_A \bar{\phi}_B .$$

Energy-Balance Equations in the NP-version :

As a sequel to the Bianchi identities the electromagnetic stress tensor (2.15) satisfies the condition

$$T^{ab}{}_{;b} = 0 \qquad \ldots \quad (2.17a)$$

since, electromagnetism is the sole contributor to the source term, the Equations (2.17a) imply the vanishing of the Lorentz force $F^a_b J^b$

$$\text{i.e.,} \quad T^{ab}{}_{;b} = F^a_b J^b = 0 \qquad \ldots \quad (2.17b)$$

These equations in the null formalism are expressed as (Tariq and Tupper, 1976) :

$$2\,\mathrm{Re}\phi_1 .\, I_2 - \phi_2 .\, I_1 - \bar{\phi}_2\, \bar{I}_1 = 0 ,$$

$$2\,\mathrm{Re}\phi_1 .\, I_0 - \phi_0\, \bar{I}_1 - \bar{\phi}_0\, \bar{I}_1 = 0 ,$$

$$2\,i\,\mathrm{Im}\phi_1 .\, \bar{I}_1 - \phi_2 .\, I_0 + \bar{\phi}_0 I_2 = 0 ,$$

$$2\,i\,\mathrm{Im}\phi_1 .\, I_1 + \bar{\phi}_2 .\, I_0 - \phi_0 . I_2 = 0 . \qquad \ldots \quad (2.18)$$

For a Kerr Newman Black Hole, $I_0 = I_2 = 0$; thus we have

$$-\phi_2 . I_1 - \bar{\phi}_2 \bar{I}_1 = 0$$

$$\phi_0 \bar{I}_1 - \bar{\phi}_0 \bar{I}_1 = 0$$

$$2 i \, Im\phi_1 . \bar{I}_1 = 0$$

$$2 i \, Im\phi_1 . I_1 = 0 . \quad \ldots (2.19)$$

For source-free non-null fields we have $J^a = 0$ and $|F_{ab}| \neq 0$. It follows that there are no Type A fields with source. In the case of non-null fields with sources we have $J^a \neq 0$ and the equation (2.17b) are true if $|F_{ab}| = 0$. (see Waite 1961).

The Nijenhuis tensor field in Relativistic electrodynamics :

The Nijenhuis tensor for an electromagnetic field in the general theory of relativity is defined by Radhakrishna and Khade (1973) as :

$$N_{ab}^{\cdot\cdot c} = F_k^{\cdot c} (F_{a;b}^{\cdot k} - F_b^{\cdot k}{}_{;a}) + F_a^{\cdot k} F_{b;k}^{\cdot c} - F_b^{\cdot k} F_{a;k}^{\cdot c} . \quad \ldots (2.20)$$

where the skew-symmetric electromagnetic field tensor F_{ab} satisfies Maxwell's equations (2.13)

We can write (2.20) as

$$N_{ab}^{\cdot\cdot c} = F^{kc}(F_{ab;k}) + F_a^{\cdot k} (F_{b;k}^{\cdot c}) - F_b^{\cdot k}(F_{a;k}^{\cdot c}) \quad \ldots (2.21)$$

by utilizing (2.13).

We note that $N_{ab}^{\cdot\cdot c} = - N_{ba}^{\cdot\cdot c}$ and so there exist 24 components of $N_{ab}^{\ c}$ in the space-line of the General Theory of Relativity.

The null tetrad approach to the electromagnetic fields is developed by Debney and Zund (1971), Zund (1973), Tariq and Tupper (1976). We utilize their results for investigating the role of the Nijenhuis tensor in relativistic electrodynamics.

The Nijenhuis Tensor field :

We denote the Nijenhuis tensor field for the non-singular field of the Type A by $\underset{(BH)}{N}{}_{ab}^{\cdot\cdot c}$, using (equation 5.4, vide Chapter 1) and (2.21) we obtain

$$
\begin{aligned}
\underset{(BH)}{N}{}_{ab}^{\cdot\cdot c} = {} & (2R.l^{[k}n^{c]} + 2\,iI.m^{[k}\bar{m}^{c]})\,(2R_{;k}l_{[a}n_{b]} + 2\,iI_{;k}m_{[a}\bar{m}_{b]} + \\
& + 2Rl_{[a}n_{b];k} + 2iI.m_{[a}\bar{m}_{b];k}) + [R(l_a n^k - n_a l^k) + \\
& + iI(m_a\bar{m}^k - \bar{m}_a m^k)][R_{;k}(l_b n^c - n_b l^c) + iI_{;k}(m_b\bar{m}_c - \bar{m}_b m^c) + \\
& + R(l_{b;k}n^c + l_b n^c{}_{;k} - n_{b;k}l^c - n_b l^c - n_b l^c{}_{;k}) + \\
& + iI(m_{b;k}\bar{m}^c + m_b\bar{m}^c{}_{;k} - \bar{m}_{b;k}m^c - \bar{m}_b m^c{}_{;k})] - [R(l_b n^k - n_b l^k) + \\
& + iI(m_b\bar{m}^k - \bar{m}_b m^k)][R_{;k}(l_a n^c - n_a l^c) + iI_{;k}(m_a\bar{m}_c - \bar{m}_a m^c) + \\
& + R(l_{a;k}n^c + l_a n^c{}_{;k} - n_{a;k}l^c - n_a l^c{}_{;k}) + iI(m_{a;k}\bar{m}^c + \\
& + m_a\bar{m}^c{}_{;k} - \bar{m}_{a;k}m^c - \bar{m}_a m^c{}_{;k})\,] \qquad \ldots (2.22)
\end{aligned}
$$

Here in the above expression,

R = Real part of ϕ_1 and I = Imaginary part of ϕ_1 of the electromagnetic field tensor F_{ab} (vide chapter 1;3.10). Using the orthonormal properties of the complex null tetrad vectors and using intrinsic derivatives symbols we write the expression as :

$$\underset{(BH)}{N^{\cdot\cdot c}_{ab}} = 4\ RDR\ \underline{l}_{[a}n_{b]}n^c - 4\ R\Delta R\ \underline{l}_{[a}n_{b]}\underline{l}^c + 2\ (\mu Q - \bar{\mu}\bar{Q} + iR\Delta I)\bar{m}_{[a}m_{b]}\underline{l}^c + 2\ (\varrho Q - \bar{\varrho}\bar{Q} - iRDI)\bar{m}_{[a}m_{b]}n^c - E_{ab}{}^c - \bar{E}_{ab}{}^c\ , \qquad \ldots (2.23)$$

where,

$$E_{ab}{}^c = 4\ \nu\bar{P}\ \underline{l}_{[a}m_{b]}\underline{l}^c + 4\lambda Q\ \underline{l}_{[a}m_{b]}m^c + 4\bar{k}P\ m_{[a}n_{b]}n^c + 4\ o^{-}\bar{Q}\ m_{[a}n_{b]}m^c - 2(\ \pi\bar{P} + \bar{\tau}P + iI\bar{\delta}R)\underline{l}_{[a}n_{b]}m^c - 2\ (\pi\bar{P} - \bar{\tau}P - i\ I\bar{\delta}R)\underline{l}_{[a}m_{b]}n^c + 2\ (\pi\bar{P} - \bar{\tau}P + iI\bar{\delta}R)m_{[a}n_{b]}\underline{l}^c - 2(\mu Q + \bar{\mu}\bar{Q} + iR\ \ I)\underline{l}_{[a}m_{b]}\bar{m}^c - 2(oQ + \bar{o}\bar{Q} + iRDI)m_{[a}n_{b]}\bar{m}^c - 4I\bar{\delta}I\ m_{[a}\bar{m}_{b]}m^c$$

Here $P = R^2 + iRI$ and $Q = I^2 + iRI$ and $\bar{P}$, $\bar{Q}$ denote the complex conjugates of the scalar functions P, Q respectively.

<u>Characterization of the vanishing $N_{ab}{}^c$ for the type A field in the NP-formalism</u> :

We establish the following theorem for the non-singular electromagnetic field. We note that here $J^a = 0$ always.

<u>Theorem</u> :

The necessary and sufficient conditions for the vanishing of the Nijenhuis tensor field for the Type A field are

(i) the optical scalars of the two real null congruences $\underline{l}^a$ and n^a vanish and

(ii) either congruence is parallelly propogated along the other.

<u>Proof</u> : We claim that

$$\underset{(BH)}{N_{ab}{}^{c}} = 0 \quad \text{iff} \quad k = \rho = \sigma = \nu = \mu = \lambda = 0$$
$$\tau = \pi = 0 \qquad \ldots (2.24)$$

from the expression (2.23), we have,

$$\underset{(BH)}{N_{ab}{}^{c}} = 0 = DR = \Delta R = \delta I = \bar{\delta} I = 0 \,.$$

$$\nu = \lambda = k = \sigma = 0 \quad \text{for} \quad P, Q \neq 0 \,.$$

$$\rho Q - \bar{\rho}\bar{Q} - iRDI = 0 \,,$$
$$\rho Q + \bar{\rho}\bar{Q} + iRDI = 0 \,,$$
$$\rho Q + \bar{\rho}\bar{Q} - iRDI = 0 \,,$$
$$\mu Q + \bar{\mu}\bar{Q} - iR\Delta I = 0 \,,$$
$$\mu Q - \bar{\mu}\bar{Q} + iR\Delta I = 0 \,,$$
$$\pi P - \bar{\tau} P - iI\bar{\delta} R = 0 \,,$$
$$\pi \bar{P} + \bar{\tau} P + iI\delta R = 0 \,,$$
$$\pi \bar{P} + \bar{\tau} P + iI\bar{\delta} R = 0 \,. \qquad \ldots (2.25)$$

solving (2.25) we obtain,

$$\underset{(BH)}{N_{ab}{}^{c}} = 0 \Rightarrow k = \rho = \sigma = 0 \,,$$
$$\nu = \mu = \lambda = 0 \,,$$
$$\tau = \pi = 0 \,. \qquad \ldots (2.26)$$

The converse of the theorem follows from (2.23) and (2.26).

Hence,

$$\underset{(BH)}{N_{ab}{}^{c}} = 0 \,, \quad \text{iff} \quad k = \sigma = \rho = 0 \,, \qquad \ldots (2.27a)$$
$$\nu = \mu = \lambda = 0 \,, \qquad \ldots (2.27b)$$
$$\tau = \pi = 0 \qquad \ldots (2.27c)$$

The equation (2.27a) means that the $\underline{l}^a$ is geodesic (k=0), besides the vanishing of the expansion ($\rho+\bar{\rho}$), the twist ($\rho-\bar{\rho}$), the shear (σ). Analogous result for the n^a congruence are contained in (2.27b). Equations (2.27c) give,

$$\underline{l}_{a;b}n^b = n_{a;b}\underline{l}^b = 0 .$$

This completes the proof of the. theorem.

3. ZILCH TENSOR FIELD :

Introduction

While studying the "time-periodic" electromagnetic fields in vacuum, Lipkin (1964) observed that

$$1/2\ iP\ ({}^*\vec{E} \times \vec{E} + {}^*\vec{H} \times \vec{H}) = 0$$

by virtue of the Maxwell equations.

Here, $iP = \frac{\partial}{\partial T}$ in 'complex phasor' notation, P is the propagation constant of the wave, ${}^*\vec{E}$ and ${}^*\vec{H}$ denote the duals of the electric field $\vec{E}$ and magnetic field H. The vector field $iP\ ({}^*\vec{E} \times \vec{E} + {}^*\vec{H} \times \vec{H})$ is asumed to be capable of representing the time-averaged flux of a physical quantity that is conserved.

Z^{abc} in the Minkowski Space-time :

In the Minkowski space-time, Lipkin's expression for the Zilch tensor field is,

$$Z^{abc} = \Big\{ 1/4\ [\eta^{ar}\eta^{cn}\epsilon^{bpmq} + \eta^{br}\eta^{cn}\epsilon^{apmq} + \eta^{ar}\eta^{bq}\epsilon^{cpmn} +$$
$$+ \eta^{br}\eta^{aq}\epsilon^{cpmn}] - 1/2\ [\ \eta^{nq}\eta^{am}\epsilon^{bcrp} + \eta^{nq}\eta^{bm}\epsilon^{acrp} +$$

$$+ \eta^{nq}{}_{n}{}^{ap}\epsilon^{bcrm} + \eta^{nq}{}_{n}{}^{bp}\epsilon^{acrm}]\} F_{mn}F_{pq,r}$$

$$a,b,c \equiv 1,2,3,4. \qquad \ldots (3.1)$$

where,

$\eta_{ab} \equiv (-1, -1, -1, 1)$ is the Minkowski metric, ϵ^{abcd} is the completely antisymmetric tensor density for 4-dimension, F_{ab} is the electromagnetic field tensor.

Properties :

(1) $Z^{abc} = Z^{bac}$ ... (3.2a)

(2) $\eta_{ab}Z^{abc} = \eta_{ab}Z^{cab} = \eta_{ab}Z^{acb} = 0$ ... (3.2b)

(3) $Z^{abc}_{\ldots,c} = 0$ ... (3.2c)

On the nomenclature of 'ZILCH' :

The physical significance of the divergence equation $Z^{abc}_{\ldots,c} = 0$ is not yet known. However, the tensor components Z^{ab4} are interpreted as the spatial densities of the conserved quantities and the remaining tensor components of Z^{abc} are interpreted as the fluxes of the conserved quantities (Lipkin, 1964). Z^{abc} has 18 independent components because of the properties (3.2a,3.2b,3.2c). Thus the testimonical representation (3.1) yields nine conservation laws which are unfamiliar in form. This circumstance has prompted Lipkin to name this tensor as 'ZILCH' (meaning nil as given in a Dictionary of Scang and unconventional English, Vol.II,Patridge,E.).

The Zilch tensor field in the NP-formalism :

We start with the definition of $Z^{ab}_{\cdot\cdot c}$ in the form

$$Z^{ab}_{\cdot\cdot c} = 2\, i\, (\bar{N}^{a}_{\cdot k}\, N^{kb}_{\ ;c} - N^{a}_{\cdot k}\, \bar{N}^{kb}_{\ ;c}) \qquad \ldots (3.3)$$

where the antiself dual bivector N^{ab} is

$$N^{ab} = (F^{ab} + i\, {}^{*}F^{ab}) / 2$$

For a null field, we have

$$N^{ab} = -\,\phi_1\, (l^{[a} n^{b]} - m^{[a} \bar{m}^{b]}) \qquad \ldots (3.4)$$

We observe that

$$4\, (\bar{N}^{a}_{\cdot k}\, N^{kb}_{\ ;c} - N^{a}_{\cdot k}\, \bar{N}^{kb}_{\ ;c} = [\, \bar{\phi}_1\, (\phi_{1;c}) - \phi_1 (\bar{\phi}_{;c})] \times$$
$$\times [l^{(a} n^{b)} + m^{(a} \bar{m}^{b)}\,] + 2\, \phi_1 \bar{\phi}_1\, [(n^k \bar{m}_{k;c})(l^{(a} m^{b)}) +$$
$$+ (\bar{m}^k l_{k;c}) m^{(a} n^{b)} + (m^k n_{k;c}) \bar{m}^{(a} l^{b)} + (l^k m_{k;c}) n^{(a} \bar{m}^{b)}\,]$$
$$\ldots (3.5)$$

making use of maxwell equations and using intrinsic derivatives of the Newman penrose formalism to eliminate the covariant derivatives, we obtain finally the expression for the Zilch tensor field as :

$$Z^{ab}_{\cdot\cdot c} = 2\, i \{ |\phi_1|^2\, [l^{(a} n^{b)} + m^{(a} \bar{m}^{b)}]\, [(\bar{\mu} - \mu) l_c + (\rho - \bar{\rho}) n_c +$$
$$+ (\pi - \bar{\tau}) m_c + (\bar{\pi} - \tau) \bar{m}_c] + [\, l^{(a} m^{b)} (\nu l_c - \lambda m_c - \mu \bar{m}_c +$$
$$+ \pi n_c) + m^{(a} n^{b)}\, (\bar{\tau} l_c - \sigma m_c - \bar{\rho} \bar{m}_c + \bar{k} n_c)\,] \} + C.C.$$

where, $\qquad \ldots (3.6)$

$$\phi_1\, \bar{\phi}_1 = |\phi_1|^2 \,.$$

$Z^{ab}_{\cdot\cdot c}$ for the Kerr Newman Black Hole :

The most spectacular exact solution of Einsteins field equations for a non-null electromagnetic field is the one which predicts the existence of a black hole.

Substituting the spin coefficients values of KNBH (vide Chapter 1; eq. 5.1) into the expression for $Z^{ab}_{\cdot\cdot c}$ (3.6) we get the Zilch tensor field of the Kerr Newman Black Hole as :

$$Z^{ab}_{\cdot\cdot c} = 2\, i\, |\phi|^2 \{ [\underline{l}^{(a} n^{b)} + m^{(a} \bar{m}^{b)}] [(\frac{\Delta(\bar{\varrho}^2 \varrho)}{2} - \frac{\Delta(\varrho^2 \bar{\varrho})}{2}) \underline{l}_c + $$
$$+ (\bar{r} + ia\cos\theta)^{-1} - (r - ia\cos\theta)^{-1} n_c +$$
$$+ (\frac{ia\sin\theta\varrho^2}{\sqrt{2}} + \frac{ia\sin\theta\bar{\varrho}\varrho}{\sqrt{2}}) m_c + \{ \underline{l}^{(a} m^{b)} [\frac{ia\sin\theta\varrho^2}{\sqrt{2}} n_c -$$
$$- \frac{\Delta(\varrho^2\bar{\varrho})}{2} \bar{m}_c] + m^{(a} n^{b)} [\frac{ia\sin\theta\varrho\bar{\varrho}}{\sqrt{2}} \underline{l}_c +$$
$$+ (\bar{r} + ia\cos\theta)^{-1} \bar{m}_c] \} + C.C. \qquad \ldots (3.7)$$

In the special case $a = 0$; we obtain (3.7) as

$$Z^{ab}_{\cdot\cdot c} = 2\, i\, |\phi_1|^2 \; [\underline{l}^{(a} n^{b)} + m^{(a} \bar{m}^{b)}] [(\frac{\Delta\bar{\varrho}^2\varrho}{2} - \frac{\Delta\varrho^2\bar{\varrho}}{2}) \underline{l}_c -$$
$$- (\bar{r} - r)^{-1} n_c] - \frac{\Delta(\varrho^2\bar{\varrho})}{2} \underline{l}^{(a} m^{b)} \bar{m}_c + (\bar{r})^{-1} m^{(a} n^{b)} \bar{m}_c + C.C. \qquad \ldots (3.8)$$

which represents the Zilch field for electron.

Remarks :

(i) Kerr-Newman space time is asymptotically flat and of Petrov type D.

(ii) $Z^{abc}_{\ldots;c} = 0$ iff $\Psi_1 = 0$, $\Psi_3 = 0$, $\mathrm{Im}\Psi_2 = 0$.

In the case of Kerr Newman Black Hole, the relation

$$\Psi_2 = \rho^3 (m + e^2 \bar{\rho}) \qquad \text{(from chapter 1; eqn. 5.2)}$$

yields,

$$\Psi_2 - \bar{\Psi}_2 = m(\rho-\bar{\rho})(\rho^2+\rho\bar{\rho}+\bar{\rho}^2)+e^2(\rho-\bar{\rho})(\rho+\bar{\rho})(\rho^2+\bar{\rho}^2)$$

obviously, for $\rho = \bar{\rho}$ (Irrotational l^a) we get

$$\mathrm{Im}\Psi_2 = 0 .$$

Zilch scalars for black hole :

The Zilch scalars of the KNBH for Z_{abc} are enumerated below.

We take,

$$Z_{ABC} \equiv Z^a_A Z^b_B Z^c_C Z_{abc} , \qquad Z^a_A = l^a, n^a, m^a, \bar{m}^a .$$

$$(A,B,C = 1,2,3,4) .$$

(i) $Z_{221} = Z_{212} = i\phi_1{}^2(\Delta\bar{\rho}^2\rho - \Delta\rho^2\bar{\rho})$,

(ii) $Z_{121} = Z_{112} = - i\phi_1{}^2[(r-ia\cos\theta)^{-1} - (r+ia\cos\theta)^{-1}]$,

(iii) $Z_{321} = Z_{312} = - i\phi_1^2 \left[\frac{ia\sin\theta\rho\bar{\rho}}{\sqrt{2}} - \frac{ia\sin\theta\bar{\rho}^2}{\sqrt{2}} \right]$,

(iv) $Z_{231} = Z_{213} = \phi_1^2\, a\sin\theta\rho\bar{\rho} / \sqrt{2}$,

(v) $Z_{432} = Z_{423} = - i\phi_1^2 \Delta (\bar{\rho}^2\rho) / \sqrt{2}$,

(vi) $Z_{123} = i\, \phi_1^2 \sin\theta\bar{\rho}^2 / \sqrt{2}$.

4. SPECIAL TRANSPORTS IN KNBH :

(I) Jaumann Transport of the stress energy tensor for a Black Hole:

Carmeli (1977) has described the Kerr-Newman Black Hole. The discovery by Kerr (1963) of the 2-parameter family of metrices associated with his name is one of the principal landmarks in the development of the General Theory of Relativity. We recall the physical components of Weyl tensor, Maxwell scalars and the spin coefficients for the rotating black hole with charge e and mass m (vide Chapter 1, Section 5). In this section we shall examine the relevance of the Jaumann derivative to the relativistic electro-dynamics. We now prove the following :

Theorem 1 :

For the stress energy momentum tensor of the Type A field, the following two statements are equivalent.

(i) $\underset{l}{J}\, T_{ab} = 0$.

(ii) $\rho + \bar{\rho} = \bar{\tau} - \alpha - \bar{\beta} = 0$.

Proof : For the stress-energy momentum tensor

$$T_{ab} = \phi_{11}\,(\underline{l}_{(a}n_{b)} + m_{(a}\bar{m}_{b)}) ,$$

of the Type A field, we obtain by using the definition of Jaumann derivative,

$$\underset{\underline{l}}{J}\, T_{ab} = 2\,(\rho+\bar{\rho})T_{ab} + 1/2\ \phi_{11}\,(\alpha+\bar{\beta}-\bar{\tau})\underline{l}_{(a}m_{b)} + \text{C.C.} \quad \ldots (4.1)$$

we readily get,

$$\underset{l}{J}\, T_{ab} = 0 \quad \text{iff} \quad \rho + \bar{\rho} = 0$$
$$\alpha + \bar{\beta} - \bar{\tau} = 0 \qquad \ldots (4.2)$$

i.e., the stress-energy momentum tensor of the Type A field is Jaumann transported with respect to the eigen vector $\underline{l}^a$ iff $\underline{l}^a$ is expansion-free and the other two vector fields m^a, n^a form a surface orthogonal to $\underline{l}^a$.

Following the notation of Carmeli (1977), we infer that

$\bar{\tau} = \alpha + \bar{\beta}$ implies that

$$\text{Im}\rho = 0 \quad \text{(from equation 5.1, chapter 1)}$$

This gives,

$$a = 0 .$$

In other words, the Jaumann propogated Kerr-Newmann Black Hole loses its angular momentum and hence degenerates into the Reissner-Nordstrom black-hole.

Jaumann Transport relative to m^a

The following result can be similarly obtained for KNBH.

Theorem 2 :

For the stress-energy momentum tensor field we obtain,

$$\underset{m}{J}\, T_{ab} = 0 \quad \text{iff} \quad \bar{\pi} - \tau = 0 ,$$
$$\text{Im}\gamma = \text{Im}\epsilon = 0 \qquad \ldots (4.3)$$

Accordingly

$$\underset{m}{J}\, T_{ab} = 0 \quad \text{iff} \quad a = 0 ,$$
$$\text{Im}\,(\Delta\rho + r) = 0 \qquad \ldots (4.4)$$

Thus the Jaumann propogated Kerr-Newman Black-Hole with respect to m^a degenerates into the Reissner-Nordstrom black hole. However, such a loss of angular momentum does not happen when the black-hole is transported relative to the congruence n^a. Specifically we have

$$\underset{n}{J}\, T_{ab} = 0 \quad \text{if} \quad \mu + \bar{\mu} = 0 \,, \quad \pi = \alpha + \bar{\beta} \,.$$

i.e., $\underset{n}{J}\, T_{ab} = 0$ iff $\mathrm{Re}\,\varrho = 0$.

We can express this result alternately as

$$\underset{n}{J}\, T_{ab} = 0 \quad \text{iff} \quad u(\mathrm{Re}r^2 - a^2 + \underline{e}^2 - 2m\mathrm{Re}r) - v(\mathrm{Im}r^2 - 2m\mathrm{Im}r) = 0 \qquad \dots (4.5)$$

where,

$$u = \mathrm{Re}\varrho \quad \text{and} \quad v = \mathrm{Im}\varrho \,.$$

Thus when we put Carmeli's values (1977) of 'μ' then

$$\mu + \bar{\mu} = 0 \quad \text{implies} \quad r = 0 \,.$$

which is an incompatible situation.

(II) Fermi Walker transportation of the NP-tetrad and the KNBH

(1) Fermi Walker transportation of the NP-tetrad with respect to the flow congruence u^a :

(a) For the choice of $u^a = 2^{-1/2}\,(\underline{l}^a + n^a)$, we get

$$F_u \underline{l}^a = -F_u n^a = -2^{-3/2}\,[f\, m^a + c.c] \qquad \dots (4.6a)$$

$$F_u m^a = 2^{3/2}\,[\bar{f}(\underline{l}^a - n^a) + 4i\,\mathrm{Im}(h)\,] \qquad \dots (4.6b)$$

where $f = \pi + \nu + \bar{k} + \bar{\tau}$, $h = \epsilon + \gamma$ $\qquad \dots (4.6c)$

Hence, we obtain :

$$F_u Z^a_{(\alpha)} = 0 \iff \bar{\pi} + \bar{\nu} + k + \tau = 0 ,$$
$$\epsilon + \gamma + \bar{\epsilon} - \bar{\gamma} = 0 \qquad \ldots (4.7)$$

without loss of generality, we assume that $\underline{l}^a$ and n^a are geodesic (This does not imply that u^a geodesic).

i.e., $\epsilon + \bar{\epsilon} = k = 0 , \quad \gamma + \bar{\gamma} = \nu = 0$ ... (4.8)

By virtue of (4.7), (4.8) we have

$$F_u Z^a_{(\alpha)} = 0 \iff \bar{\pi} + \tau = \epsilon + \gamma = 0 \qquad \ldots (4.9)$$

Note :

Non-geodesic flow congruence :

The NP-version of the acceleration of the flow congruence u^a for the choice of $u^a = 2^{-1/2}(\underline{l}^a + n^a)$ is given by

$$\dot{u}^a = 2^{-1}[(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})(\underline{l}^a-n^a) + (\pi+\nu-\bar{k}-\bar{\tau})m^a + c.c.\ ]$$
$$= 2^{-1}[(\pi - \bar{\tau})m^a + c.c.\ ],$$

when

$$F_u Z^a_{(\alpha)} = 0 \qquad \ldots (4.10)$$

For Fermi Walker transport of $Z^a_{(\alpha)}$,

$$\dot{u}^a \neq 0$$

i.e., $\pi \neq \bar{\tau}$ ... (4.11)

Now, we recall that the Kerr Newman black hole is asymptotically flat and so $\rho \neq 0$.

In this case, from (4.9) we get

$$F_u Z^a_{(\alpha)} = 0 \quad \Leftrightarrow \quad \gamma = \pi + \bar{\tau} = 0 \qquad \ldots (4.12)$$

From (4.11), we get

$$a \sin\theta \rho \, (\rho - \bar{\rho}) \neq 0 \quad \text{and so} \quad a \neq 0 \,.$$

For the black hole $a \neq 0$ and so (4.12) implies that

$F_u Z^a_{(\alpha)} = 0$ iff $\rho + \bar{\rho} = 0$ ($\underline{l}^a$ is expansion free).

i.e., $$\Delta \rho + r - m = 0 \qquad \ldots (4.13)$$

(b) <u>On the Incompatibility of Fermi Walker Transportation of the NP-tetrad $Z^a_{(\alpha)}$ for the choice of $u^a = 2^{-1/2}(m^a - \bar{m}^a)$ in Kerr-Newman black hole</u>

Here, we obtain,

$$\dot{u}^a = \mathrm{Re}(\lambda - \mu)\underline{l}^a + \mathrm{Re}(\rho - \sigma)n^a + \mathrm{Re}(\beta - \alpha)(m^a + \bar{m}^a) \qquad \ldots (4.14)$$

and

$$F_u \underline{l}^a = i2^{-1/2}[2 \,\mathrm{Im}(\bar{\alpha} + \beta)\underline{l}^a + \mathrm{Im}(\rho - \sigma)(m^a + \bar{m}^a)],$$

$$F_u n^a = i2^{-1/2}[2 \,\mathrm{Im}(\alpha + \bar{\beta})n^a + \mathrm{Im}(\mu - \lambda)(m^a + \bar{m}^a)],$$

$$F_u m^a = F_u \bar{m}^a = i2^{-1/2}[\mathrm{Im}(\mu - \lambda)\underline{l}^a + \mathrm{Im}(\rho - \sigma)n^a].$$

consequently,

$$F_u Z^a_{(\alpha)} = 0 \quad \Longleftrightarrow \quad \mathrm{Im}(\mu - \lambda) = \mathrm{Im}(\bar{\alpha} + \beta) = \mathrm{Im}(\rho - \sigma) = 0 \qquad \ldots (4.15)$$

The geodesic nature of m^a (without loss of generality) implies,

$$\bar{\lambda} = \bar{\alpha} - \beta = \sigma = 0 \qquad \ldots (4.16)$$

Therefore,

$$F_u Z^a_{(\alpha)} = 0 \quad \Leftrightarrow \quad \mathrm{Im}(\rho) = \mathrm{Im}(\mu) = \mathrm{Im}(\alpha) = 0.$$

provided $\dot{u}^a \neq 0$. i.e., $\mathrm{Re}(\mu) \neq 0$, $\mathrm{Re}(\rho) \neq 0$ simultaneously.

Remark :

For the Kerr-Newman black hole, $a \neq 0$, but the condition

$\rho - \bar{\rho} = 0$ implies $a = 0$.

Hence, the incompatibility results.

(2) The null projection operator :

The two dimensional projection operator introduced by Jordon, Ehlers and Sachs (1961) is

$$\gamma_{ab} = g_{ab} + \frac{k_a k_b}{(u_c k^c)^2} - \frac{2\, u_{(a} k_{b)}}{u_{(c} k_{c)}} \qquad \ldots (4.17)$$

where $u_a u^a = 1$, $k_a k^a = 0$

$$u_{a;b} k^b = 0 \qquad \ldots (4.18)$$

Supplementary condition :

The condition (4.18) which means that the time-like congruence u^a is parallelly propagated with respect to the null congruence k^a is referred as the supplementary condition in the following discussions.

(i) For the null congruence l^a and n^a :

where $u^a = 2^{-1/2} (l^a + n^a)$, equation (4.18) yields

$$u_{a;b} l^b = 0 \iff \epsilon + \bar{\epsilon} = \pi - \bar{k} = 0 \qquad \ldots (4.19)$$

$$u_{a;b} n^b = 0 \iff \gamma + \bar{\gamma} = \nu - \bar{\tau} = 0 \qquad \ldots (4.20)$$

Note :

Since $u_c m^c = 0$, the null projection operator, γ_{ab} is not defined for $k^a = m^a$.

(ii) The complex null congruence m^a :

We choose

$$u^a = 2^{-1/2}(m^a - \bar{m}^a).$$

In this case, the supplementary conditions (4.18) reduce to,

$$\lambda = \mu = \sigma = \rho = \bar{\alpha} - \beta = 0 \qquad \ldots (4.21)$$

(3)(a) Effect of Fermi Walker transportation of the NP-tetrad on the optical parameters of a Black Hole :

(i) For the real null congruence l^a :

The kinematical tensor quantities expansion, shear and rotation for l^a (using supplementary conditions (4.19() are :

$$\theta = -(\rho+\bar{\rho}) \qquad \ldots (4.22a)$$

$$\underset{(l)}{\sigma_{ab}} = (\gamma+\bar{\gamma})l_a l_b + [\sigma m_a m_b - (\alpha+\bar{\beta}+\bar{\tau})l_{(a}m_{b)} - \bar{k}\, m_{(a}n_{b)}] + C.C., \qquad \ldots (4.22b)$$

$$\underset{(l)}{w_{ab}} = (\rho-\bar{\rho})\bar{m}_{[a}m_{b]} + [(\bar{\tau} - \alpha - \bar{\beta})l_{[a}m_{b]} - \bar{k}\, m_{[a}n_{b]}] + C.C., \qquad \ldots (4.22c)$$

Remarks :

(1) The shear and rotation are affected but not the expansion under Fermi Walker transportation of the NP-tetrad for the choice of $u^a = 2^{-1/2}(l^a + n^a)$.

(2) Optical parameters for the black hole :

In this case, the optical parameters (4.22a,b,c) under

$F_u Z^a_{(\alpha)} = 0$ yield that $\underline{l}^a$ is rigid, i.e.,

$$\underset{(\underline{l})}{\theta} = 0 , \qquad \underset{(\underline{l})}{\sigma_{ab}} = 0 ,$$

and

$$\underset{(\underline{l})}{w_{ab}} = 2 \varrho \left[m_{[a}\bar{m}_{b]} - 2^{-1/2} i a \varrho \sin\theta \, \underline{l}_{[a}\bar{m}_{b]} + \text{C.C.} \right] .$$

(ii) <u>For the real null congruence n^a</u> :

We record the kinematical tensor quantities expansion, shear and rotation for n^a as

$$\underset{(n)}{\theta} = (\mu + \bar{\mu}) \qquad \text{... (4.23a)}$$

$$\underset{(n)}{\sigma_{ab}} = [-\lambda m_{(a}m_{b)} + \nu m_{(a}\underline{l}_{b)} + (\bar{\pi}+\alpha+\bar{\beta}) m_{(a}n_{b)}] + \text{C.C.} - $$
$$- (\epsilon + \bar{\epsilon}) n_a n_b \qquad \text{... (4.23b)}$$

$$\underset{(n)}{w_{ab}} = (\bar{\mu}-\mu) m_{(a}\bar{m}_{b)} + [\nu m_{[a}\underline{l}_{b]} + (\pi-\alpha-\bar{\beta}) m_{[a}n_{b]}] + \text{C.C.} \qquad \text{...(4.23c)}$$

<u>For a black hole we have,</u>

$$\underset{(n)}{\theta} = \varrho \bar{\varrho} \, \text{Re}(\Delta \varrho) ,$$
$$\underset{(n)}{\sigma_{ab}} = 2^{1/2} i a \sin\theta \varrho^2 \, m_{(a}n_{b)} + \text{C.C.} ,$$
$$\underset{(n)}{w_{ab}} = - i \varrho \bar{\varrho} \, \text{Im}(\Delta \varrho) \, m_{[a}\bar{m}_{b]} .$$

Thus the angular momentum affects only the shear and not θ, W_{ab} .

(iii) <u>For the complex null congruence m^a</u> :

The expansion, the shear and rotation for the complex null congruence m^a are listed below :

$$\underset{(m)}{\theta} = \bar{\pi} - \tau \,, \qquad \ldots (4.24a)$$

$$\underset{(m)}{\sigma_{ab}} = \bar{\nu} l_a l_b - k n_a n_b + (\gamma - \bar{\gamma}) l_{(a} m_{b)} + (\epsilon - \bar{\epsilon}) m_{(a} n_{b)} \,, \qquad \ldots (4.24b)$$

$$\underset{(m)}{w_{ab}} = (\bar{\gamma} - \gamma) l_{[a} m_{b]} + (\bar{\pi} + \tau) l_{[a} n_{b]} + (\epsilon - \bar{\epsilon}) m_{[a} n_{b]} \qquad \ldots (4.24c)$$

We observe that the above parameters are unaffected under Fermi Walker transport of $Z^a_{(\alpha)}$ for the choice of $u^a = 2^{-1/2}(m^a - \bar{m}^a)$.

(b) Effect of Fermi Walker Transport of the NP-tetrad on the kinematical parameters of the flow for the choice of $u^a = 2^{-1/2}(l^a + n^a)$:

Under $F_u Z^a_{(\alpha)} = 0$, the kinematical parameters take the form,

$$\underset{(u)}{\theta} = 2^{1/2}\, \mathrm{Re}(\mu - \rho) \qquad \ldots (4.25a)$$

$$3(2)^{1/2} \underset{(u)}{\sigma_{ab}} = \mathrm{Re}(\mu - \rho)\, [l_a l_b + n_a n_b - 2\, l_{(a} n_{b)} - 2\, m_{(a} \bar{m}_{b)}\,] + $$
$$+ 3(\alpha + \bar{\beta})\,[m_{(a} n_{b)} - l_{(a} m_{b)}\,] + \mathrm{C.C.} + $$
$$+ 3\,(\sigma - \lambda) m_a m_b + \mathrm{C.C.}\,, \qquad \ldots (4.25b)$$

$$2^{1/2} \underset{(u)}{w_{ab}} = -\, 2\, i\mathrm{Im}(\mu + \rho) \bar{m}_{[a} m_{b]} - (\alpha + \bar{\beta})\,[l_{[a} m_{b]} + m_{[a} n_{b]}\,] \qquad \ldots (4.25c)$$

Remarks :

(1) All the three kinematical parameters are affected by Fermi Walker transport of $Z^a_{(\alpha)}$.

(2) <u>Parameters for the black hole</u> :

In this case, the parameters reduce to

$$\underset{(\mu)}{\Theta} = 2^{-1/2}\,\rho\,\bar{\rho}\,m,$$

6 $$\underset{(\mu)}{\sigma_{ab}} = 2^{-1/2}\,\rho\bar{\rho}m\,[l_a l_b + n_a n_b - 2\,l_{(a}n_{b)} - 2\,m_{(a}\bar{m}_{b)}\,] + \\ + 3i\rho^2 a\,\sin\theta\,[m_{(a}n_{b)} - l_{(a}m_{b)}] + C.C.,$$

2 $$\underset{(\mu)}{w_{ab}} = 2^{1/2}\rho\,(\bar{\rho}\bar{r} + 2)m_{[a}\bar{m}_{b]} - ia\sin\theta\rho^2[l_{[a}m_{b]} + m_{[a}n_{b]}] + C.C.$$

Here we observe that the rotation of Black Hole is independent of its mass.

APPENDIX

The Ruse-Synge Classification of Electromagnetic Fields :

We follow the Ruse-Synge classification which has been extensively investigated by using Newman-Penrose formalism. We summarise this classification in the following table:(Debney and Zune,1971)

Class	Type of fields	Classification of the electromagnetic field	ϕ_0	ϕ_1	ϕ_2	Electromagnetic field tensor
Non singular	A	$k \neq 0$,	0	-	0	$F_{ab}=-Re\phi_1 \underline{l}_{[a}n_{b]} + 2iIm\phi_1 m_{[a}\bar{m}_{b]}$
	A'	$Re k \neq 0,\ Im k \neq 0$	0	$\phi_1=\bar{\phi}_1$	0	$F_{ab}= -2\phi_1 \underline{l}_{[a}n_{b]}$.
	A'''	$Re k \neq 0,\ Im k = 0$	0	$\phi_1=-\bar{\phi}_1$	0	$F_{ab}= 2\phi_1 m_{[a}\bar{m}_{b]}$.
	A'''	$Re k = 0,\ Im k \neq 0$	0	$\phi_1=+i\bar{\phi}_1$	0	$F_{ab}=-2Re\phi_1(\underline{l}_{[a}n_{b]} \pm m_{[a}\bar{m}_{b]})$.
Singular	B	$k = 0$	0	0	-	$F_{ab}=\phi_2 \underline{l}_{[a}m_{b]}+\underline{C.C}.$
	C	$k = 0$	-	0	0	$F_{ab}=\bar{\phi}_0 m_{[a}n_{b]}+\underline{C.C}.$

Notation :

The vanishing of the Maxwell's scalar is indicated by '0'.

A dash '-' indicates that the Maxwell scalar is unrestricted .

The stress-energy momentum tensor for the different types can be enumerated as :

Type A : $T_{ab} = \phi_{11} (\underline{l}_{(a}n_{b)} + m_{(a}\bar{m}_{b)})$, Type B : $T_{ab} = 1/2\ \phi_{22}\ \underline{l}_a \underline{l}_b$.

Type C : $T_{ab} = 1/2\ \phi_{00}\ n_a n_b$.

The non-singular electromagnetic field tensor of the Type A admits two principal null directions, $\underline{l}^a$ and n^a and corresponds to a general non-radiating electromagnetic field.