CHAPTER - II

Non-Static Conformally Flat Spherically Symmetric Perfect Fluid Distributions in Einstein-Cartan Theory

" Einstein-Cartan theory is an even more beautiful theory than Einstein's general relativity because of its relation to the Poincare Group. "

... HEHL, F.W.

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1) INTRODUCTION

The Einstein-Cartan theory of space-time has attracted a lot of interest in recent years. From the cosmological stand point the interest stems from the fact that non-singular cosmological models in Einstein-Cartan theory have been constructed explicitly. In most of these models the spin of the particles composing the fluid is assumed to be aligned along a particular direction.

The general theory of relativity is bedevilled by a number of unknown functions - the ten components of g_{ij}. Hence there is a little hope of getting physically interesting results without making reduction in their number. In conformally flat-space-time the number of unknown functions is reduced to one. The conformally flat metrics are of particular interest since all of the homogeneous and isotropic osmological models of the universe can be cast in conformally Minkowskian form. Due to the equality of conformal curvature tensors of two conformally related spaces it is clear that corresponding to the two existing physical systems, one can generate the others by introducing a conformal transformation.

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A physically significant space-times which are conformally flat are the Sohwarzschid internal sclution and Lemaitre universe.

Buchdahl [2] has shown that the only static distribution of the fluid with positive density and pressure which would generate a conformally flat metric through the Einstein's equations without cosmological term is that described by the Schwarzschild inferior solution. Burman [3] discussed the motion of the particles in conformally flat space-time. Singh and Abdussattar [22] has obtained a non-static generalization of the Schwarzschild interior solution which is conformal to flat space-time, and it has also shown that the model admits of distribution of discrete particles and disordered radiation. Nduka [13] generated a closed analytic solution to the Einstein's equations for a uniformally charged fluid sphere by a method similar to that used by Adler [1]. Zaĭcev and Šikin [27] have obtained conformally flat non-static solutions in general relativity theory and scalar-tensor-theories of gravitation. Collinson [5] has shown that every conformally flat axisymmetric stationary space-time is static. He has also proved that if the source is a perfect fluid the space-time is the interior Schwarzchild field. Gupta [6] has observed that if a conformally flat space-time describes a perfect fluid distribution of matter with $\varrho \neq 0$, then it is necessarily of embedding class one and the lines of flow are normal to the hypersurface $\varrho(x^i)$ = constant. Gurses [7] has shown that the Schwarzschild interior metric is the only conformally flat static solution of the Einstein field equations with perfect fluid distribution.

Roy and Raj Bali [20] have obtained the solutions of Einstein's field equations representing non-static spherically symmetric perfect fluid distribution which is conformally flat. Prasanna [15] has described the Einstein-Cartan equations with special reference to a perfect fluid distribution following the work of Trautman and then obtained three solutions adopting Hehl's [8,9] approach and Tolman's [23] technique. He has shown that a space-time metric similar to the interior Schwarzschild solution will no longer represent a homogeneous fluid sphere in the presence of spin density.

Recently Kallyanshetti and Waghmode [11] considered the static fonformally flat spherically symmetric perfect fluid-distribution in the frame-work of Einstein-Cartan theory and obtained the field equations. They have solved these field equations and discussed the reality conditions in the view of their solutions. They have observed that the density ρ will not be constant as observed by Narlikar [12] for conformally flat spherically symmetric perfect fluid distribution. But they have shown that $\bar{\rho}$ will be constant.

In this chapter we consider the non-static conformally flat sperically symmetric perfect-fluid distribution in Einstein-Cartan theory and obtain the field equations. These field equations are solved.

In Section 2, we have considered the metric in non-static conformally flat spherically symmetric perfect-fluid distribution.

By using Cartan structural equations, we have obtained the curvature forms $\Omega^i_{\ j}$, Ricci tensors R_{ij} and the curvature scalar R. In Section 3, field equations are obtained. In Section 4, solutions of the field equations are obtained by adopting Hehl's [8,9] approach. In Section 5, the solutions obtained are compared with those solutions obtained by Singh and Abdussattar [22], Kallynshetti and Waghmode [11].

2) METRIC AND THE CURVATURE

Let M be a C^∞ four-dimensional oriented connected Hausdorff differential manifold and g be a Lorentz metric defined on it. The metric g and the connection w are described with respect to the co-frame θ^i chosen by the metric components g_{ij} and by a set of one forms $w^i_{\ j}$.

Therefore we have

$$ds^2 = g_{ij}\theta^i \otimes \theta^j \qquad \dots (2.1)$$

where $w^i_{\ j}$ are determined by

$$w^i_{\ j} = \Gamma^{\ i}_{kj}\, \theta^k \, . \qquad \dots (2.2)$$

The Cartan structural equations are

$$\begin{aligned} \Theta^i &= D\theta^i \\ &= d\theta^i + w^i_{\ j} \wedge \theta^j \\ &= \tfrac{1}{2} Q^i_{\ jk}\, \theta^j \wedge \theta^k \end{aligned} \qquad \dots (2.3)$$

$$\Omega^i{}_j = dw^i{}_j + w^i{}_k \wedge w^k{}_j$$
$$= \frac{1}{2} R^i{}_{jkl} \theta^k \wedge \theta^l \quad \ldots (2.4)$$

$$Q^i{}_{jk} - \delta^i_j Q^l_{lk} - \delta^i_k Q^l_{jl} = - kS^i_{jk} \quad \ldots (2.5)$$

where D denotes the exterior covariant derivative and Q^i_{jk} and R^i_{jkl} are the torsion and the curvature tensors respectively.

The classical description of spin is defined by the relation

$$S^i_{jk} = u^i S_{jk} \quad \text{with} \quad S_{jk} u^k = 0 \quad \ldots (2.6)$$

where u^i is the four velocity vector and S_{jk} is the intrinsic angular momentum tensor.

We consider a non-static conformally flat spherically symmetric perfect-fluid distribution represented by the space-time metric :

$$ds^2 = e^{2\lambda} (- dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2) \quad \ldots (2.7)$$

where λ is a function of r and t alone. The energy-momentum tensor for a perfect-fluid distribution is given by

$$T_{ij} = (p + \rho)V_i V_j - p g_{ij},$$
$$\text{together with} \quad g^{ij} V_i V_j = 1, \quad \ldots (2.8)$$

where p is the pressure, ρ is the density and $V_i = (V_1, 0, 0, V_4)$ is the flow-vector which describes the radial motion of the fluid.

We have then the orthonormal tetrad

$$\theta^1 = e^{\lambda}dr, \qquad \theta^2 = re^{\lambda}d\theta$$
$$\theta^3 = r\sin\theta e^{\lambda}d\phi, \qquad \theta^4 = e^{\lambda}dt \qquad \ldots (2.9)$$

The metric (2.7) now becomes

$$ds^2 = - \left\{(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 - (\theta^4)^2\right\} \qquad \ldots (2.10)$$

so that

$$g_{ij} = \text{diag}\,(-1, -1, -1, 1).$$

We suppose that spins of the particles composing the fluid are all aligned in the radial direction alone. Therefore the only independent non-zero component of the spin S_{ij} is $S_{23} = K$ (say). From this the non-static condition, we have the velocity four vector $u^i = \delta^i_j$, $i,j = 1,2$.

Thus the non-zero components of $s^i{}_{jk}$ are

$$S^1_{23} = - S^1_{32} = u^1S_{23} = 1.\ K = K.$$
$$S^4_{23} = - S^4_{32} = u^4S_{23} = 1.\ K = K.$$

Therefore from Cartan's equations

$$Q^i_{jk} - \delta^i_j\, Q^l_{lk} - \delta^i_k\, Q^l_{jl} = - kS^i_{jk}$$

we have the non-zero components of Q^i_{jk}

$$Q^1_{23} = - k\,K \text{ and } Q^4_{23} = - k\,K \qquad \ldots (2.11)$$

Using (2.11) in (2.3), we get

$$\textcircled{H}^1 = \textcircled{H}^4 = -\frac{1}{2} k \, \mathcal{K} \theta^2 \wedge \theta^3$$
$$\textcircled{H}^2 = \textcircled{H}^3 = 0 \qquad \ldots (2.12)$$

From (2.9), we obtain

$$d\theta^1 = e^{-\lambda} \dot{\lambda} \, \theta^4 \quad \theta^1 \qquad \ldots (2.13)$$

$$d\theta^2 = e^{-\lambda}(\lambda' + r^{-1})\theta^1 \wedge \theta^2 + e^{-\lambda} \dot{\lambda}\theta^4 \wedge \theta^2 \qquad \ldots (2.14)$$

$$d\theta^3 = e^{-\lambda}(\lambda' + r^{-1})\theta^1 \wedge \theta^3 + e^{-\lambda}\dot{\lambda}\theta^4 \wedge \theta^3$$
$$+ e^{-\lambda} r^{-1} \cot\theta \theta^2 \wedge \theta^3 \qquad \ldots (2.15)$$

$$d\theta^4 = e^{-\lambda}\lambda' \theta^1 \wedge \theta^4 \, . \qquad \ldots (2.16)$$

where a dash and a dot over λ denote differentiation with respect to r and t respectively.

Here (2.13), (2.14), (2.15) and (2.16) are Cartan's first structural equations. Comparing these equations with the equations (2.3), we obtain the non-zero components of $w^i{}_j$ as

$$w_1^2 = -w^1{}_2 = \bar{e}^{\lambda}(\lambda' + r^{-1})\theta^2 - \frac{1}{2} k \, \mathcal{K}\theta^3$$

$$w_1^3 = -w^1{}_3 = \bar{e}^{\lambda}(\lambda' + r^{-1})\theta^3 + \frac{1}{2} k \, \mathcal{K}\theta^2$$

$$w^1{}_4 = w^4{}_1 = \bar{e}^{\lambda}\dot{\lambda}\theta^1 + \bar{e}^{\lambda}\lambda'\theta^4$$

$$w^3{}_2 = -w^2{}_3 = \bar{e}^{\lambda} r^{-1} \cot\theta \theta^3 - \frac{1}{2} k \, \mathcal{K}\theta^4 + \frac{1}{2} k \, \mathcal{K}\theta^1$$

$$w^2{}_4 = w^4{}_2 = \bar{e}^{\lambda}\dot{\lambda}\theta^2 + \frac{1}{2} k \, \mathcal{K}\theta^3 \qquad \ldots (2.17)$$

$$w^4{}_3 = w^3{}_4 = \bar{e}^{\lambda}\dot{\lambda}\theta^3 - \frac{1}{2} k \, \mathcal{K}\theta^2$$

Now from the Cartan's second structural equations

$$\Omega^i{}_j = dw^i{}_j + w^i{}_k \wedge w^k{}_j$$

we have

$$\Omega^1{}_1 = dw^1{}_1 + w^1{}_1 \wedge w^1{}_1 + w^1{}_2 \wedge w^2{}_1 + w^1{}_3 \wedge w^3{}_1 + w^1{}_4 \wedge w^4{}_1$$

$$= d(0) + 0 + [-\bar{e}^{\lambda}(\lambda' + r^{-1})\theta^2 + \tfrac{1}{2} k \,K\theta^3] \wedge [\bar{e}^{\lambda}(\lambda' + \bar{r}^1)\theta^2 +$$

$$- \tfrac{1}{2} k \,K\theta^3] + [-\bar{e}^{\lambda}(\lambda' + \bar{r}^1)\theta^3 - \tfrac{1}{2} k \,K\theta^2] \wedge [\bar{e}^{\lambda}(\lambda' + \bar{r}^1)\theta^3 +$$

$$+ \tfrac{1}{2} k \,K\theta^2] + [\bar{e}^{\lambda}\dot{\lambda}\theta^1 + \bar{e}^{\lambda}\lambda'\theta^4] \wedge [\bar{e}^{\lambda}\dot{\lambda}\theta^1 + \bar{e}^{\lambda}\lambda'\theta^4]$$

$$= 0 + 0 + 0 + 0$$

(Since $\theta^i \wedge \theta^i = 0$ and $\theta^i \wedge \theta^j = -\theta^j \wedge \theta^i$)

$$\therefore \Omega^1{}_1 = 0 .$$

$$\Omega^1{}_2 = dw^1{}_2 + w^1{}_1 \wedge w^1{}_2 + w^1{}_2 \wedge w^2{}_2 + w^1{}_3 \wedge w^3{}_2 + w^1{}_4 \wedge w^4{}_2$$

$$= d[-\bar{e}^{\lambda}\lambda'\theta^2 - \bar{e}^{\lambda}r^{-1}\theta^2 + \tfrac{1}{2} k \,K\theta^3] + 0 + 0$$

$$+[-\bar{e}^{\lambda}\lambda'\theta^3 - \bar{e}^{\lambda}r^{-1}\theta^3 - \tfrac{1}{2} k \,K\theta^2] \wedge [\bar{e}^{\lambda}r^{-1}\cot\theta\theta^3 - \tfrac{1}{2}k \,K\theta^4$$

$$+ \tfrac{1}{2}k \,K\theta^1] + [\bar{e}^{\lambda}\dot{\lambda}\theta^1 + \bar{e}^{\lambda}\lambda'\theta^4] \wedge [\bar{e}^{\lambda}\dot{\lambda}\theta^2 + \tfrac{1}{2}k \,K\theta^3]$$

$$= -\bar{e}^{\lambda}(-\lambda')dr \wedge \lambda'\theta^2 - \bar{e}^{\lambda}(-\dot{\lambda})dt \wedge \lambda'\theta^2 - \bar{e}^{\lambda}\lambda'' dr \wedge \theta^2 -$$

$$- \bar{e}^{\lambda}\dot{\lambda}' dt \wedge \theta^2 - \bar{e}^{\lambda}\lambda' d\theta^2 - \bar{e}^{\lambda}(-\lambda')dr \wedge r^{-1}\theta^2 -$$

$$- \bar{e}^{\lambda}(-\dot{\lambda})dt \wedge r^{-1}\theta^2 - e^{-\lambda}(-\bar{r}^2)dr \wedge \theta^2 - \bar{e}^{\lambda}r^{-1}d\theta^2 +$$

$$+ \tfrac{1}{2} k \,K' dr \wedge \theta^3 + \tfrac{1}{2} k \,\dot{K} dt \wedge \theta^3 + \tfrac{1}{2} k \,K d\theta^3$$

$$+ \tfrac{1}{2} k \,K\bar{e}^{\lambda}\lambda'\theta^3 \wedge \theta^4 - \tfrac{1}{2} k \,K\bar{e}^{\lambda}\dot{\lambda}\theta^3 \wedge \theta^1 +$$

$$+ \tfrac{1}{2} k \,K\bar{e}^{\lambda}r^{-1}\theta^3 \wedge \theta^4 - \tfrac{1}{2} k \,K\bar{e}^{\lambda}r^{-1}\theta^3 \wedge \theta^1 -$$

$$- \frac{1}{2} k \, K \bar{e}^{\lambda} r^{-1} \cot \theta \theta^2 \wedge \theta^3 + \frac{1}{4} k^2 K^2 \theta^2 \wedge \theta^4$$

$$- \frac{1}{4} k^2 K^2 \theta^2 \wedge \theta^1 + \bar{e}^{2\lambda} (\dot{\lambda})^2 \theta^1 \wedge \theta^2 + \frac{1}{2} k \, K \bar{e}^{\lambda} \dot{\lambda} \theta^1 \wedge \theta^3 +$$

$$+ \bar{e}^{2\lambda} \lambda' \dot{\lambda} \theta^4 \wedge \theta^2 + \frac{1}{2} k \, K \bar{e}^{\lambda} \lambda' \theta^4 \wedge \theta^3$$

Using (2.9), (2.14) and (2.15), we get

$$\Omega^1{}_2 = [\, \bar{e}^{2\lambda} \{ \lambda'' + \lambda' r^{-1} - (\dot{\lambda})^2 \} - \frac{1}{4} k^2 K^2] \theta^2 \wedge \theta^1 +$$

$$+ [\, \bar{e}^{2\lambda} (\lambda' \dot{\lambda} - \dot{\lambda}') - \frac{1}{4} k^2 K^2] \theta^4 \wedge \theta^2$$

$$+ \frac{1}{2} k \, \bar{e}^{\lambda} [\, K' + 2 K \lambda' + 2 K r^{-1} + K \dot{\lambda}] \theta^1 \wedge \theta^3$$

$$+ \frac{1}{2} k \, \bar{e}^{\lambda} [\, \dot{K} + K \dot{\lambda} - K r^{-1}] \theta^4 \wedge \theta^3 .$$

Similarly we can find the other components of $w^i{}_j$.

Thus the non-zero components of curvature form $\Omega^i{}_j$ are

$$\Omega^1{}_2 = [\, \bar{e}^{2\lambda} \{ \lambda'' + \lambda' r^{-1} - (\dot{\lambda})^2 \} - \frac{1}{4} k^2 K^2] \theta^2 \wedge \theta^1$$

$$+ [\, \bar{e}^{2\lambda} (\lambda' \dot{\lambda} - \dot{\lambda}') - \frac{1}{4} k^2 K^2] \theta^4 \wedge \theta^2$$

$$+ \frac{1}{2} k \, \bar{e}^{\lambda} [\, K' + 2 K \lambda' + 2 K r^{-1} + K \dot{\lambda}] \theta^1 \wedge \theta^3$$

$$+ \frac{1}{2} k \, \bar{e}^{\lambda} [\, \dot{K} + K \dot{\lambda} - K r^{-1}] \theta^4 \wedge \theta^3$$

$$\Omega^1{}_3 = [\, \bar{e}^{2\lambda} \{ \lambda'' + \lambda' \bar{r}^1 - (\dot{\lambda})^2 \} - \frac{1}{4} k^2 K^2] \theta^3 \wedge \theta^1$$

$$+ [\, \bar{e}^{2\lambda} (\lambda' \dot{\lambda} - \dot{\lambda}') - \frac{1}{4} k^2 K^2] \theta^4 \wedge \theta^3$$

$$+ \frac{1}{2} k \, \bar{e}^{\lambda} [\, K' + 2 K \lambda' + 2 K r^{-1} + K \dot{\lambda}] \theta^2 \wedge \theta^1$$

$$+ \frac{1}{2} k \, \bar{e}^{\lambda} [\, \dot{K} \div K \dot{\lambda} - K r^{-1}] \theta^2 \wedge \theta^4$$

...(2.18)

$$\Omega^{1}{}_{4} = \bar{e}^{2\lambda}(\ddot{\lambda} - \lambda'')\,\theta^4\wedge\theta^1 + k\,\mathbb{K}\,\bar{e}^{\lambda}(\dot{\lambda} + \lambda' + r^{-1})\,\theta^3\wedge\theta^2.$$

$$\Omega^{2}{}_{3} = \bar{e}^{2\lambda}\left[(\lambda')^2 + 2\lambda' r^{-1} - (\dot{\lambda})^2\right]\theta^3\wedge\theta^2 + \frac{1}{2}k\,\bar{e}^{\lambda}\left[\mathbb{K}' + \mathbb{K}\lambda' + \dot{\mathbb{K}} + \mathbb{K}\dot{\lambda}\right]\theta^1\wedge\theta^4$$

$$\Omega^{2}{}_{4} = \left[\bar{e}^{2\lambda}\{\ddot{\lambda} - (\lambda')^2 - \lambda' r^{-1}\} - \frac{1}{4}k^2\,\mathbb{K}^2\right]\theta^4\wedge\theta^2 + \left[\bar{e}^{2\lambda}(\lambda'\dot{\lambda} - \dot{\lambda}') - \frac{1}{4}k^2\,\mathbb{K}^2\right]\theta^2\wedge\theta^1 + \frac{1}{2}k\,\bar{e}^{\lambda}\left[\mathbb{K}' + \mathbb{K}\lambda' + \mathbb{K}r^{-1}\right]\theta^1\wedge\theta^3 + \frac{1}{2}k\,\bar{e}^{\lambda}\left[\dot{\mathbb{K}} + 2\,\mathbb{K}\dot{\lambda} + \mathbb{K}\lambda'\right]\theta^4\wedge\theta^3 .$$

$$\Omega^{3}{}_{4} = \left[\bar{e}^{2\lambda}\{\ddot{\lambda} - (\lambda')^2 - \lambda' r^{-1} \quad - \frac{1}{4}k^2\,\mathbb{K}^2\right]\theta^4\wedge\theta^3 + \left[\bar{e}^{2\lambda}(\lambda'\dot{\lambda} - \dot{\lambda}') - \frac{1}{4}k^2\,\mathbb{K}^2\right]\theta^3\wedge\theta^1 + \frac{1}{2}k\,\bar{e}^{\lambda}\left[\mathbb{K}' + \mathbb{K}\lambda' + \mathbb{K}r^{-1}\right]\theta^2\wedge\theta^1 + \frac{1}{2}k\,\bar{e}^{\lambda}\left[\dot{\mathbb{K}} + 2\,\mathbb{K}\dot{\lambda} + \mathbb{K}\lambda'\right]\theta^2\wedge\theta^4 .$$

Comparison of these results with

$$\Omega^{i}{}_{j} = \frac{1}{2}R^{i}{}_{jkl}\,\theta^k\wedge\theta^l$$

immediately yields the following components of the Riemann tensor :

$$R^{1}{}_{221} = R^{1}{}_{331} = \bar{e}^{2\lambda}\left[\lambda'' + \lambda' r^{-1} - (\dot{\lambda})^2\right] - \frac{1}{4}k^2\,\mathbb{K}^2$$

$$R^{1}{}_{242} = R^{1}{}_{343} = R^{2}{}_{421} = R^{3}{}_{431} = \bar{e}^{2\lambda}\left[\lambda'\dot{\lambda} - \dot{\lambda}'\right] - \frac{1}{4}k^2\,\mathbb{K}^2$$

$$R^{1}{}_{213} = R^{1}{}_{321} = \frac{1}{2}k\,\bar{e}^{\lambda}\left[\mathbb{K}' + 2\,\mathbb{K}\lambda' + 2\,\mathbb{K}\bar{r}^{1} + \mathbb{K}\dot{\lambda}\right]$$

$$R^1{}_{243} = R^1{}_{324} = \frac{1}{2} k\, e^{-\lambda} \,[\, \dot{K} + K\dot{\lambda} - K r^{-1} \,]$$

$$R^1{}_{441} = e^{-2\lambda} \,[\, \ddot{\lambda} - \lambda'' \,]$$

$$R^1{}_{432} = k\, K e^{-\lambda} \,[\, \lambda' + \dot{\lambda} + r^{-1} \,] \qquad \ldots(2.19)$$

$$R^2{}_{332} = e^{-2\lambda} \,[\, (\lambda')^2 + 2\lambda' r^{-1} - (\dot{\lambda})^2 \,]$$

$$R^2{}_{314} = \frac{1}{2} k\, e^{-\lambda} \,[\, K' + K\lambda' + \dot{K} + K\dot{\lambda} \,]$$

$$R^2{}_{442} = R^3{}_{443} = e^{-2\lambda} \,[\, \ddot{\lambda} - (\lambda')^2 - \lambda' r^{-1} \,] - \frac{1}{4} k^2 K^2$$

$$R^2{}_{413} = R^3{}_{421} = \frac{1}{2} k\, e^{-\lambda} \,[\, K' + K\lambda' + K r^{-1} \,]$$

$$R^2{}_{443} = R^3{}_{424} = \frac{1}{2} k\, e^{-\lambda} \,[\, \dot{K} + 2 K\dot{\lambda} + K\lambda' \,]$$

The Ricci tensor R_{ij} is defined as

$$R_{ij} = g^{kl} R_{kijl} \qquad \ldots (2.20)$$

where $R_{hijk} = g_{ha} R^a{}_{ijk}$.

Also we have $g_{ij} = \mathrm{diag}\,(-1, -1, -1, 1)$.

$$\therefore \; R_{11} = g^{11} R_{1111} + g^{22} R_{2112} + g^{33} R_{3113} + g^{44} R_{4114}$$

$$= 0 + e^{-2\lambda} \,[\, \lambda'' + \lambda' r^{-1} - (\dot{\lambda})^2] - \frac{1}{4} k^2 K^2$$

$$+ e^{-2\lambda} \,[\lambda'' + \lambda' r^{-1} - (\dot{\lambda})^2] - \frac{1}{4} k^2 K^2 - e^{-2\lambda} [\ddot{\lambda} - \lambda'']$$

$$\therefore \; R_{11} = e^{-2\lambda} \,[3\lambda'' + 2\lambda' r^{-1} - 2(\dot{\lambda})^2 - \ddot{\lambda} \,] - \frac{1}{2} k^2 K^2$$

Similarly we can obtain the other components of Ricci tensor R_{ij} .

Therefore the non-zero Ricci tensor R_{ij} are given as follows :

$$R_{11} = \bar{e}^{2\lambda} [\, 3\lambda'' + 2\lambda' r^{-1} - 2(\dot{\lambda})^2 - \ddot{\lambda} \,] - \frac{1}{2} k^2 K^2$$

$$R_{22} = R_{33} = \bar{e}^{2\lambda} [\, \lambda'' + 2(\lambda')^2 + 4\lambda' r^{-1} - 2(\dot{\lambda})^2 - \ddot{\lambda} \,]$$

$$R_{44} = \bar{e}^{2\lambda} [\, 3\ddot{\lambda} - \lambda'' - 2(\lambda')^2 - 2\lambda' \bar{r}^{1} \,] - \frac{1}{2} k^2 K^2$$

$$R_{14} = 2\, \bar{e}^{2\lambda} [\, \dot{\lambda}' - \lambda' \dot{\lambda} \,] + \frac{1}{2} k^2 K^2 \qquad \ldots (2.21)$$

Now the scalar of curvature R is given by $R = g^{ij} R_{ij}$

$$\therefore R = - \bar{e}^{2\lambda} [\, 6\lambda'' + 6(\lambda')^2 + 12\lambda' r^{-1} - 6(\dot{\lambda})^2 - 6\ddot{\lambda} \,] \ldots (2.22)$$

3) THE FIELD EQUATIONS

A non-static conformally flat spherically symmetric perfect fluid is considered by the space-time metric :

$$ds^2 = e^{2\lambda} (- dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2) \qquad \ldots (3.1)$$

where λ is a function of r and t alone. The energy-momentum tensor for a perfect-fluid distribution is given by

$$T_{ij} = (p + \rho) V_i V_j - p g_{ij} \qquad \ldots (3.2)$$

together with $g^{ij} V_i V_j = 1$

where p is the pressure, ρ is the density and $V_i = (V_1, 0, 0, V_4)$ is the flow-vector which describes the radial motion of the fluid.

The Einstein-Cartan equations are

$$R_{ij} - \frac{1}{2} Rg_{ij} + \Lambda g_{ij} = - 8\pi T_{ij} \qquad \ldots (3.3)$$

$$Q^{i}_{jk} - \delta^{i}_{j} Q^{l}_{lk} - \delta^{i}_{k} Q^{l}_{jl} = - kS^{i}_{jk} \qquad \ldots (3.4)$$

Hence the field equations (3.3) for the metric (3.1) by using (2.21), (2.22) and (3.2), we have

$$R_{11} - \frac{1}{2} Rg_{11} + \Lambda g_{11} = - 8\pi T_{11}$$

$$\therefore \quad \bar{e}^{2\lambda} [3\lambda'' + 2\lambda' r^{-1} - 2(\dot{\lambda})^2 - \ddot{\lambda}] - \frac{1}{2} k^2 K^2$$

$$- \frac{1}{2} \bar{e}^{2\lambda} [6 \lambda'' + 6 (\lambda')^2 + 12 \lambda' r^{-1} - 6(\dot{\lambda})^2 - 6\ddot{\lambda}] + \Lambda(-1)$$

$$= - 8\pi [(p + q)V_1^2 - p(-1)]$$

$$\therefore \quad 3(\lambda')^2 + 4 \lambda' r^{-1} - 2\ddot{\lambda} - (\lambda)^2 + e^{2\lambda} + \frac{1}{2} k^2 K^2 e^{2\lambda}$$

$$= 8 \pi [(p + q)V_1^2 + p e^{2\lambda}]$$

Similarly we can easily obtain the other field equations.

Thus the field equations are

$$3(\lambda')^2 + 4 \lambda' r^{-1} - 2\ddot{\lambda} - (\dot{\lambda})^2 + \Lambda e^{2\lambda} + \frac{1}{2} k^2 K^2 e^{2\lambda}$$

$$= 8 \pi[(p + q)V_1^2 + p e^{2\lambda}] \qquad \ldots (3.5)$$

$$2 \lambda'' + (\lambda')^2 + 2 \lambda' r^{-1} - 2 \ddot{\lambda} - (\dot{\lambda})^2 + \Lambda e^{2\lambda}$$

$$= 8 \pi p e^{2\lambda} \qquad \ldots (3.6)$$

$$- 2\lambda'' - (\lambda')^2 - 4\lambda' r^{-1} + 3(\dot{\lambda})^2 - \Lambda e^{2\lambda} + \frac{1}{2} k^2 K^2 e^{2\lambda}$$

$$= 8\pi \, [(p + \rho)V_4^2 - pe^{2\lambda}] \qquad \ldots (3.7)$$

$$- 2\dot{\lambda}' + 2\lambda'\dot{\lambda} - \frac{1}{2} k^2 K^2 e^{2\lambda} = 8\pi(p + \rho)V_1V_4 \qquad \ldots (3.8)$$

4) SOLUTION OF THE FIELD EQUATIONS

Eliminating the cosmological constant term $-\Lambda$, we get the field equations as

$$- 2\lambda'' + 2(\lambda')^2 + 2\lambda' r^{-1} + \frac{1}{2} k^2 K^2 e^{2\lambda} = 8\pi(p + \rho)V_1^2 \qquad \ldots (3.9)$$

$$- 2\lambda' r^{-1} + 2(\dot{\lambda})^2 - 2\ddot{\lambda} + \frac{1}{2} k^2 K^2 e^{2\lambda} = 8\pi(p + \rho)V_4^2 \qquad \ldots (3.10)$$

$$- 2\dot{\lambda}' + 2\lambda'\dot{\lambda} - \frac{1}{2} k^2 K^2 e^{2\lambda} = 8\pi(p + \rho)V_1V_4 \qquad \ldots (3.11)$$

Following Hehl's [8,9] approach by redefining pressure and density as

$$\bar{p} = (p - 2\pi K^2), \quad \bar{\rho} = (\rho - 2\pi K^2)$$

the above field equations can be written as

$$- 2\lambda'' + 2(\lambda')^2 + 2\lambda' r^{-1} = 8\pi \, (\bar{p} + \bar{\rho})V_1^2 \qquad \ldots (3.12)$$

$$- 2\lambda' r^{-1} + 2(\dot{\lambda})^2 - 2\ddot{\lambda} = 8\pi \, (\bar{p} - \bar{\rho})V_4^2 \qquad \ldots (3.13)$$

$$- 2\dot{\lambda}' + 2\lambda'\dot{\lambda} = 8\pi \, (\bar{p} + \bar{\rho})V_1V_4 \qquad \ldots (3.14)$$

Eliminating $\bar{p}$, $\bar{\rho}$, V_1 and V_4 from the equations (3.12) to (3.14), we obtain the differential equation

$$[\dot{\lambda} + \frac{\lambda'}{r}] [\lambda'' - \frac{\lambda'}{r} - (\lambda')^2] + (\dot{\lambda})^2 [\frac{\lambda'}{r} - \lambda''] = \dot{\lambda}' (\dot{\lambda}' - 2\dot{\lambda}\lambda') \quad \ldots (3.15)$$

The solution of this differential equation can be written in the form

$$\lambda = A(r^2 - t^2) + Bt + C \quad \ldots (3.16)$$

where A, B, C are arbitrary constants.

Thus the metric (3.1) can be written as

$$ds^2 = e^{[A(r^2-t^2)+Bt+C]} [-dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2]. \quad \ldots (3.17)$$

5) DISCUSSION

(a) If we have not considered the torsion and spin, then the model is reduced to a conformally non-static spherically symmetric perfect-fluid distribution studied by Singh and Abdussattar [22]. In that case they have shown that the pressure and density are given by

$$8\pi p = e^{-[A(r^2-t^2)+Bt]} [A^2(r^2-t^2) + ABt + 6A - \frac{B^2}{4}] + \Lambda$$

$$8\pi \rho = e^{-[A(r^2-t^2)+Bt]} [-3A^2(r^2-t^2) - 3ABt - 6A + \frac{3B^2}{4}] - \Lambda$$

(b) If we take $t = 0$

then (3.16) becomes

$\lambda = Ar^2 + C$.

This shows that λ is a function of r only. In this case the model reduces to a static. This further shows that the study done by Kalyanshetti and Waghmode [11] is a particular case.

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