CHAPTER - II

STARLIKE NORMALISATIONS

ABSTRACT

Let $E = \{ z : |z| < 1 \}$ be the unit disc. Let S designate the family of holomorphic univalent functions with normalisations $f(o) = 0 = f'(o) - 1$. Let $S^*(\alpha)$ denote the class of functions

Repetition

$$S(z) = z + az^3 + \sum_{n=4}^{\infty} a_n z^n$$

which are holomorphic, univalent and starlike of order α, $(0 \leq \alpha < 1)$ with second missing coefficients. Putting to use this class, some results related to regions of univalence are gained on the lines of Causey and Merkes.[5] The Class $V_a(\alpha, \beta)$, the class of functions $f(z)$ having Taylor Series expansion

$$f(z) = z + \sum_{2}^{\infty} a_n z^n, \text{ which can be expressed as}$$

$f(z) = 1/2 \mid g(z) + zg'(z) \mid$ where $g(z) \in S_a^*(\alpha, \beta)$, has been discussed from the point view of radii of Starlikeness.

Reference

The Class $D(\alpha, \beta, \varsigma)$ studied by Kulkarni S.R. has been generalised to $D(\alpha, \beta, \varsigma, \nu, \lambda)$ pacifying the condition

$$\left| \frac{f'(z) - 1}{2\varsigma\left[f'(z) - \alpha + (1-\nu)\cos\lambda \cdot e^{-i\lambda}\right] - (f'(z) - 1)} \right| < \beta$$

..2

 Specify !

with appropriate restrictions on α, β, ς, ν, λ has been looked into from the point view of negative coefficients and varieties of results pertaining to linear combinations, radius of convexity etc. have been attempted.

In section III this idea of region of univalence has been interwoven via Gupta - Ahmad literature. [ ]

The univalent functions with second missing coefficient have been extensively used in generalising the results of Kulkarni - Swamy. [ ]

In the concluding section IV, some novel results leading to the regions of univalency have been obtained for the class $S(\alpha, \beta, \varsigma)$ introduced by Kulkarni, S.R.[10] satisfying

$$\left| \frac{Zf'/f - 1}{\left[2\varsigma\left(Zf'/f - \alpha \right) - \left(Zf'/f - 1 \right) \right]} \right| < \beta$$

$\beta \in (0,1)$, $1/2 \leqslant \varsigma \leqslant 1$, $0 \leqslant \alpha < 1/2\varsigma$ 10

Particular cases and sharp results are listed wherever plausible.

STARLIKE NORMALISATIONS

SECTION I

1. INTRODUCTION :

Let $E = \{ z : |z| < 1 \}$ and $S\ (\alpha)$ are class of univalent starlike functions of order α, $0 \leqslant \alpha < 1$. $S(z) = z + az^3 + \ldots$, Starlike functions of order α, with second missing coefficients were first introduced by M.S.Robertson [16].

We have the following definitions for our investigations.

Definition 1.1 Let $S^*(\alpha)$ denote the class of functions

$$W = S(z) = z + az^3 + \sum_{n=4}^{\infty} a_n z^n,$$ which are holomorphic,

Repetition

univalent and starlike of order α, for $|z| < 1$,

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)} \right] > \alpha, \text{ for } |z| < 1, \text{ where } 0 \leqslant \alpha < 1.$$

S^α

Definition 1.2 Let S^* denote the class of functions

Earlier denoted differently subclass

$$W = f(z) = z + az^3 + \sum_{n=4}^{\infty} a_n z^n,$$ which are holomorphic and univalent for $|z| < 1$, onto a starshaped domain. The class $S^* = S^*_0$ i.e.

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)} \right] > 0 \text{ for } |z| < 1., \quad z \in E.$$

..4

The functions in $S^*(\alpha)$ are univalent in E.

J.S.Ratti, has attained in [15] the discs of univalence for certain classes of functions f(z) holomorphic in E. Some of his consequences required that either

$$\operatorname{Re}\left\{\frac{g(z)}{z}\right\} > 0 \text{ or } \quad \operatorname{Re}\left\{\frac{g(z)}{z}\right\} > 1/2$$

be satisfied in E, where $\operatorname{Re}\left\{ f(z)/g(z)\right\} > 0$. Causey and Merkes [5] obtained such types of results mere by replacing the expression $g(z)/z$ by $g(z)/S(z)$ ~~or~~ $S(z)/g(z)$ where $S(z) \in S^*(\alpha)$, defined above.

In the second section of this chapter we have continued the same discussion on Starlike normalisations, for S(z) belonging to the class $V_a(\alpha, \beta)$, we define this Class $V_a(\alpha, \beta)$ as follows :

Let $p(\alpha, \beta) = q(z) = 1 + q_1 z + q_2 z^2 + \ldots$

$$\left| \frac{q(z) - 1}{(2\beta-1)q(z) + 1 - 2\alpha\beta} \right| < 1$$

Is it not E?

for $0 \leq \alpha < 1, \quad 0 < \beta \leq 1, \quad z \in D = \{z : |z| < 1\}$

Let $P_a(\alpha, \beta) = q(z) \in P(\alpha, \beta) : q'(0) = 2a\beta(1-\alpha), 0 \leq a \leq 1$

$$S_a^*(\alpha, \beta) = f(z) = z + 2a\beta(1-\alpha)z^2 + \ldots, \frac{zf'(z)}{f(z)} \in P_a(\alpha, \beta)$$

$$0 \leq a \leq 1 \quad .$$

The class $V_a(\alpha, \beta)$ consists of the functions f(z) which can be expressed as

$f(z) = 1/2 \left| g(z) + z g'(z) \right|$, where $g(z) \in S_a^*(\alpha, \beta)$

1.2. SOME LEMMAS :

construction

The following Lemmas we require during our course of research.

The class $P(\alpha)$ is not defined

Lemma 1.2.1 If $h(z)$ is in $P(\alpha)$, then

$$\mathrm{Re}\left\{h(z)\right\} \geqslant \frac{1-(1-2\alpha)\,|z|}{1+|z|}$$

This readily succeeds from the fact that

$$h(z) = \frac{1-(1-2\alpha)\;z\;\phi(z)}{1+z\,\phi(z)},$$

where $\phi(z)$ is holomorphic and $|\phi(z)| \leqslant 1$, in E.

Lemma 1.2.2 If $h(z)$ is in $P(0)$, then

$$\mathrm{Re}\left\{\frac{z\,h'(z)}{h(z)}\right\} \geqslant \frac{-2\,|z|}{1-|z|^2}$$

This is followed from Lemma 2.1.

Lemma 1.2.3 Let $h(z)$ be in $P(1/2)$. Then

$$\mathrm{Re}\left\{\frac{z\,h'(z)}{h(z)}\right\} \leqslant \frac{|z|}{1-|z|}, \quad \text{for } z \in E.$$

This can be found in [5]

Lemma 1.2.4 If $f(z) = z + az^3 + \sum_{n=4}^{\infty} a_n z^n$ belongs to

$S^*(\alpha)$,

then

$$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant \frac{(1-\alpha) + a|z| - a(1-2\alpha)|z|^2 - (1-2\alpha)(1-\alpha)|z|^3}{(1-\alpha) + a|z| + a|z|^2 + (1-\alpha)|z|^3}$$

$0 \leqslant a \leqslant 1$. This result is sharp for the functions

$$f_{a,\alpha}(z) = z\left[\frac{(1-\alpha)^{1-\alpha}}{(1+z)^{1-\alpha-a}\left[(1-\alpha)z^2 - (1-\alpha-a)\,z + (1-\alpha)\right]^{1-\alpha}}\right]^{\frac{2(1-\alpha)}{3(1-\alpha)-a}}$$

wherever $a/1-\alpha \leqslant 1$

This can be found in [5]

1·3. <u>REGION OF UNIVALENCY FOR STARLIKENESS</u> :

<u>Theorem 1·3.1</u> Suppose $f(z)$ and $g(z)$ belong to A, where A is the class of functions $f(z) = Z + \sum_{2}^{\infty} a_n z^n$, holomorphic in E,

$S(z) \in S^*(\alpha)$, pacifying the condition ,

$\operatorname{Re}\left\{\frac{g(z)}{S(z)}\right\} > 0$, $z \in E$, then $f(z)$ is univalent and

Starlike, where the radius of Starlikeness is given by the expression -

$$r^5\left[(1-2\alpha)(1-\alpha)\right]+r^4\left[-4(1-\alpha)+a(1-2\alpha)\right]+r^3\left[-4a-(1-2\alpha)(1-\alpha)-a\right]$$

$$+r^2\left[-4a-a(1-2\alpha)-(1-\alpha)\right]+r\left[-4(1-\alpha)+a\right]+(1-\alpha)=0$$

<u>Proof</u> The hypothetical condition $\operatorname{Re}\left[g(z)/S(z)\right]>0$ implies that $g(z)/S(z) = p_1(z)$ ie. $g(z) = S(z) \cdot p_1(z)$

and $\operatorname{Re}\left[\frac{f(z)}{g(z)}\right]>0 \Rightarrow \frac{f(z)}{g(z)} = p_2(z)$ i.e. $f(z)$

$= g(z) \cdot p_2(z).$

What are $p_1(z)$ and $p_2(z)$? Specify.

Consequently we can express :

(3.2) $f(z) = p_1(z) \cdot p_2(z) \cdot S(z).$

Logarithmic differentiation of (3.2) yields,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{z\,p_1'(z)}{p_1(z)}\right\} + \operatorname{Re}\left\{\frac{z\,p_2'(z)}{p_2(z)}\right\} +$$

$$+\operatorname{Re}\left\{\frac{z\,S'(z)}{S(z)}\right\}$$

Application of Lemmas 2.1, 2.2 and 2.4 bring forth

$$\operatorname{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\} \geqslant \frac{-2r}{1-r^2} + \frac{-2r}{1-r^2} +$$

What about ?

$$+\frac{(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)\,r^3}{(1-\alpha)+ar+ar^2+(1-\alpha)\,r^3} \geqslant 0$$

Hence f(z) is starlike, the radius r_o of Starlikeness is given by equation.

$$r^5\left[(1-2\alpha)(1-\alpha)\right]+r^4\left[-4(1-\alpha)+a(1-2\alpha)\right]+r^3\left[-4a-(1-2\alpha)(1-\alpha)-a\right]$$

$$+r^2\left[-4a-a(1-2\alpha)-(1-\alpha)\right]+r\left[-4(1-\alpha)+a\right]+(1-\alpha)=0$$) / (1)

We notice that $f(o)=(1-\alpha)>0$ and // how ?

$f(1)=-8a-8(1-\alpha)<0$, which evidently shows that r_o lies between 0 and 1. // What is r_o ?

The result is sharp for the following functions.

$$g(z)=\frac{z(1+z)}{(1-z)^{3-2\alpha}}\ ,\qquad f(z)=\frac{z(1+z)^2}{(1-z)^{4-2\alpha}}\ ,$$

$$S(z)=z\left[\frac{(1-\alpha)^{1-\alpha}}{(1+z)^{1-\alpha-a}\left[(1-\alpha)z^2-(1-\alpha-a)z+(1-\alpha)\right]^{1-\alpha}}\right]^{\frac{2(1-\alpha)}{3(1-\alpha)-a}}$$

wherever $a/(1-\alpha)\leqslant 1.$

<u>Particular Cases</u> : For $\alpha=o$, we get the radius of Starlikeness for 2nd missing coefficients. // how ?

$$r^5 - r^4 (a-4) - r^3 (5a+1) - r^2(5a+1)$$
$$+ r(a-4) + (1-\alpha) = 0$$

Theorem 3.2. Let $f(z)$ and $g(z)$ be in A, $S(z) \in S^*(\alpha)$, $0 \leqslant \alpha < 1$, If $\operatorname{Re}\left\{\frac{g(z)}{S(z)}\right\} > 1/2$, $\operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0$, $z \in E$, then $f(z)$ is univalent and starlike for $|z| < r_\alpha$, where r_α is given by the equation :

$$\frac{(1-\alpha)+ar - a(1-2\alpha)r^2 -(1-2\alpha)(1-\alpha)r^3}{(1-\alpha) + ar + ar^2 + (1-\alpha)r^3} - \frac{2r}{(1-r^2)} - \frac{r}{1+r} = 0$$

Proof :

The facts Re $\left\{\frac{g(z)}{S(z)}\right\} > 1/2$ implies that $g(z) = p_1(z)\cdot S(z)$ where $p_1(z) \in P(1/2)$ also Re $\left\{\frac{f(z)}{S(z)}\right\} > 0$ means that

$f(z) = p_2(z).\ g(z)$, where $p_2(z) \in P(0)$.

Therefore combining these two conditions we obtain,

$$f(z) = p_1(z) \cdot p_2(z) \cdot S(z).$$

Differentiating logarithmically, we achieve,

$$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{z p_1'(z)}{p_1(z)}\right\} + \operatorname{Re}\left\{\frac{z p_2'(z)}{p_2(z)}\right\} + $$

$$+ \operatorname{Re}\left\{\frac{z S'(z)}{S(z)}\right\}$$

Making use of Lemmas 2.2, 2.3, and 2.4, we get,

$$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant \frac{-r}{1+r} + \frac{-2r}{1-r^2} +$$

$$+ \frac{(1-\alpha) + ar - a(1-2\alpha)r^2 - (1-2\alpha)(1-\alpha)r^3}{(1-\alpha) + ar + ar^2 + (1-\alpha)r^3} > 0$$

Hence f(z) is starlike, the radius r_0 of which is given by the equation.

$$H(r) = \frac{(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3} - \frac{2r}{(1-r^2)} - \frac{r}{1+r} = 0$$

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geq \frac{(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3} - \frac{2r}{(1-r^2)} - \frac{r}{1+r} > 0$$

The result is sharp for the following functions.

$$S(z) = Z\left[\frac{(1-\alpha)^{1-\alpha}}{(1+z)^{1-\alpha-a}\left[(1-\alpha)z^2-(1-\alpha-a)Z+(1-\alpha)\right]^{1-\alpha}}\right]^{\frac{2(1-\alpha)}{3(1-\alpha)-a}}$$

$$p_1(z) = \frac{1}{1-z}$$

Particular Cases - for $\alpha = 0$, we get the result of radius of Starlikeness for starlike functions having second missing coefficients and can be restated as follows :

Corollary:

Let $f(z)$ and $g(z)$ be in A, $S(z) \in S^*(\alpha)$

If $\operatorname{Re}\left\{g(z)/S(z)\right\} > 1/2$, $\operatorname{Re}\left\{f(z)/g(z)\right\} > 0$, $Z \in E$, then $f(z)$ is univalent and starlike for $|z| < r$, where 'r' is given by the equation. $\frac{1+ar-ar^2-r^3}{1+ar+ar^2+r^3} - \frac{2r}{1-r^2} - \frac{r}{1+r} = 0$

Theorem :1.3.3

Let $f(z)$ and $g(z)$ be in A, $S(z) \in S^*(\alpha)$,

If $\operatorname{Re}\left\{\frac{g(z)}{S(z)}\right\} > 0$ and $\left|\frac{f(z)}{g(z)} - 1\right| < 1$, for $Z \in E$, then

$f(z)$ is univalent and Starlike, the radius of starlikeness is given by the equation,

$$H(r) = \frac{(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3} - \frac{2r}{(1-r^2)} - \frac{r}{(1-r)} = 0$$

Proof :

We first note that $\left| f/g - 1 \right| < 1$ if and only if $\operatorname{Re}\left\{ g/f \right\} > 1/2$. // why?

$\operatorname{Re}\left\{ \frac{g(z)}{S(z)} \right\} > 0$ implies that,

$g(z) = S(z) . P_2(z)$, where $P_2(z) \in \mathcal{P}(0)$ and

$\operatorname{Re}\left\{ g/f \right\} > 1/2$ implies that $g(z) = f(z) . p_1(z)$, where $p_1(z) \in \mathcal{P}(1/2)$.

Hence these two conditions together yield,

$$f(z) = \frac{S(z) . P_2(z)}{p_1(z)} .$$

Thus, differentiating logarithmically, we get,

$$\operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} = \operatorname{Re}\left\{ \frac{zS'(z)}{S(z)} \right\} + \operatorname{Re}\left\{ \frac{zP_2'(z)}{P_2(z)} \right\} - \operatorname{Re}\left\{ \frac{z\,p_1'(z)}{p_1(z)} \right\}$$

small z

In view of Lemmas 2.2 , 2.3 and 2.4, the above expression reduces to

$$\operatorname{Re}\left\{ \frac{z\,f'(z)}{f(z)} \right\} \geqslant \frac{(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3} - \frac{2r}{(1-r^2)} - \frac{r}{(1-r)} \geqslant 0$$

13

Hence f(z) is starlike and the radius of Starlikeness is given by the equation $H(r) = 0$, where small

$$H(r) = \frac{(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3} - \frac{2r}{(1-r^2)} - \frac{r}{(1-r)} = 0$$

We can easily verify that $H(0) > 0$ and $H(1) < 0$, hence we assure that the root lies between 0 and 1.

The result is sharp for the following functions.

$$S(z) = z\left[\frac{(1-\alpha)^{1-\alpha}}{(1+z)^{1-\alpha-a}\left[(1-\alpha)z^2-(1-\alpha-a)z+(1-\alpha)\right]^{1-\alpha}}\right]^{2(1-\alpha)/3(1-\alpha)-a}$$

$$p_1(z) = {}^1/(1-z)\ , \quad p_2(z) = \frac{1-z}{1+z}$$

<u>Particular Case</u> :

For $\alpha = 0$, we obtain the radius of Starlikeness for the starlike functions with 2nd missing coefficient.

<u>Corollary</u> :

Let f(z) and g(z) be in A, $S(z) \in S^*$. If small

$\mathrm{Re}\left\{\frac{g(z)}{S(z)}\right\} > 0$ and $|\,f(z)/_{g(z)} - 1| < 1$, for $z \in E$, then f(z) is univalent and starlike, the radius of Starlikeness small is given by the equation,

$$\frac{1+ar-ar^2-r^3}{1+ar+ar^2+r^3} - \frac{2r}{(1-r^2)} - \frac{r}{(1-r)} = 0$$

Theorem 3.4

Let f(z) and g(z) be in A and S(z) be in $S^*(\alpha)$, If

$\text{Re}\left\{\frac{g(z)}{S(z)}\right\} > \frac{1}{2}$ and $\left|\frac{f(z)}{g(z)} - 1\right| < 1$, for $z \in E$, then f(z) is univalent and starlike the radius of which is given by the polynomial, $K(r) = 0$, as stated,

$$K(r) = r^5(1-2\alpha)(1-\alpha) + r^4(a(1-2\alpha)-2(1-\alpha)) - r^3(3a+(1-2\alpha)(1-\alpha)) + r^2(a(1-2\alpha)-3(1-\alpha)-2a) + r(a-2(1-\alpha)) + (1-\alpha).$$

Proof :

By the aforesaid reasoning as in Theorem 3.3,

$g(z) = f(z)\ p_1(z)$ where $p_1(z) \in \mathcal{P}(1/2)$. Also $g(z) = S(z).\ p_2(z)$ where $p_2(z) \in \mathcal{P}(1/2)$. Hence by lemmas 2.3 and 2.4 it follows that

$$\text{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant \frac{(1-\alpha)+ar-a(1-2\alpha)r^2-(1-2\alpha)(1-\alpha)r^3}{(1-\alpha)+ar+ar^2+(1-\alpha)r^3} - \frac{r}{1+r} - \frac{r}{1-r} \geqslant 0$$

Hence f(z) is starlike, the radius of starlikeness, is given by the equation $K(r) = 0$, where K(r) is

..15

$$K(r) = r^5(1-2\alpha)(1-\alpha) + r^4\left\{ a(1-2\alpha) - 2(1-\alpha)\right\}$$
$$- r^3\left\{3a + (1-2\alpha)(1-\alpha)\right\} + r^2\left\{ a(1-2\alpha) - 3(1-\alpha) - 2a\right\}$$
$$+ r\left\{a-2(1-\alpha)\right\} + (1-\alpha)$$

Again we have $K(0) = (1-\alpha) > 0$

and $K(1) < 0$

Hence confirming that the root lies between 0 and 1.

This result is sharp for the following functions.

$$S(z) = Z\left[\frac{(1-\alpha)^{1-\alpha}}{(1+z)^{1-\alpha-a}\left[(1-\alpha)z^2 - (1-\alpha-a)z + (1-\alpha)\right]^{(1-\alpha)}}\right]^{2(1-\alpha)/3(1-\alpha)-a}$$

$$p_1(z) = 1/(1+z)$$

$$p_2(z) = 1/(1-z)$$

<u>Particular Cases</u> :

For $\alpha = 0$, we get the radius of starlikeness of $f(z)$ but for 2nd missing coefficient, which is not found in literature.

$$K(r) = r^5 + (a-2)r^4 - (3a+1)r^3 - (a+3)r^2$$
$$+ (a-2)r + (1-\alpha)$$

We continue our discussion of finding the regions of univalency, particularly that of starlikeness, within the same frame of Causey and Merkes[5]. Here we obtain these results for the Class $V_a(\alpha, \beta)$, the Class of functions $f(z)$ which can be expressed as

$$f(z) = 1/2 \left[g(z) + zg'(z) \right] \text{ where } g(z) \in S_a^*(\alpha, \beta)$$

$$0 \leqslant \alpha < 1, \quad 0 < \beta \leqslant 1, \quad \text{and} \quad 0 \leqslant a < 1.$$

We consider the following class of function

$$P(\alpha, \beta) = \left\{ q(z) = 1 + q_1(z) + q_2(z^2) + \ldots \right.$$

$$\left. : \left| \frac{q(z) - 1}{(2\beta - 1)\, q(z) + 1 - 2\alpha\beta} \right| \right\}$$

For $0 \leqslant \alpha < 1$, $0 < \beta \leqslant 1$ and $z \in D = \{ z : |z| < 1 \}$

Let $P_a(\alpha, \beta) = \left\{ q(z) \in P(\alpha, \beta) : q'(0) = 2a\beta(1 - \alpha) \right.$

$$\left. 0 \leqslant a \leqslant 1 \right\}$$

$$S_a^*(\alpha, \beta) = \left\{ f(z) = z + 2a\beta(1 - \alpha) z^2 + \ldots, \right.$$

$$\left. \frac{zf'(z)}{f(z)} \in P_a(\alpha, \beta) \quad 0 \leqslant a \leqslant 1 \right\}$$

We necessitate the following result for our discussion of region or univalency. This can be found in [18]

Theorem :

Let $f(z) \in Va(\alpha, \beta)$, then

$$\text{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant \frac{2(1-2\beta)}{\beta(1-\alpha)} + \frac{1}{\beta(1-\alpha)} \cdot \frac{2}{(1-r^2)} \quad x$$

$$\left[\sqrt{(\alpha\beta+\beta)\,(1-(\alpha\beta+\beta-1)r^2)\left[(4\beta-1-2\alpha\beta)(1-r^2)+1-(2\beta-1)^2r^2\right]}\right.$$

$$\left. -\,(1-(2\beta-1)(\alpha\beta+\beta-1)\,r^2\right], \text{ when } Ra \leqslant R^*$$

$$\left[\,1+4\alpha\beta ar + \left(2\alpha\beta^2(\alpha+1)\,a^2 + \beta(5\alpha-1) - .2\right)r^2\right.$$

$$\left. + 2a(1+\alpha)\,\beta\,(2\alpha\beta-1)r^3 + (\alpha\beta+\beta-1)\,(2\alpha\beta-1)\,r^4\right] /$$

$$\left(1+ra(\alpha\beta+\beta) + (\alpha\beta+\beta-1)\,r^2\right)\left(1+2a\beta r + (2\beta-1)\,r^2\right)$$

when $Ra \geqslant R^*$

$$\text{where } Ra = \frac{1+(\alpha+1)\beta ar + (\alpha\beta+\beta-1)\,r^2}{1+2\beta ar + (2\beta-1)\,r^2}$$

$$R^* = \sqrt{\frac{(\alpha+1)(1-(\alpha\beta+\beta-1)\,r^2}{2\,(2-\alpha+(\alpha-2\beta)\,r^2)}}$$

4. SOME THEOREMS :

Theorem-4.1

Suppose $f(z)$ and $g(z)$ are in A and $\text{Re}\left\{g(z)/S(z)\right\}>0$, $z \in E$, $S(z) \in Va(\alpha, \beta)$. If $\text{Re}\left\{f(z)/g(z)\right\} > 0$, $z \in E$, then $f(z)$ is univalent and starlike in $|z| < r_\alpha$, where r_α is given by the polynomial

small

how

$$H(r) = \frac{-2r}{(1-r^2)} - \frac{2r}{(1-r^2)} + \frac{2(1-2\beta)}{\beta(1-\alpha)} + \frac{1}{\beta(1-\alpha)} \cdot \frac{2}{(1-r^2)} \; x$$

$$\left[\sqrt{(\alpha\beta + \beta)\left[1-(\alpha\beta+\beta-1)r^2\right]\left[(4\beta-1-2\alpha\beta)(1-r^2) + 1-(2\beta-1)^2 r^2\right]} - \left[1-(2\beta-1)(\alpha\beta+\beta-1)r^2\right] = 0\right.$$

$(Ra \leqslant R^*)$ what are these? Are they the same as that are used by Soni?

<u>Proof</u> :

$$\operatorname{Re}\left[\frac{g(z)}{S(z)}\right] > 0 \Rightarrow \frac{g(z)}{S(z)} = p_1(z)$$

$$\therefore \; g(z) = p_1(z) \cdot S(z)$$

$$\operatorname{Re}\left[\frac{f(z)}{g(z)}\right] > 0 \qquad \frac{f(z)}{g(z)} = p_2(z)$$

$$f(z) = g(z) \cdot p_2(z)$$

Hence $f(z) = p_1(z) \cdot p_2(z) \cdot S(z)$.

Logarithmic differentiation and application of relevant construction Lammas in Section I and in this section we obtain e

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{-2r}{(1-r^2)} - \frac{2r}{(1-r^2)} + \frac{2(1-2\beta)}{\beta(1-\alpha)} + \frac{1}{\beta(1-\alpha)} \cdot \frac{2}{(1-r^2)}$$

$$\sqrt{(\alpha\beta + \beta)\left[1-(\alpha\beta+\beta-1)r^2\right]\left[(4\beta-1-2\alpha\beta)(1-r^2) + 1-(2\beta-1)^2 r^2\right]}$$

$$- \left[1-(2\beta-1)(\alpha\beta+\beta-1)r^2\right] \quad Ra \leqslant R^*$$

?

and

$$\text{and} \geqslant \frac{-2r}{(1-r^2)} - \frac{2r}{(1-r^2)} + \Big[1 + 4\alpha\beta ar + \big(2\alpha\beta^2(\alpha+1)a^2 + \beta(5\alpha-1)-2\big)r^2$$

$$+ 2a(1+\alpha)\beta(2\alpha\beta-1)r^3 + (\alpha\beta+\beta-1)(2\alpha B-1)r^4)/$$

$$\big(1+ra(\alpha\beta+\beta) + (\alpha\beta+\beta-1)r^2\big)\big(1+2a\beta r+(2\beta-1)r^2\big)$$

when $R_a \geqslant R^*$.

Hence $f(z)$ is univalent and starlike in $|z| < r_\alpha$ where r_α is given by the equation $H(r) = 0$. // What does this mean ?

<u>Theorem 1.4.2</u>

Let $f(z)$ and $g(z)$ be in A, $S(z) \in Va(\alpha, B)$ / 0

If $\operatorname{Re}\{g(z)/S(z)\} > 1/2$, $\operatorname{Re}\{f(z)/g(z)\} > 0$, $\cancel{z} \in E$. z/ small

Then $f(z)$ is univalent and starlike for $|z| < r_\alpha$.

where r_α is the root given by the equation $H(r) = 0$.

$$H(r) = -\frac{2r}{1-r^2} - \frac{2r}{1-r^2} + \big[1+4\alpha\beta ar + (2\alpha\beta^2(\alpha+1)a^2 + \beta(5\alpha-1)-2)r^2 + 2a(1+\alpha)\beta(2\alpha\beta-1)r^3 + (\alpha\beta+\beta-1)(2\alpha\beta-1)r^4 \big/ \big(1+ra(\alpha\beta+\beta) + (\alpha\beta+\beta-1)r^2\big)\big(1+2a\beta r+(2B-1)r^2\big)$$

Proof : Arguing exactly on the same lines as in section I. and using the appropriate lemmas and therem we arrive at // Sp

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{r}{1-r} - \frac{2r}{1-r^2} + \frac{2(1-2B)}{\beta(1-\alpha)} +$$

$$\frac{1}{\beta(1-\alpha)} \cdot \frac{2}{(1-r^2)} \frac{\Big[\sqrt{(\alpha\beta+\beta)\,[1-(\alpha\beta+\beta-1)r^2]}.}{\big[(4\beta-1-2\alpha\beta)(1-r^2) + (1-(2\beta-1)^2 r^2\big]}$$

$$- \big[1-(2\beta-1)(\alpha\beta+\beta-1)\,r^2\big]$$

$$Ra \leqslant R^*$$

$$\text{and} \geqslant \frac{-2r}{(1-r^2)} - \frac{2r}{(1-r^2)} + \Big[1 + 4\alpha\beta a r + (2\alpha\beta^2(\alpha+1)\, a^2 +$$

$$\beta\,(5\alpha-1)) - 2\,)\, r^2 + 2a(1+\alpha)\beta\,(2\alpha\beta-1) r^3 +$$

$$(\alpha\beta+\beta-1)\,(2\alpha\beta-1) r^4 /$$

$$(1 + ra\,(\alpha\beta+\beta) + (\alpha\beta+\beta-1)\, r^2)\, x$$

$$(1 + 2a\beta r + (2\beta-1)\, r^2) \text{ when } Ra \geqslant R^*.$$

Hence $f(z)$ is univalent and starlike in $|z| < r_\alpha$, where r_α is given by the equation $H(r) = 0$.

Theorem 4.3 :

Let $f(z)$ and $g(z)$ be in A and $S(z) \in Va\,(\alpha, B)$ λ / 0

If $\text{Re}\left\{\frac{g(z)}{S(z)}\right\} > 0$, and $\left|\frac{f(z)}{g(z)} - 1\right| < 1$, for $z \in E$, then z / small
$f(z)$ is univalent and starlike for $|z| < r_\alpha$ where r_α is given by the equation

$H(r) = 0$ as a root

Proof :

Proceeding exactly on the same lines of Theorem in Section I and using the relevant lemmas, we obtain, finally

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{-2r}{(1-r^2)} - \frac{r}{(1-r)} + \frac{2(1-2\beta)}{\beta\,(1-\alpha)} + \frac{1}{\beta\,(1-\alpha)} \cdot \frac{2}{(1-r^2)}$$

$$\Big[\sqrt{(\alpha\beta+\beta)\,[1-(\alpha\beta+\beta-1)\, r^2]\,[((4\beta-1-2\alpha\beta)(1-r^2)+(1-(2\beta-1)^2 r^2]}$$

$$-\,[\,1-(2\beta-1)(\alpha\beta+\beta-1)\, r^2\,] \;\Big\|\; Ra \leqslant R^* \Big\|$$

and $\geqslant \frac{-2r}{(1-r^2)} - \frac{2r}{(1-r^2)} + \Big[1+ 4\alpha\beta ar + (2\alpha B^2(\alpha+1)a^2 +$

$\beta(5\alpha-1)-2)\ r^2 + 2a\ (1+\alpha)\ \beta(\ 2\alpha\beta-1)r^3 + (\alpha\beta + \beta - 1)$

$(2\alpha\beta-1)\ r^4/\ (1+ra(\alpha\beta + \beta) + (\alpha\beta + \beta-1)\ r^2)$

$(\ 1 + 2a\,\beta r + (2\,\beta - 1)\ r^2\ )\ ]$

when $Ra \geqslant R^*$ N/O

Hence f(z) is univalent and starlike in $|z| < r_\alpha$, where r_α is given by the equation $H(r) = 0$.

Theorem 1.4.4

Let f(z) and g(z) $\in$ A, S(z) $\in$ Va (α, β).
If $\operatorname{Re}\left\{\frac{g(z)}{S(z)}\right\} > 1/2 \quad |\ f(z)/g(z) - 1\ | < 1,$ for $z \in E$,
Then f(z) is univalent and starlike for $|z| < r_\alpha$
r_α is given by the polynomial $H(r) = 0$ N/O

Proof : Carrying out exactly the same procedure as in section I, we write down, with the help of relevant lemmas.

$$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant \frac{r}{1-r} - \frac{r}{1+r} + \frac{2\ (1-2\,\beta)}{\beta\ (1-\alpha)} +$$

$$\frac{1}{\beta(1-\alpha)} \cdot \frac{2}{(1-r^2)}\ \Big[\sqrt{(\alpha\beta + \beta)\ [\ 1 - (\alpha\beta + \beta-1\ )r^2\ ]}$$

$$|(4\beta - 1 - 2\alpha\beta)\ (1-r^2) + (1-(2\,\beta-1)^2\ r^2\ \big]$$

$$-\ [\ 1-(2\,\beta-1)(\alpha\beta+\beta-1)\ r^2\ ] \qquad Ra \leqslant R^*$$

and

$$\geqslant \frac{r}{(1-r)} - \frac{r}{(1+r)} + \Big[1 + 4\alpha\beta ar + (2\alpha B^2(\alpha+1)\ a^2 + \beta(5\alpha - 1) - 2) r^2$$

$$+ 2a(1+\alpha)\ \beta(2\alpha\beta - 1) r^3 + (\alpha\beta + \beta - 1)(2\alpha\beta - 1)\ r^4 /$$

$$(1 + ra\ (\alpha\beta + \beta) + (\alpha\beta + \beta - 1)\ r^2)\ x$$

$$(1 + 2a\ \beta r + (2\ \ \beta - 1) r^2 \Big] \quad \text{when} \quad Ra \geqslant R^*.$$

$\therefore$ f(z) is univalent, and starlike for $|z| < r_\alpha$, where r_α is given by the equation $H(r) = 0$

The results stated in all these theorems are new in literature, however these results are not sharp.

SECTION II

Kulkarni, S.R.[10] has studied the various properties of the class $D(\alpha, \beta, \varsigma)$, which he defined and introduced as follows :-

$D(\alpha, \beta, \varsigma)$ is the class of holomorphic, normalised univalent function in the unit disc in the complex plane satisfying the condition.

$$\left| \frac{f'(z) - 1}{[2\varsigma(f'(z)-\alpha)-(f'(z) - 1)]} \right| < \beta$$

where $\beta \in (0, 1]$, $1/2 \leq \varsigma \leq 1$, $0 \leq \alpha < 1/2\varsigma$

This class has been generalised in the following way

$D(\alpha, \beta, \varsigma, \nu, \lambda)$ is a subfamily of S of normalised univalent functions f that are holomorphic in the open unit disc E and pacifying the inequality

$$\left| \frac{f'(z) - 1}{2\varsigma[f'(z) - \alpha + (1-\nu)\cos\lambda . e^{-i\lambda}]-(f'(z) - 1)} \right| < \beta$$

$0 \leq \nu < 1$, $-\lambda < \pi/2 < \lambda$ the remaining restrictions same as above stated.

We shall specialise our considerations for those members of $D(\alpha, \beta, \varsigma, \nu, \lambda)$ that have negative coefficients. This motivation to conduct such a study develops from the recent investigations carried out by Silvarman[19], Gupta and Jain [8]

Let T be the subclass of S holomorphic functions in E, that have the following power series representation,

$$f(z) = Z - \sum_{2}^{\infty} | a_n | z^n.$$

Our putting restrictions of non-negativity on the coefficients results in getting rather pleasing conclusions,. Thus for those holomorphic functions which lie in both families $D(\alpha, \beta, \varsigma, \nu, \lambda)$ and T we obtain several refined results. We see that the family $P^*(\alpha, \beta, \varsigma, \nu, \lambda) = D(\alpha, \beta, \varsigma, \nu, \lambda) \cap T$ gives us a setting so that the results assume very pleasing forms. We shall see how nice are the results concerning coefficient bounds, distortion theorems and radius of convexity for members of $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$. Let us begin with a profound characterisation of members of $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$.

2·5. <u>THEOREMS</u> : <u>Theorem2·5.1</u>

A holomorphic function

$f(z) = z - \sum_{2}^{\infty} | a_n | z^n$ is in $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$ if and only if

$$\sum_{2}^{\infty} n |a_n| \left\{ 1 + \beta(2\varsigma - 1) \right\} \leqslant 2\beta\varsigma (1 - \alpha) + 2\beta\varsigma (1 - \nu) \cos\lambda \, e^{-i\lambda}$$

This result is sharp.

Proof : Let $|z| = 1$, then

$$[f'(z) - 1] - \beta \left[\left\{ 2\varsigma (f'(z) - \alpha) + (1 - \nu) \cos\lambda \, e^{-i\lambda} - (f'(z) - 1) \right\} \right]$$

$$= \left| -\sum n\,|a_n|\, z^{n-1} \right| - \beta \left| 2\varsigma \left[1 - \sum n |a_n|\, z^{n-1} - \alpha + (1-\nu) \cos\lambda .\, e^{-i\lambda} + \sum n |a_n|\, z^{n-1} \right| \right.$$

$$= \sum n\,|a_n| - \beta \left| 2\varsigma - 2\varsigma \sum n |a_n| - 2\varsigma\alpha + 2\varsigma (1-\nu) \cos\lambda .\, e^{-i\lambda} + n\,|a_n| \right|$$

$$= \sum n |a_n| \left\{ 1 + (2\varsigma - 1)\,\beta \right\} - 2\,\beta\varsigma\,(1-\alpha) - 2\,\beta\varsigma\,(1-\nu) \cos\lambda\, e^{-i\lambda}$$

$$\leqslant 0 \text{ , by hypothesis.}$$

$$\therefore \sum n |a_n| \left\{ 1 + (2\varsigma - 1)\,\beta \right\} \leqslant 2\,\beta\varsigma\,(1-\alpha) + 2\,\beta\varsigma\,(1-\nu) .\, \cos\lambda .\, e^{-i\lambda}$$ λ/ⓘ

Thus by maximum modulus theorem,

$f \in D^{*}(\alpha, \beta, \varsigma, \nu, \lambda)$

For the converse, let us assume that,

$$\left| \frac{f'(z) - 1}{2\varsigma \left\{ f'(z) - \alpha + (1-\nu) \cos\lambda\, e^{-i\lambda} \right\} - (f'(z) - 1)} \right| < \beta$$

or $|z| < 1$. Since $|\operatorname{Re}(z)| \leqslant |z|$, for all z, we have choosing the values of ~~Z~~ on the real axis, so that $f'(z)$ ~~is~~ /z/small is real. Letting $z \to 1$ through real values we get

$$\sum n\,|a_n| \leqslant 2\,\beta\varsigma\,(1-\alpha) - \beta\,(2\varsigma - 1) \sum n\,|a_n| + 2\,\beta\varsigma\,(1-\nu) \cos\lambda\, e^{-i\lambda}$$ λ/ⓘ

The result is sharp and the extremal function is given by

$$f(z) = \cancel{z} - \frac{2\,\beta\varsigma\,(1-\alpha) + 2\,\beta\varsigma\,(1-\nu) \cos\lambda .\, e^{-i\lambda}}{n\,(1 + (2\varsigma - 1)\,\beta)} .$$

z/

..26

We state, below some special cases,

Putting $\nu = 1$, $\lambda = 0$ we get the class $D(\alpha, \beta, \varsigma)$ studied by Kulkarni S.R [10]

<u>Corollary</u> 1 :

A holomorphic function $f(z) = z - \sum_{2}^{\infty} |a_n| z^n$ is in $P^*(\alpha, \beta, \varsigma)$ if and only if

$$\sum_{2}^{\infty} n |a_n| \{ 1 + \beta (2\varsigma - 1)\} \leqslant 2 \beta \varsigma (1 - \alpha) .$$

The class $D(\alpha) = D(0, \alpha, 1, 1, 0)$ is precisely the class of functions in E, studied by Caplinger . We state coefficient inequality for this class.

The holomorphic functions which are in $D(\alpha)$ and T are characterised as follows.

T ∈ D(α)

<u>Corrollary 2</u> :

A holomorphic function f in $T \in D(\alpha) \equiv D(0, \alpha, 1, 1, 0)$ if and only if $\sum_{2}^{\infty} n | a_n | (1 + \alpha) \leqslant 2 \alpha$

The characterisation for members of $T \cap D(\alpha, \beta)$, where $D(\alpha, \beta)$ is the family investigated by Juneja-Mogra [9]. construction

<u>Corollary</u> 3 :

A holomorphic function f in $T \in D(\alpha, \beta) \equiv D(\alpha, \beta, 1, 1, 0)$ is in $T \cap D(\alpha, \beta)$ if and only if

$$\sum_{2}^{\infty} n |a_n| \{ 1 + \beta\} \leqslant 2 \beta (1 - \alpha).$$

..27

The motivation is found from the study by Gupta and Jain, for $\varsigma = 1$.

Corollary 4 : A holomorphic function $f(z) = z - \sum_{2}^{\infty} |a_n| z^n$ is in $P^*(\alpha, \beta)$ if and only if.

$$\sum_{2}^{\infty} n\,|a_n|\,(1+\beta) \leqslant 2\beta\,(1-\alpha)$$

We shall further, see the sharpening of our considerations for members of $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$. We have in the next result bounds on $|f|$ and $|f'|$.

Theorem 2.5.2 :

If $f \in P^*(\alpha, \beta, \varsigma, \nu, \lambda)$, then for $|z| = r$, $0 < r < 1$, we have

(i) $$r - \frac{\beta\varsigma(1-\alpha) + \beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}}{1+\beta(2\varsigma-1)} r^2 \leqslant |f(z)|$$

$$\leqslant r + \frac{\beta\varsigma(1-\alpha) + \beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}}{1+\beta(2\varsigma-1)} r^2$$

(ii) $$1 - \frac{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}}{1+(2\varsigma-1)\beta} r \leqslant |f'(z)|$$

$$\leqslant 1 + \frac{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}}{1+\beta(2\varsigma-1)} r$$

Proof : (1) we have

$$\sum |a_n| \leqslant \frac{\beta\varsigma_1(1-\alpha) + \beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}}{1+\beta(2\varsigma-1)}$$

.28

Hence $|f(z)| \leqslant r + r^2 \sum_2^{\infty} |a_n|$

$$\leqslant r + \frac{\beta\varsigma(1-\alpha) + \beta\varsigma(1-\nu)\operatorname{Cos}\lambda . e^{-i\lambda}}{1+\beta(2\varsigma-1)} r^2$$

and $|f(z)| \geqslant r - \dfrac{\beta\varsigma(1-\alpha) + \beta\varsigma(1-\nu)\operatorname{Cos}\lambda\, e^{-i\lambda}}{1+\beta(2\varsigma-1)} r^2$

(ii) In same stripe, we have

$$|f'(z)| \leqslant 1 + r\sum |a_n|$$

$$|f'(z)| \leqslant 1 + \frac{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu) . \cos\lambda e^{-i\lambda}}{1+(2\varsigma-1)\beta} r$$

and $|f'(z)| \geqslant 1 - r\sum |a_n|$

$$\geqslant 1 - \frac{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}}{1+\beta(2\varsigma-1)} r$$

The bounds of $f'(z)$ are sharp for the function

$$f(z) = z - \frac{\beta\varsigma(1-\alpha) + \beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}}{1+\beta(2\varsigma-1)} z^2$$ small

for $|z| = r$

We state some particular cases.

Replacing $\alpha = 0$, $\varsigma = 1$ and β replaced by α, ν by 1 and $\lambda = 0$, We get a result of capłinger [4].

Corollary 1 :

Let f be holomorphic function both in T and D(α).

Then for $|z| = r, \quad 0 < r < 1,$

(i) $r - \dfrac{\alpha}{1+\alpha} r^2 \leqslant |f| \leqslant r + \dfrac{\alpha}{1+\alpha} r^2$

(ii) $1 - \dfrac{2\alpha}{1+\alpha} r \leqslant |f'| \leqslant 1 + \dfrac{2\alpha}{1+\alpha} r.$

Further, $D(\alpha, 1, \beta)$ is the family $D(\alpha, \beta)$ investigated by Juneja - Mogra for that we have, construction

Corollary 2 :

Suppose $f \in D(\alpha, \beta) \cap T$. Then for $|z| = r$, we have

$$r - (1-\alpha) r^2 \leqslant |f| \leqslant r + (1-\alpha) r^2,$$

$$1 - 2(1-\alpha) r \leqslant |f'| \leqslant 1 + 2(1-\alpha) r.$$

Corollary 3 :

A holomorphic function $f(z) = z - \sum_{2}^{\infty} |a_n| z^n$ is in $P^*(\alpha, \beta)$ Then for $|z| = r$.

(i) $r - \dfrac{\beta(1-\alpha)}{(1+\beta)} r^2 \leqslant |f| \leqslant r + \dfrac{\beta(1-\alpha)}{1+\beta} r^2$

(ii) $1 - \dfrac{2\beta(1-\alpha)}{1+\beta} r \leqslant |f'| \leqslant 1 + \dfrac{2\beta(1-\alpha)}{1+\beta} r$

This is due to Gupta and Jain [8].

Collary 4 :

A holomorphic function $f(z) = z - \sum_{2}^{\infty} |a_n| z^n$

is in $P^*(\alpha, \beta, \varsigma) = D(\alpha, \beta, \varsigma) \cap T$ Then for $|z| = r$, $0 < r < 1$,

(i) $r - \dfrac{\beta\varsigma(1-\alpha)}{1+\beta(2\varsigma-1)} r^2 \leqslant |f| \leqslant r + \dfrac{\beta\varsigma(1-\alpha)}{1+\beta(2\varsigma-1)} r^2$

$$\text{(ii)} \quad 1 - \frac{2\beta\varsigma(1-\alpha)}{1+\beta(2\varsigma-1)}\, r \leqslant |f'| \leqslant 1 + \frac{2\beta\varsigma(1-\alpha)}{1+\beta(2\varsigma-1)}\, r$$

This can be found in Kulkarni S.R. [10]

We now concern ourselves with the problem of determining the radius of Convexity for members of $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$

Theorem 2.5.3

* If $f \in P^*(\alpha, \beta, \varsigma, \nu, \lambda)$ then f is convex in the disc $|z| < r = r(\alpha, \beta, \varsigma, \nu, \lambda)$

where

y_{n-1}

$$r(\alpha, \beta, \varsigma, \nu, \lambda) = \inf_n \left\{ \frac{1 + \beta(2\varsigma - 1)}{2n\beta\varsigma(1-\alpha) + 2n\beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}} \right\}$$

The result is sharp and the extremal function is

$$f(z) = z - \frac{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda\; e^{-i\lambda}}{n\left\{1 + (2\varsigma-1)\beta\right\}}$$

Proof :

To prove that f is convex, it is sufficient to prove

$$\left| zf''/f' \right| < 1, \quad \text{for } |z| < 1.$$

$$\left|\frac{zf''}{f'}\right| \leqslant \sum_2^\infty \frac{n(n-1)|a_n| z^{n-1}}{1 - \sum n|a_n| z^{n-1}}$$

$$\therefore \sum n(n-1)|a_n|\,|z|^{n-1} \leqslant 1 - \sum n|a_n|\,|z|^{n-1}$$

$$\sum n^2 |a_n|\,|z|^{n-1} \leqslant 1.$$

which

Now from Theorem we have

$$\sum n|a_n| \left\{1 + (2\varsigma-1)\beta\right\} \leqslant 2\varsigma\beta(1-\alpha) + 2\varsigma\beta(1-\nu)\cos\lambda . e^{-i\lambda}$$

$$\Rightarrow |a_n| \leqslant \frac{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda \quad e^{-i\lambda}}{n\{1 + (2\varsigma-1)\beta\}}$$

$$\Rightarrow \sum n^2 |z|^{n-1} \leqslant \frac{n\{1 + (2\varsigma-1)\beta\}}{2\beta\varsigma\{(1-\alpha) + (1-\nu)\cos\lambda . e^{-i\lambda}}$$

$$|z|^{n-1} \leqslant \frac{1+\beta(2\varsigma-1)}{2n\beta\varsigma\{(1-\alpha) + (1-\nu)\cos\lambda . e^{-i\lambda}\}}$$

$$|z| \leqslant \left| \frac{1+\beta(2\varsigma-1)}{2n\beta\varsigma\{(1-\alpha) + (1-\nu)\cos\lambda . e^{-i\lambda}\}} \right|^{1/(n-1)}$$

cap We further proceed to account special cases of this result involving radius of convexity. We first state for members in $T \cap D(\alpha)$ and the resulting conclusion seems to be new. contd

Corollary 1 :

Suppose that $f \in T \cap D(\alpha)$, then f is convex in the disc $|z| < r = r(0, \alpha, 1, 1, 0)$ where

$$r = \operatorname*{Inf}_n \left(\frac{1+\alpha}{2n\alpha}\right)^{1/(n-1)} . \quad n = 2, 3, \ldots$$

The result is sharp.

Equivalently, we also state a result for those holomorphic functions considered by Juneja and Mogra [9] and that have negative coefficients. This result is also not found in the literature.

Corollary 2 :

The holomophic function $f \in T \cap D(\alpha, B))$, then f is convex in the disc $|z| < r = r(\alpha, \beta, 1, 1, 0)$ where

$$r = \operatorname*{Inf}_{n} \left\{ \frac{1}{n(1-\alpha)} \right\}^{1/(n-1)} \qquad n = 2, 3, \ldots ;$$

the result is sharp.

We have the following known result originally due to Gupta and Jain [8] .

Corollary 3 :

Let $f \in P^*(\alpha, \beta)$, then f is convex in the disc

$|z| < r = r(\alpha, \beta, 1, 1, 0)$, where

$$r = \operatorname*{Inf}_{n} \left\{ \frac{1+\beta}{2\beta n(1-\alpha)} \right\}^{1/(n-1)}$$

This last result can be found in Kulkarni S.R. [10]

Corollary 4 :

If $f \in P^*(\alpha, \beta, \varsigma)$ then f is convex in the disc

$|z| < r = r(\alpha, \beta, \varsigma, 1, 0)$ where

$$r = \operatorname*{Inf}_{n} \left\{ \frac{1+(2\varsigma-1)\beta}{2n\beta\varsigma(1-\alpha)} \right\}^{1/(n-1)}$$

We now explicitely show that the family $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$ is closed under the formation of arithmetic means.

Theorem 2.5.4

z small If $f(z) = z - \sum_{2}^{\infty} |a_n| z^n$ and

z small $g(z) = z - \sum_{2}^{\infty} |b_n| z^n$ z small

are in $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$ then

$$h(z) = \cancel{z} - \tfrac{1}{2}\sum \left| a_n + b_n \right| \cancel{z}^n$$ is also in z/small / z/small

$P^*(\alpha, \beta, \varsigma, \nu, \lambda)$.

Proof :

Since f & $g \in P^*(\alpha, \beta, \varsigma, \nu, \lambda)$, we have

$$\sum n|a_n| \left\{1 + (2\varsigma - 1)\beta\right\} \leqslant 2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}$$

and $\sum n|b_n| \left\{1 + (2\varsigma - 1)\beta\right\} \leqslant 2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda\, e^{-i\lambda}$ λ/⊙

For h to be a member of $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$ it is adequate to show that,

$$\tfrac{1}{2}\sum \left\{ n \left[1 + (2\varsigma - 1)\beta \right] \right\} |a_n + b_n|$$

$$\leqslant 2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda \cdot e^{-i\lambda}$$

which follows immediately by the use of the above two inequalities.

Y/small Therefore $h(z) = z - \tfrac{1}{2}\sum |a_n + b_n| \cancel{z}^n$ is also in $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$.

As special cases, we note that the arithmetic means of function that are in $T \cap D(\alpha)$ are again in $T \cap D(\alpha)$. The same statement holds for functions in $T \cap D(\alpha, \beta)$ $P^*(\alpha, \beta)$ and $P^*(\alpha, \beta, \varsigma)$ ... The first two consequences seems to be new.

Corollary 1 :

small/Small Let $f(z) = \cancel{z} - \sum_{2}^{\infty} |a_n| z^n$ and $g(z) = \cancel{z} - \sum_{2}^{\infty} |b_n| z^n$

small be in $D(\alpha) \cap T$, then $h(z) = z - \tfrac{1}{2}\sum |a_n + b_n| \cancel{z}^n$ is also in $D(\alpha) \cap T$

Corollary 2 :

Suppose $f(z) = \cancel{z} - \sum_{2}^{\infty} |a_n| z^n$ and $g(z) = z - \sum_{2}^{\infty} |b_n| z^n$

... 34

be in $D(\alpha, \beta) \cap T$, then $h(z) = z - 1/2 \sum_{2}^{\infty} |a_n + b_n| z^n$ is also in $D(\alpha, \beta) \cap T$.

Corollary 3 :

Let $f(z) = z - \sum |a_n| z^n$, $g(z) = z - \sum |b_n| z^n$ be in $P^*(\alpha, \beta)$, then $h(z) = z - 1/2 \sum |a_n + b_n| z^n$ is also in $P^*(\alpha, \beta)$ λ/⊙

Corollary 4 :

Let $f(z) = z - \sum |a_n| z^n$ and $g(z) = z - \sum |b_n| z^n$ be in $P^*(\alpha, \beta, \varsigma)$

then $h(z) = z - 1/2 \sum |a_n + b_n| z^n$ is also in $P^*(\alpha, \beta, \varsigma)$ λ/⊙

Finally, we show that the convex linear combination of members of $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$ is again a member of $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$. Thus we show that the family $P^*(\alpha, \beta, \varsigma, \nu, \lambda)$ is closed under the formation of convex linear combinations.

Theorem 2.5.5 :

Let
$$f_n(z) = \frac{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda \cdot e^{-i\lambda}}{b\{1 + (2\varsigma - 1)\beta\}} \cdot z^n$$

For $n = 2, 3, \ldots$

Then $f \in P^*(\alpha, \beta, \varsigma, \nu, \lambda)$ if and only if it can be expressed in the form

$$f(z) = z - \sum_n \lambda_n f_n(z) \quad \text{where } \lambda_n \geq 0.$$

$(n = 1, 2, \ldots)$ and $\sum \lambda_n = 1$.

Proof : Let us suppose that

$$f(z) = z - \sum \lambda_n f_n(z)$$

Small $\quad \Big| \; \cancel{z} - \dfrac{\sum 2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\cos\lambda \, e^{-i\lambda}}{n[1 + \beta(2\varsigma - 1)]} \lambda_n z^n$ λ/⊙

Then $\sum_{2}^{\infty} \left\{ \frac{n\ (1 + (2\varsigma - 1)\ \beta)}{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\ \mathrm{Cos}\,\lambda\ e^{-i\lambda}}\lambda_n\right.$

$$\left.\frac{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\ \mathrm{Cos}\,\lambda\ e^{-i\lambda}}{n\ [\,1 + (2\varsigma-1)\ \beta\,]}\right\} \leq 1$$

B/cat

by coefficient inequality theorem, we conclude that $f \in P^*(\alpha, \beta, \varsigma, \nu, \lambda)$.

which ?

~~Conversly~~, let us suppose that $f \in P^*(\alpha, \beta, \varsigma, \nu, \lambda)$.

Therefore, we have in view of the coefficient inequality

$$|a_n| \leq \frac{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\ \mathrm{Cos}\,\lambda e^{-i\lambda}}{n\ (1 + (2\varsigma - 1)\ \beta)}$$

λ / ?

Let us set $\lambda_n = \dfrac{n\ [\,1 + (2\varsigma - 1)\ \beta\,]}{2\beta\varsigma(1-\alpha) + 2\beta\varsigma(1-\nu)\ \mathrm{Cos}\,\lambda . e^{-i\lambda}}\ |a_n|$

$n = 2, 3, \ldots$

then we have $\sum_{n=2}^{\infty} \lambda_n \leq 1, \quad \lambda_n \geq 0$

$$\lambda_1 = 1 - \sum_{2}^{\infty}\lambda_n$$

So that we have $f(z) = z - \sum_{2}^{\infty} \lambda_n f_n(z)$ and the proof is complete.

We enumerate a few special cases while coming to end to this section.

With $\alpha = 0$, $\beta = \alpha$, $\varsigma = \nu = \frac{1}{2}$ and $\lambda = 0$, we shall get a corresponding result for holomorphic functions that are in $D(\alpha) \cap T$.

..

<u>Corollary 1</u> :

Let $f_n(z) = \frac{2\alpha}{n(1+\alpha)} \cdot z^n$, $n = 2, 3, \ldots$ Then $f \in D(\alpha) \cap T$ if and only if it can be expressed in the form $f(z) = z - \sum_{2}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geqslant 0$, ($n = 2,3..$) $\sum_{1}^{\infty} \lambda_n = 1$. This result appears to be new.

Next we have aresult involving holomorphic functions that are in D $(\alpha, \beta) \cap T$. Thus we have a new result, namely

<u>Corollary 2</u> :

Let $f_n(z) = \frac{1-\alpha}{n} \cdot z^n$, $n = 2, 3, \ldots$ Then $f \in D(\alpha, \beta) \cap T$ if and only if it can be expressed in the form

$$f_n(z) = z - \sum_{2} \lambda_n f_n(z), \text{ where}$$

z small

$$\lambda_n \geqslant 0 \quad (n = 1,2, \ldots\ldots), \quad \sum_{1}^{\infty} \lambda_n = 1.$$

We have the following well - known consequence, See Gupta and Jain [8]. For $\varsigma = 1 = \nu$, $\lambda = 0$ this yields.

<u>Corollary 3</u> :

Let $f_n(z) = \frac{2\beta(1-\alpha)}{n(1+\beta)} \cdot z^n$, $n = 2, 3 \ldots$

z small

Then $f \in P^*(\alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = z - \sum_{2}^{\infty} \lambda_n f_n(z), \quad \lambda_n \geqslant 0, \quad n = 1, 2, \ldots,$$

and $\sum_{1}^{\infty} \lambda_n = 1$

A 9711

Finally, we have the following known outcome, See Kulkarni, S.R. [10], obtained by replacing $\nu = 1$ and $\lambda = 0$.

Corollary 4:

Let $$f_n(z) = \frac{2\beta\varsigma(1-\alpha)}{n\{1+(2\varsigma-1)\beta\}} \cdot z^n.$$

$n = 2, 3, \ldots$ Then $f \in P^*(\alpha, \beta, \varsigma)$ if and only if it can be expressed in the form $f(z) = Z - \sum_{2}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geqslant 0$

$$(n = 1, 2, 3, \ldots.) \quad \text{and} \quad \sum_{1}^{\infty} \lambda_n = 1.$$

SECTION - 3

Let S denote the class of functions f holomorphic and univalent in the open disc $E = \{ z : |z| < 1 \}$ and normalised by $f(o) = 0 = f'(0)-1$. Let $S^*(\alpha)$ designate the set of starlike functions of order α $(0 \leqslant \alpha < 1)$ in S. In this section, we have generalised the results obtained by Gupta and Ahmad [7];

We need the following defini tion and lemma for our investigation.

Definition :

Let $p(z) = a \prod_{k=1}^{n} (z - z_k)$ be a polynomial of degree n, where n is a positive integer whose all the n zeros lie outside or on the circle with centre at the ~~O~~rigin and radius R $(\geqslant 1)$. Here a will always be a constant to be appropriately selected so that the functions involved turn out to be normalised and we say that $p(z) \in \mathcal{P}(n, R)$.

Lemma 3.1.1

Let $z = re^{i\theta}$ and $z_1 = Re^{i\phi}$ where $0 < r < R$, then

$$\frac{-r}{R-r} \leqslant \operatorname{Re}\left\{ \frac{z}{z - z_1} \right\} \leqslant \frac{r}{R+r}$$

Equality holds in the first inequality if and only if $z = \frac{r}{R} z_1$ and in the second inequality if and only if $z = -r/_R \, z_1$.

..39

Let $R > 1$ and $\rho \in \mathcal{P}(n, R)$. Then allowing $r \to 1$, we have

Corollary :

If $z = r\,e^{i\theta}$, $z_1 = R.e^{i\,\phi}$, where $0 \leq r < 1$, and $R > 1$, then

$$\frac{-1}{R-1} \leq \mathrm{Re}\left\{\frac{z}{z-z_1}\right\} \leq \frac{1}{R+1}$$

2. STATEMENTS OF THE RESULTS :

Theorem 3.2.1

Let $f \in S^*(\alpha)$, the starlike functions of order α, $0 \leq \alpha \leq 1$, $g \in S^*(\lambda)$, $h \in S^*(\nu)$ and F be defined by

$$F(z)^a = \frac{a+c}{g(z)^c} \int_0^z h(t)^{c-1} f(t)^a \left[\frac{M(t)}{N(t)}\right]^{\nu} dt$$

$a, C \in N$

where $M(z)$ and $N(z)$ be the polynomials belonging to $\mathcal{P}(m, R_1)$ and $\mathcal{P}(n, R_2)$, with $m \geq 1$, $n \geq 0$ and $R_1, R_2 > 1$, 'ν' is a fixed non negative number.

9/8

Then F belongs to $S^*(\beta)$. for $|z| < r_0$, where r_0 is given by the equation $Ar^2 + Br + C = 0$ where

$$A = b(1-a\beta) + abd + b\nu\left(M/R_1 - 1 + N/R_2 + 1\right)$$
$$B = 2 - a\beta(1+b) + a(b+d) - b + \nu(1+b)$$
$$C = \nu\left(M/R_1 - 1 + N/R_2 + 1\right) + a(1-\beta)$$

Proof :

We have $F(z)^a = \dfrac{a+c}{g(z)^c} \int_0^z h(t)^{c-1} f(t)^a \left[\dfrac{M(t)}{N(t)}\right]^{\nu} dt.$

T/cap Therefore $a\dfrac{F'(z)}{F(z)} + \dfrac{c\,g'(z)}{g(z)} = \dfrac{(a+c)\,h(z)^{c-1} f(z)^a}{F(z)^a g(z)^c}\left[\dfrac{M(z)}{N(z)}\right]^{\nu}$ λ/C

Hence $F(z)^a \cdot \dfrac{1}{(a+c)}\left[\dfrac{azF'(z)}{F(z)} + \dfrac{c\,z\,g'(z)}{g(z)}\right] =$

$$\frac{zh(z)^{c-1} f(z)^a}{g(z)^c} \cdot \left(\frac{M(z)}{N(z)}\right)^{\nu}.$$

By hypothetical Conditions $f \in S^*(\alpha)$, $g \in S^*(\lambda)$, and

hence $\dfrac{zf'(z)}{f(z)} \in P(\alpha)$, the set of positive real parts having real part greater than α

and $\dfrac{z\,g'(z)}{g(z)} \in P(\lambda)$ λ/C // conditions on α, λ

By a well known representation formula we have

$$\frac{1}{(a+c)}\left[\frac{az\,f'(z)}{f(z)} + \frac{c\,z\,g'(z)}{g(z)}\right] = \frac{1+bw(z)}{1+w(z)}$$ /c/ Small

where $b = \dfrac{a(2\alpha-1) + c(2\lambda-1)}{a+c}$ λ/C

Hence $(F(z))^a = \dfrac{z\,h(z)^{c-1} f(z)^a}{g(z)^c}\left[\dfrac{M(z)}{N(z)}\right]^{\nu}\left[\dfrac{1+w(z)}{1+bw(z)}\right]$ λ/C

Logarithmic differentiation yields,

..41

$$\frac{azF'(z)}{F(z)} = 1 + (c-1)\frac{zh'(z)}{h(z)} + \frac{az\,f'(z)}{\cancel{F}(z)} - \frac{c\,zg'(z)}{g(z)} +$$

f/ Small

$$\nu\left[\frac{zM'(z)}{M(z)} - \frac{zN'(z)}{N(z)}\right] + \frac{z\,w'(z)}{1+w(z)} - \frac{b\,z\,w'(z)}{1+bw(z)}$$

λ/O

Therefore

$$\frac{zF'(z)}{F(z)} = \gamma a + \frac{\cancel{c}-1}{a}\left[\frac{1+(2\nu-1)\,w(z)}{1+w(z)}\right] + \left[\frac{1+(2\alpha-1)\,w(z)}{1+w(z)}\right]$$

c/ Small

$$-c/a\left[\frac{1+(2\lambda-1)\,w(z)}{1+w(z)}\right] + \nu/a\left[\frac{M}{R_1-1} + \frac{N}{R_2+1}\right]$$

$$+\frac{(1-b)\,z\,w'(z)}{a[1+w(z)]\,[1+bw(z)]}$$

λ/O

Now $\left(\frac{c-1}{a}\right)\left[\frac{1+(2\nu-1)\,w(z)}{1+w(z)}\right] + \left[\frac{1+(2\alpha-1)\,w(z)}{1+w(z)}\right]$

$$-\frac{c}{a}\left[\frac{1+(2\lambda-1)\,w(z)}{1+w(z)}\right]$$

$$= \frac{1}{1+w(z)}\Big[\left(\frac{c-1}{a}\right) + (2\nu-1)\,w(z)\left(\frac{c-1}{a}\right) + 1 + (2\alpha-1)w(z) - c/a$$

$$- c/a\,(2\lambda-1)\,w(z)\Big]$$

$$= \frac{1}{1+w(z)}\Big[(c/a - \gamma a + 1 - c/a) + w(z)\Big\{(2\nu-1)\left(\frac{\cancel{c}-1}{a}\right) + (2\alpha-1)$$

c/ Small

$$-\frac{c}{a}(2\lambda-1)\Big\}\Big]$$

λ/O

$$= \frac{1}{1+w(z)}\left[(1-\gamma a) + \frac{w(z)}{a}\left\{(2\nu-1)\ (c-1) + a(2\alpha-1) - c(2\lambda-1)\right\}\right]$$

$$= \frac{1}{1+w(z)}\ (1-\gamma a) + \frac{dw(z)}{1+w(z)},$$

where $d = \gamma a \{ (2\nu-1)\ (c-1) + a(2\alpha-1) - c\ (2\lambda-1)$

$$\therefore\ \frac{zF'(z)}{F(z)} = \frac{1}{a} + (1-\gamma a)\frac{1}{1+w(z)} + \frac{dw(z)}{1+w(z)}$$

$$+ \frac{\nu}{a}\left[\frac{M}{R_1-1} + \frac{N}{R_2+1}\right] + \frac{(1-b)z\ w'(z)}{a[1+w(z)]\,[1+bw(z)]}$$

implies that

$$\frac{zF'(z)}{F(z)} - \beta = \frac{1}{1+w(z)}\ [1-\gamma a + dw\ (z)] - \beta + \gamma a$$

$$+ \frac{\nu}{a}\left[\frac{M}{R_1-1} + \frac{N}{R_2+1}\right]$$

$$+ \frac{(1-b)\ z\ w'(z)}{a[\ 1+w(z)\ ]\,[1+bw(z)]}$$

What is R_1?
What is R_2?

$$\mathrm{Re}\left\{\frac{zF'(z)}{F(z)} - \beta\right\} \geqslant 0, \quad \text{for } F \in S^*(\beta)$$

...

$$\frac{1}{(1+r)}\left[1-\gamma a + dr\right] - \beta + \gamma a + \frac{\nu}{a}\left[\frac{M}{R_1-1} + \frac{N}{R_2+1}\right] + \frac{(1-b)r}{a(1+r)(1+br)} \geqslant 0$$

$$= r^2\left[b(1-aB) + abd + b\nu\left(M/R_1-1 + N/R_2+1\right)\right]$$

$$+ r\left[2 - a\beta(1+b) + a(b+d) - b + \nu(1+b)\right] + \left[a(1-\beta) + \nu\left(M/R_1-1 + N/R_2+1\right)\right] \geqslant 0$$

$$= Ar^2 + Br + C = 0 \quad \text{where}$$

$$A = b(1-a\beta) + abd + b\nu\left(M/R_1-1 + N/R_2+1\right)$$

$$B = 2 - a\beta(1+b) + a(b+d) - b + \nu(1+b)$$

$$C = \nu\left(M/R_1-1 + N/R_2+1\right) + a(1-\beta).$$

...

Thus $F \in S^*(\beta)$ for $|z| < r_o$, where r_o is given by the root of the equation $Ar^2 + Br + C = 0$

Special Cases :

For $\nu = 0$ we get the result of Gupta and Ahmed [7]

Corollary :

Let $f \in S^*(\alpha)$, $g \in S^*(\lambda)$, $h \in S^*(\nu)$

and F be defined by c / small

$$F(z)^a = \frac{a+c}{g(z)^a} \int_0^z h(t)^{c-1} f(t)^a \, dt, \quad a, c \in N.$$

Then F belongs to $S^*(\beta)$, for $|z| < r_o$, where r_o is given by

$$r_o = \left\{ \frac{(1 + C - \delta - \lambda C) + \left((\delta - 1 - C + C\lambda)^2 - (2 - C - a\beta)(a\beta - 2\delta + 2C\lambda - C) \right)^{1/2}}{a\beta - 2\delta + 2\lambda C - C} \right\}$$

when $a\beta - 2\delta + 2\lambda C - C \neq 0$

$$= \frac{(2 - c - a\beta)}{2(1 + C - C\lambda - \delta)}, \text{ when } a\beta - 2\delta + 2\lambda C - C = 0$$

when $\delta = a\alpha + (C-1)\nu$

Theorem : 3.2.2

Let $F \in S^*(\alpha)$, $g \in S^*(\lambda)$, $h \in S^*(\nu)$ and f be defined by

$$F(z)^a = \frac{a+c}{g(z)^c} \int_0^z h(t)^{c-1} f(t)^a \left[\frac{M(t)}{N(t)} \right]^{\nu} dt,$$

Then f belongs to $S^*(\beta)$

<u>Proof</u> :

$$F(z)^a = \frac{a+c}{g(z)^c} \int_0^z h(t)^{c-1}\, f(t)^a \left[\frac{M(t)}{N(t)} \right]^{\nu} dt,$$

$$\Rightarrow \frac{aF'(z)}{F(z)} + \frac{\cancel{c}g'(z)}{g(z)} = \frac{(a+c)\ h(z)^{c-1}}{F(z)^a\ g(z)^c}\, f(z)^a \left[\frac{M(z)}{N(z)} \right]^{\nu}$$

$$f(z)^a = \frac{F(z)^a\ g(z)^c}{z(a+c)\ h(z)^{c-1}} \left[\frac{M(z)}{N(z)} \right]^{\nu} \left[\frac{a z\, F'(z)}{F(z)} + \cancel{c}\, z \frac{g'(z)}{g(z)} \right]$$

$$F \in S^*(\alpha), \quad g \in S^*(\lambda)$$

$$\Rightarrow \frac{zF'(z)}{F(z)} \in P(\alpha), \quad \frac{z\, g'(z)}{g(z)} \in P(\lambda).$$

$$\frac{1}{(a+c)} \left[\frac{az\, F'(z)}{F(z)} + \frac{\cancel{c}z\, g'(z)}{g(z)} \right] = \frac{1 + b\, w(z)}{1 + w(z)}$$

where $b = \dfrac{a\,(2\alpha - 1) + c\,(2\lambda - 1)}{(a + c)}$

$$f(z)^a = \frac{F(z)^a\ g(z)^c\ h(z)^{-c+1}}{z} \left[\frac{N(z)}{M(z)} \right]^{\nu} \left[\frac{1 + bw(z)}{1 + w(z)} \right]$$

Logarithmic differentiation gives us

$$\frac{az\, f'(z)}{f(z)} = \frac{azF'(z)}{F(z)} + \frac{czg'(z)}{g(z)} + (1 - c)\, \frac{zh'(z)}{h(z)} - 1$$

$$+ \nu \frac{z\, N'(z)}{N(z)} - \nu \frac{z\, M'(z)}{M(z)} + \frac{bz\, w'(z)}{1 + bw(z)} - \frac{z\, w'(z)}{1 + w(z)}$$

$$\frac{azf'(z)}{f(z)} = a \left[\frac{1 + (2\alpha - 1)\ w(z)}{1 + w(z)} \right] + c \left[\frac{1 + (2\lambda - 1)\ w(z)}{1 + w(z)} \right]$$

$$+ (1-C)\left[\frac{1+(2\nu-1)\,w(z)}{1+w(z)}\right] - \frac{(1-b)Zw'(z)}{[1+bw(z)][1+w(z)]}$$

$$-\nu\left[\frac{M}{R_1-1} + \frac{N}{R_2+1}\right]$$

$$\frac{zf'(z)}{f(z)} - \beta = \frac{1}{(1+w(z))}\left[1+dr+\frac{1}{a}\right] - \beta - \frac{(1-b)}{a}\,\frac{Zw'(z)}{[1+bw(z)][1+w(z)]}$$

$$-\frac{\nu}{a}\left[\frac{M}{R_1-a} + \frac{N}{R_2+1}\right] \quad \lambda/0$$

$\therefore$ Re $\{zf'/f - \beta\} \geqslant 0$ for $f \in S^*(\beta)$

$$\frac{1}{1+r}\left[1+dr+1/a\right] - \beta - \left(\frac{1-b}{a}\right)\frac{zw'(z)}{(1+br)(1+r)}$$

$$-\frac{\nu}{a}\left[\frac{M}{R_1-1} + \frac{N}{R_2+1}\right] \geqslant 0$$

Where $d = (2\alpha-1) + 1/a\,[C(2\lambda-1) + (1-c)(2\nu-1) - 1]$ $\lambda/0$

$\therefore$ f(z) is starlike of order β, the radius of which is given by

$$r^2\left[abd - a\beta b - b\nu\left(\frac{M}{R_1-1} + \frac{N}{R_2+1}\right)\right]$$

$$+r\left[ad \pm ab + b - a\beta(1+b) - (1-b) - \nu(1+b)\left(\frac{M}{R_1-1} + \frac{N}{R_2+1}\right)\right]$$

$$+\left[a+1 - a\beta - \nu\left(M/_{R_1-1} + N/_{R_2+1}\right)\right] \geqslant 0 \quad \lambda/0$$

<u>Special Case</u> :

Special Case :

Putting $\nu = a$ we get the result derived by Gupta and Ahmed [7]

Corollary :

Let $F \in S^*(\alpha)$, $g \in S^*(\lambda)$, $h \in S^*(\nu)$

and f be defined by the relation,

$$F(z)^a = \frac{a+c}{g(z)^c} \int_0^z (h(t))^{c-1} (f(t))^a \, dt$$

Therefore $f \in S^*(\beta)$, for $|z| < r$, where r is the root of the equation,

$$a(1-\beta)\, b\, D\, r^2 + [\,1-b-a(1-\beta)(b-D)\,]\, r - a(1-\beta) = 0$$

where $b = [\,a(2\alpha-1) + c(2\nu-1)\,]/(a+c)$

$$D = |\,d-\beta\,|/(1-\beta)$$

$$d = (2\alpha-1) + \frac{c(2\lambda-1) + (1-c)(2\nu-1) - 1}{a}$$

Lastly, we have

Theorem 3.2.3 :

Let $F \in S^*(\alpha)$, $g \in S^*(\lambda)$, $h \in S^*(\nu)$

and f be given by

$$F(z)^a = \frac{a+c}{z^c} \int_0^z (f(t))^a \left[\frac{g(t)}{h(t)}\right]^n \left[\frac{M(t)}{N(t)}\right]^{\nu} dt.$$

Then $f(z) \in S^*(\beta)$ and the radius of Starlikeness is given by

$$r^2 \left[abd + b(c-1-a\beta) - \frac{\nu b}{a}\left(\frac{M}{R_1-1} + \frac{N}{R_2+1}\right)\right]$$

$$+ r\left[(bn+ad) + (b+1)(c-1-a\beta) - (1-b) - \frac{\nu}{a}(1+b)\left(\frac{M}{R_1-1} + \frac{N}{R_2+1}\right)\right]$$

$$+ (n+c-1-a\beta) - \frac{\nu}{a}\left(\frac{M}{R_1-1} + \frac{N}{R_2+1}\right) = 0$$

Proof :

Differentiating and arranging the expression, we obtain

$$f(z)^a = \frac{F(z)^a . z^{\cancel{c}-1}}{(a+c)} \left[\frac{h(z)}{g(z)}\right]^n \left[\frac{N(z)}{M(z)}\right]^{\nu}$$

$$\cdot \left[az \frac{F'(z)}{F(z)} + c\right]$$

$$F \in S^*(\alpha) \Rightarrow \frac{zF'(z)}{F(z)} \in P(\alpha)$$

$$\therefore \frac{1}{(a+c)} \left[az \frac{F'(z)}{F(z)} \cancel{+ c}\right] = \left(\frac{1}{a+c}\right)$$

$$\left\{ a \left[\frac{1+(2\alpha-1)\ w(z)}{1+w(z)}\right] + c\right\}$$

$$= \frac{1}{(a+c)} \left[\frac{a + (2\alpha - 1)\, aw(z) + \cancel{C} + c\, w(z)}{1 + w(z)} \right]$$

$$= \frac{1 + bw(z)}{1 + w(z)}, \text{ where } b = \frac{2a\alpha - a + c}{a + c}$$

Hence

$$f(z)^a = F(z)^a \cdot z^{c-1} \left[\frac{h(z)}{g(z)} \right]^n \left[\frac{N(z)}{M(z)} \right]^{\nu} \left[\frac{1 + bw(z)}{1 + w(z)} \right]$$

Logarithmic differentiation yields.

$$\frac{zf'(z)}{f(z)} = \frac{zF'(z)}{F(z)} + \cancel{\frac{(C-1)}{a}} + \frac{n}{a} \left\{ \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \right\}$$

$$+ \frac{\nu}{a} \left[\frac{zN'(z)}{N(z)} - \frac{zM'(z)}{M(z)} \right] - \frac{(1-b)}{a} \frac{zw'(z)}{[1 + bw(z)]\,[1 + w(z)]}$$

In view of the Lemmas, this expression reduces to

$$\frac{zf'(z)}{f(z)} = \frac{n/a + dw(z)}{1 + w(z)} + \cancel{\frac{C-1}{a}} - \nu/a \left[\frac{M}{R_1 - 1} + N/R_2 + 1 \right]$$

$$- \frac{(1-b)}{a} \frac{zw'(z)}{[1 + bw(z)]\,[1 + w(z)]}, \quad d = (2\alpha - 1) + \frac{2n}{a}(\nu - \lambda)$$

$$\text{Re}\left\{ \frac{zf'(z)}{f(z)} - \beta \right\} \geqslant 0 \qquad \text{for } f \in S^*(\beta)$$

..48

$$\therefore \quad \mathrm{Re}\left\{ \frac{z f'(z)}{f(z)} - \beta \right\} \geq \frac{n/a + dw(z)}{1 + w(z)} + \frac{C-1}{a} - \beta -$$

$$-\nu/a \left[\frac{M}{R_1 - 1} + \frac{N}{R_2 + 1} \right] - \frac{(1-b)}{a} \cdot \frac{z w'(z)}{[1 + bw(z)]\,[1 + w(z)]}$$

$$= \frac{n/a + dr}{1 + r} + \frac{C-1}{a} - \beta - \nu/a \left[\frac{M}{R_1 - 1} + \frac{N}{R_2 + 1} \right]$$

$$- \frac{(1-b)}{a} \cdot \frac{r}{(1 + br)\,(1 + r)} \geq 0$$

Thus $f(z)$ is starlike of order β, the radius of which is given by the equation

$$r^2 \left[abd + b(c - 1 - a\beta) - \frac{\nu b}{a} \left(\frac{M}{R_1 - 1} + \frac{N}{R_2 + 1} \right) \right]$$

$$+ r \left[(bn + ad) + (b+1)(C - 1 - a\beta) - (1 - b) - \nu/a\,(1 + b) \left(\frac{M}{R_1 - 1} + \frac{N}{R_2 + 1} \right) \right]$$

$$+ (n + c - 1 - a\beta) - \nu/a \left(\frac{M}{R_1 - 1} + \frac{N}{R_2 + 1} \right) = 0.$$

For $\nu = 0$, we get a special case of Gupta-Ahmad [7]

Corollary :

Let $F \in S^*(\alpha)$, $g \in S^*(\lambda)$, $h \in S^*(\nu)$, $p \in P(0)$ and f be given by

$$F(z)^a = \frac{a + c}{z^c} \int_0^z f(t)^a \left[\frac{g(t)}{h(t)} \right]^n p(t)\, dt. \quad n \geq 0$$

Then $f(z) \in S^*(\beta)$ for $|z| < r_2$. where r_2 is the smallest positive root of the equation,

$$A\gamma^{3} + B\gamma^{2} + C'\gamma + a' = 0,$$

where $A = -b'[a(2\alpha - 1) + 2n(\nu - \lambda) + C - 1 - a\beta)]$

$$B = b'[2a(\alpha - 1) + 2n(\nu + \lambda - 2) + 1]$$

$$- a(2\alpha - 1) - 2n(\nu - \lambda) - C + a\beta + 2$$

$$C' = a(2\alpha - 2 + b') + 2n(\nu + \lambda - 2) + b(C - 2 - a\beta) - 3,$$

$$b' = \frac{C - a + 2a\alpha}{a + C}, \quad a' = a(1 - \beta) + C - 1.$$

We carry on the discussions of functions with second missing coefficients, to obtain the discs of univalence and starlikeness for certain classes of functions. We are tempted to carryout such research work from Kulkarni-Swamy Here we enlist our results for different conditions. Vague

We require the following Lemmas for our discussions.

Lemma 3.3.1 [Shah 1972] If $p(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ is holomorphic and satisfies $\mathrm{Re}(p(z)) > \alpha$, $0 \leq \alpha < 1$, for $|z| < 1$, then we have for $|z| < 1$

$$\left| \frac{z\,p'(z)}{p(z)} \right| \leq \frac{2n\,|z|^{n}(1 - \alpha)}{(1 - |z|^{n})(1 + (1 - 2\alpha)|z|^{n})}$$

Lemma 3.3.2 : (Shah 1972) - Under the hypothesis of Lemma above, we have for $|z| < 1$

$$\mathrm{Re}(p(z)) \geq \frac{1 + (2\alpha - 1)|z|^{n}}{1 + |z|^{n}}$$

Lemma 3.3.3 : [Shah 1972] If $\phi(z) = 1 + d_n z^n + d_{n+1} z^{n+1} + ..$ is holomorphic and Re $\phi(z) \geqslant 0$ for $|z| < 1$, then

$$\left[1 - \lambda \left| \phi(z) \right|\right]^{-1} \leqslant (1 - |z|^n) / \left[(1 - |z|^n) - \lambda(1 + |z|^n)\right]$$

for $|z| < \left[(1 - \lambda) / (1 + \lambda)\right]^{1/n}$, where $0 \leqslant \lambda < 1$.

We prove the following results.

Theorem 3.4.1

Let $f(z) = \cancel{z} + a_{n+1} \cdot z^{n+1} + a_{n+2} z^{n+2} + \ldots$ z/ small

$g(z) = z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \ldots$

z/ small and $h(z) = \cancel{z} + a\cancel{z}^3 + \sum_4^\infty a_n z^n$ are holomorphic / z / small

for $|z| < 1$.

Let Re $\left\{\frac{g(z)}{S_1(z)}\right\} > 0$ for $|z| < 1$, where $S_1(z)$ is starlike of order β, $0 \leqslant \beta < 1$, $h(z)$ is starlike of order α with second missing coefficient $0 \leqslant \alpha < 1$.

If Re $\left\{\frac{z f(z)}{\lambda z f(z) + (1 - \lambda) g(z) . h(z)}\right\} > 0$ for $|z| < 1$. λ

then f(z) is univalent and starlike. The radius of starlikeness is given as in the proof.

Proof :

Let $\phi(z) = \dfrac{z f(z)}{\lambda z f(z) + (1 - \lambda) g(z) h(z)}$ λ / $\odot$

Then $\phi(z)$ is holomorphic and Re $\left\{\phi(z)\right\} > 0$ for $|z| < 1$. Now

$$[\,1 - \lambda\phi(z)\,]\; z f(z) = (1 - \lambda)\, g(z)\, h(z)\, \phi(z)$$

Differentiating logarithmically and multiplying by z. We get

$$\frac{zf'(z)}{f(z)} = \frac{z h'(z)}{h(z)} + \frac{z g'(z)}{g(z)} + \frac{z\phi'(z)/\phi(z)}{1 - \lambda\phi(z)} - 1$$

This equation is valid for those z, for which $1 - \lambda\phi(z) \neq 0$ and $|z| < 1$, Since $\phi(z) \leq \dfrac{1 + |z|^n}{1 - |z|^n}$

$1 - \lambda\phi(z) \neq 0$, in particular if $|z| < \left[\dfrac{1-\lambda}{1+\lambda}\right]^{1/n}$

Let $p(z) = g(z)/S_1(z)$

Hence $$\frac{z g'(z)}{g(z)} = \frac{z p'(z)}{p(z)} + \frac{z S_1'(z)}{S_1(z)}$$

therefore equation takes the form as which

$$\frac{z f'(z)}{f(z)} = \frac{z h'(z)}{h(z)} + \frac{z S_1'(z)}{S_1(z)} + \frac{z p'(z)}{p(z)} + \frac{z\phi'(z)/\phi(z)}{1 - \lambda\phi(z)} - 1$$

$$\therefore \operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geq \operatorname{Re}\left\{\frac{z h'(z)}{h(z)}\right\} + \operatorname{Re}\left\{\frac{z S_1'(z)}{S_1(z)}\right\} - \left|\frac{z p'(z)}{p(z)}\right| - \left|\frac{z\phi'(z)/\phi(z)}{1 - \lambda\phi(z)}\right| - 1$$

Using lemmas we obtain.

$$\geqslant \frac{(1-\alpha)\, a\,|z| - a\,(1-2\alpha)\,|z|^2 - (1-2\alpha)\,(1-\alpha)\,|z|^3}{(1-\alpha) + a\,|z| + a\,|z|^2 + (1-\alpha)\,|z|^3}$$

$$+ \frac{1 + (2\beta - 1)\,|z|^n}{1 + |z|^n} - \frac{2n\,|z|^n}{1 - |z|^{2n}}$$

$$- \frac{2n\,|z|^n}{(1 - |z|^{2n}) - \lambda\,(1 + |z|^n)^2} - 1 \geqslant 0$$

Provided that, $|z| < \left[\dfrac{1 - \lambda}{1 + \lambda}\right]^{1/n}$ $\geqslant 0$

$\therefore$ Simplifying we conclude that

$\operatorname{Re}\left\{\dfrac{z f'(z)}{f(z)}\right\} \geqslant 0$, the radius of which is given by the

equation $H(\gamma) = 0$.

For $\alpha = 0$, we get a result for the radius of starlikeness which is new in literature.

<u>Corollary</u> :

Suppose that $f(z) = z + a_{n+1} z^{n+1} + \dots$

$g(z) = z + b_{n+1} z^{n+1} + \dots$ and

$h(z) = z + az^3 + \sum_{4}^{\infty} a_n z^n$ are holomorphic for

$|z| < 1$,

Let $\operatorname{Re}\left\{ g(z)/s_1(z)\right\} > 0$ for $|z| < 1$, where $S_1(z)$ is starlike of order β, $0 \leq \beta < 1$, $h(z)$ is starlike with 2nd missing coefficient If

$$\operatorname{Re}\left\{\frac{z\, f(z)}{\lambda z\, f(z) + (1-\lambda)\, g(z).h(z)}\right\} > 0, \text{ for } |z| < 1,$$

then f(z) is univalent and starlike, the radius of which is given by the equation

$$\operatorname{Re}\left\{\frac{z(f'(z)}{f(z)}\right\} \geqslant \frac{a|z| - a|z|^2 - |z|^3}{1 + a|z| + a|z|^2 + |z|^3} + \frac{1 + (2\beta-1)|z|^n}{1 + |z|^n} - \frac{2n|z|^n}{1 - |z|^{2n}} - \frac{2n|z|^n}{(1 - |z|^{2n}) - \lambda(1 + |z|^n)^2} - 1 = 0$$

Theorem 3.4.2

Suppose $f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots$

$g(z) = z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \dots$ and

$h(z) = z + C_{n+1} z^{n+1} + C_{n+2} z^{n+2} + \dots$

Let $\operatorname{Re}\left\{\frac{g(z)}{s_1(z)}\right\} \geqslant 0$

for $|z| < 1$, where $s_1(z) = z + az^3 + \sum_{n=4}^{\infty} a_n z^n$ are holomorphic

for $|z| < 1$, and $\operatorname{Re}\left\{\frac{h(z)}{S_2(z)}\right\} > 0$, for $|z| < 1$, where

$S_2(z)$ is starlike of order β, $0 \leqslant \beta < 1$. If

$$\operatorname{Re}\left\{\frac{z\, f(z)}{\lambda z f(z) + (1-\lambda)\, g(z).\, h(z)}\right\} > 0, \text{ for } |z| < 1,$$

then f(z) is univalent and starlike, the radius of which is given by the equation

$$\frac{(1+\alpha)\, a\gamma - a(1-2\alpha)\gamma^{2} - (1-2\alpha)(1-\alpha)\gamma^{3}}{(1-\alpha) + a\gamma + a\gamma^{2} + (1-\alpha)\gamma^{3}} +$$

$$+ \frac{1+(2\beta-1)\gamma^{n}}{1+\gamma^{n}} - \frac{2n\gamma^{n}}{1-\gamma^{2n}} - \frac{2n\gamma^{n}}{1-\gamma^{2n}} -$$

r आर / not Υ

$$- \frac{2n\gamma^{n}}{(1-\gamma^{2n}) - \lambda(1+\gamma^{n})^{2}} - 1 = 0$$

<u>Proof</u> :

Let $p(z) = g(z)/S_1(z)$ and $q(z) = h(z)/S_2(z)$

Hence $\dfrac{z\,g'(z)}{g(z)} = \dfrac{z\,p'(z)}{p(z)} + \dfrac{z\,S_1'(z)}{S_1(z)}$, and

$$\frac{z\,h'(z)}{h(z)} = \frac{z\,q'(z)}{q(z)} + \frac{z\,S_2'(z)}{S_2(z)} - .$$

What are g, h, S_1, S_2 ?

$$\therefore \operatorname{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\} \geqslant \operatorname{Re}\left\{\frac{z\,S_1'(z)}{S_1(z)}\right\} + \operatorname{Re}\left\{\frac{z\,S_2'(z)}{S_2(z)}\right\}$$

$$- \left|\frac{z\,p'(z)}{p(z)}\right| - \left|\frac{z\,q'(z)}{q(z)}\right| - \left|\frac{z\,\phi'(z)/\phi(z)}{1-\phi(z)}\right| - 1$$

In view of the Lemmas, the above equation transforms into

$$\geqslant \frac{(1-\alpha)\, a\,|z| - a(1-2\alpha)\,|z|^2 - (1-2\alpha)(1-\alpha)\,|z|^3}{(1-\alpha) + a\,|z| + a\,|z|^2 + (1-\alpha)\,|z|^3}$$

$$+ \frac{1+(2\beta-1)\,|z|^n}{1+|z|^n} - \frac{2n\,|z|^n}{1-|z|^{2n}} - \frac{2n\,|z|^n}{1-|z|^{2n}} -$$

$$\frac{2n\,|z|^n}{(1-|z|^{2n}) - \lambda(1+|z|^n)} - 1 = 0$$

provided that, $|z| < [\,(1-\lambda)/(1+\lambda)\,]^{1/n}$ $\lambda/0$

$\therefore$ $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$, the radius of which is given by

the equation $H(r) = 0$ $\lambda/0$

For $\alpha = 0$, we get a result for starlike function with second missing coefficient, which is not found in the literature.

Corollary :

Suppose $f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots$

$g(z) = z + b_{n+1} z^{n+1} + \ldots$ and

$h(z) = z + C_{n+1} z^{n+1} + C_{n+2} z^{n+2} + \ldots$ Let

$\mathrm{Re}\left\{\frac{g(z)}{S_1(z)}\right\} \geqslant 0$, for $|z|<1$, where

$S_1(z) = z + az^3 + \sum a_n z^n$ are holomorphic

for $|z|<1$, and $\mathrm{Re}\left\{\frac{h(z)}{S_2(z)}\right\} > 0$ for $|z|<1$, where

$S_2(z)$ is starlike of order β, $0 \leqslant \beta < 1$. If

$$\operatorname{Re}\left\{\frac{z\,f(z)}{\lambda z\,f(z)+(1-\lambda)\;g(z)\,h(z)}\right\} > 0, \quad \text{for } |z|<1,$$

then f(z) is univalent and starlike, the radiús of which is given by the equation.

$$H(r) = \frac{a|z| - a|z|^2 - |z|^3}{1 + a|z| + a|z|^2 + |z|^3} + \frac{1+(2\beta-1)|z|^n}{1+|z|^n}$$

$$- \frac{2n|z|^n}{1-|z|^{2n}} - \frac{2n|z|^n}{1-|z|^{2n}} - \frac{2n|z|^n}{(1-|z|^{2n}) - \lambda(1+|z|^n)^2} - 1 = 0$$

Theorem 3-4.3

Suppose that $f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots$

$g(z) = z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \ldots$ and

$h(z) = z + az^3 + \sum_{n=4}^{\infty} a_n z^n$ are holomophic, for $|z|<1$.

Let $\operatorname{Re}\{g(z)/S_1(z)\} > 0$. For $|z|<1$, where $S_1(z)$ is starlike of order β, $0 \leqslant \beta < 1$, If

$$\left| \frac{z\,f(z)}{\lambda z\,f(z) + (1-\lambda)\,g(z)\,h(z)} - 1 \right| < 1$$

for $|z|<1$. Then f(z) is univalent and starlike, the radius of which is given by the polynomial $P(r) = 0$.

z/small

..57

$$\frac{(1-\alpha)\,a\gamma - a(1-2\alpha)\gamma^2 - (1-2\alpha)(1-\alpha)\gamma^3}{(1-\alpha) + a\gamma + a\gamma^2 + (1-\alpha)\gamma^3} +$$

$$\frac{1+(2B-1)\gamma^n}{1+\gamma^n} - \frac{2n\gamma^n}{1-\gamma^{2n}} - \frac{n\gamma^n}{(1-\gamma^n)(1-\lambda-\lambda\gamma)} = 0$$

Proof :

Let $$\psi(z) = \frac{z\,f(z)}{\lambda z\,f(z) + (1-\lambda)\,g(z)\,h(z)} - 1$$

By hypothesis $\psi(z)$ is holomorphic, $|\psi(z)| < 1$ for $|z| < 1$. Hence by a result of Goluzin(1945) we have for $|z| < 1$

$$|\psi'(z)| \leqslant \frac{n|z|^{n-1}(1-|\psi(z)|^2)}{(1-|z|^{2n})}$$

and by Schwarz' lemma for $|z| < 1$, $|\psi(z)| \leqslant |z|^n$

from the defined relation of $\psi(z)$ we have

$$\psi(z)\,[\lambda z f(z) + (1-\lambda)\,g(z)\,h(z)]$$

$$= z f(z) - \lambda z\,f(z) - (1-\lambda)\,g(z)\,h(z)$$

$$z\,f(z)\,[\lambda(\psi(z)+1) - 1] = -(1-\lambda)$$

$$[g(z)\;h(z)\;(1+\psi(z)]$$

$\therefore$ Logarithmic differentiation yields.

$$\frac{z f'(z)}{f(z)} = \frac{z h'(z)}{h(z)} + \frac{z g'(z)}{g(z)} + \frac{z\psi'(z)}{[1+\psi(z)]\,[1-\lambda-\lambda\psi(z)]} - 1$$

..68

$$\therefore \ \mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant \mathrm{Re}\left\{\frac{zh'(z)}{h(z)}\right\} + \mathrm{Re}\left\{\frac{z S_1'(z)}{S_1(z)}\right\}$$

$$- \left| \frac{z p'(z)}{p(z)} \right| - \left| \frac{z \psi'(z)}{(1+\psi(z))(1-\lambda-\lambda\psi(z)} \right| - 1$$

$$\geqslant \frac{(1-\alpha) a|z| - a(1-2\alpha)|z|^2 - (1-2\alpha)(1-\alpha)|z|^3}{(1-\alpha) + a|z| + a|z|^2 + (1-\alpha)|z|^3}$$

$$+ \frac{1+(2\beta-1)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{1-|z|^{2n}}$$

$$- \frac{n|z|^n(1-|\psi(z)|^2)}{(1-|z|^{2n})(|1+\psi(z)|)\,|1-\lambda-\lambda\psi(z)|} - 1$$

$$\geqslant \frac{(1-\alpha) a|z| - a(1-2\alpha)|z|^2 - (1-2\alpha)(1-\alpha)|z|^3}{(1-\alpha) + a|z| + a|z|^2 + (1-\alpha)|z|^3}$$

$$+ \frac{1+(2\beta-1)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{1-|z|^{2n}}$$

$$- \frac{n|z|^n}{(1-|z|^n)(1-\lambda-\lambda|z|)}$$

N/C

Thus f(z) is starlike, the radius of starlikeness of which is given by the equation.

$$\frac{(1-\alpha)\, a\,|z| - a\,(1-2\alpha)\,|z|^2 - (1-2\alpha)(1-\alpha)\,|z|^3}{(1-\alpha) + a\,|z| + a\,|z|^2 + (1-\alpha)\,|z|^3}$$

$$+ \frac{1+(2\beta-1)\,|z|^n}{1+|z|^n} - \frac{2n\,|z|^n}{1-|z|^{2n}} - \frac{n|z|^n}{(1-|z|^n)\,(1-\lambda-\lambda|z|)} = 0$$

Special Case :

For $\alpha = 0$ we get the result for starlike function with 2nd missing coefficient.

Corollary :

Suppose that $f(z) = z + a_{n+1}\, z^{n+1} + a_{n+2}\, z^{n+2} + \dots$

$g(z) = z + b_{n+1}\, z^{n+1} + \dots$ and $h(z) = z + az^3 + \sum_{4}^{\infty} a_n\, z^n$

are holomorphic, for $|z|<1$. Let $\mathrm{Re}\left\{\frac{g(z)}{s_1(z)}\right\} > 0$, for $|z| < 1$, small

where $s_1(z)$ is starlike of order β, $\alpha \leq \beta \leq 1$, If

$$\left|\frac{z\, f(z)}{\lambda z f(z) + (1-\lambda)\, g(z)\, h(z)} - 1\right| < 1, \quad \text{for}$$

$|z| < 1$. Then f(z) is univalent and starlike, the radius of starlikeness of which is given by the equation,

$$\frac{a|z| - a|z|^2 - |z|^3}{1 + a|z| + a|z|^2 + |z|^3} + \frac{1+(2\beta-1)\,|z|^n}{1+|z|^n} - \frac{2n\,|z|^n}{1-|z|^{2n}}$$

$$- \frac{n\,|z|^n\,(1-|\psi(z)|^2)}{(1-|z|^{2n})\,(|1+\psi(z)|\,|1-\lambda-\lambda\psi(z)|} - 1 = 0.$$

Lastly we have

Theorem 3.4.4

Suppose $f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots\cdots$

$g(z) = z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \cdots$

and $h(z) = z + C_{n+1} z^{n+1} + C_{n+2} z^{n+2} + \cdots$.

Let $\operatorname{Re}\left\{\dfrac{g(z)}{S_1(z)}\right\} > 0$

For $|z| < 1$, where

$S_1(z) = z + az^3 + \sum\limits_{4}^{\infty} a_n z^n$ are holomorphic for

$|z| < 1$ and $\operatorname{Re}\left\{\dfrac{h(z)}{S_2(z)}\right\} > 0$ for $|z| < 1$, where $S_2(z)$ is

Starlike of order β, $0 \leq \beta < 1$, If $\operatorname{Re}\left\{\dfrac{Z f(z)}{\lambda z f(z) + (1 - \lambda) g(z) h(z)}\right\}$

$> 0.$

for $|z| < 1$ then $f(z)$ is univalent and starlike, the radius of starlikeness of which is given by the equation

$p(\Upsilon) = 0.$

$$\frac{(1-\alpha)\,ar - a(1-2\alpha)\,r^2 - (1-2\alpha)(1-\alpha)r^3}{(1-\alpha) + ar + ar^2 + (1-\alpha)\,r^3} +$$

$$\frac{1 + (2\beta - 1)\,r^n}{1 + r^n} - \frac{2\,n\,r^n}{1 - r^n} - \frac{2n\,r^n}{1 - r^n} - \frac{n\,r^n}{(1-r^n)(1 - \lambda - \lambda r^n)} - 1 = 0$$

Proof :

Let $p(z) = g(z) / S_1(z)$ and

$q(z) = h(z) / S_2(z)$

Hence $$\frac{zg'(z)}{g(z)} = \frac{z\,p'(z)}{p(z)} + \frac{z\,S_1'(z)}{S_1(z)}$$

and $$\frac{z\,h'(z)}{h(z)} = \frac{z\,S_2'(z)}{S_2(z)} + \frac{z\,q'(z)}{q(z)}$$

$$\therefore \operatorname{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\} \geqslant \operatorname{Re}\left\{\frac{z\,S_1'(z)}{S_1(z)}\right\} + \operatorname{Re}\left\{\frac{z\,S_2'(z)}{S_2(z)}\right\}$$

$$- \left|\frac{z\,p'(z)}{p(z)}\right| - \left|\frac{z\,q'(z)}{q(z)}\right| - \left|\frac{z\,\phi'(z)/\phi(z)}{1-\lambda\phi(z)}\right| - 1$$

$$\geqslant \frac{(1-\alpha)\,a\,|z| - a(1-2\alpha)|z|^2 - (1-2\alpha)(1-\alpha)|z|^3}{(1-\alpha) + a|z| + a|z|^2 + (1-\alpha)|z|^3}$$

$$+ \frac{1+(2\beta-1)|z|^n}{1+|z|^n} - \frac{2.n\,|z|^n}{1-|z|^n} - \frac{2\,n\,|z|^n}{1-|z|^n}$$

$$- \frac{n\,|z|^n}{(1-|z|^n)(1-\lambda-\lambda|z|^n)} - 1$$

Small

Valid for $|z| < [(1-\lambda)/\lambda]^{1/n}$

<u>Particular Case</u> :

For $\alpha = 0$, we obtain the new result not found in the literature for starlike functions with second missing coefficient and can be stated as

<u>Corollary</u> :

Suppose $f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots..$

$g(z) = z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} \ldots$

$$h(z) = z + C_{n+1} z^{n+1} + C_{n+2} z^{n+2} + \dots$$

Let $\mathrm{Re}\left\{\frac{g(z)}{S_1(z)}\right\} > 0$, for $|z| < 1$, where

$S_1(z) = z + az^3 + \sum_4^\infty a_n z^n$ are holomorphic for $|z| < 1$ and starlike, and $\mathrm{Re}\left\{\frac{h(z)}{S_2(z)}\right\} > 0$ for $|z| < 1$,

where $S_2(z)$ is starlike of order β, $0 \leq \beta < 1$.

If $\mathrm{Re}\left\{\frac{z f(z)}{\lambda z f(z) + (1-\lambda) g(z). h(z)} - 1\right\} < 1$, for $|z| < 1$,

then f(z) is univalent and starlike, the radius of which is given by the equation $P(r) = 0$

What is $P(r)$

$$\mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geq \frac{a|z| - a|z|^2 - |z|^3}{1 + a|z| + a|z|^2 + |z|^3} + \frac{1 + (2\beta - 1)|z|^n}{1 + |z|^n}$$

$$- \frac{2n|z|^n}{1 - |z|^n} - \frac{2n|z|^n}{1 - |z|^n} \quad \frac{- n|z|^n}{(1-|z|^n)(1-\lambda-\lambda|z|^n)} - 1$$

SECTION - 4

Lastly, we Surmise the study of starlike normalisation by considering the Class defined by Bhargav - Pandey. [3] We come into possession of ~~beautiful~~ results, on Univalency, with particular cases wherever credible. Small

Let $S(m, M)$ and $K(\delta)$ denote the sub classes of S and satisfying the conditions

$$|(z f'(z)/ f(z)) - m| < M, \quad z \in D \ (m, M) \in E,$$

$$E = \{(m, M) : |m-1| < M \leq m\}$$ and

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \delta \quad , \quad z \in D, \quad 0 \leq \delta \leq 1;$$

respectively, Let $p(\mu)$ denote the Class of functions p holomorphic in D having $\mathrm{Re}\{p(z)\} \geq \mu$, $z \in D$, $0 \leq \mu \leq 1$ and normalised by $p(0) = 1$. Let $V(\delta)$ denote the class of functions g given by

$$g(z) = 1/2 \,[\, f(z) + z f'(z) \,] \ , \ f \in S^*(\delta) , \quad z \in D,$$

$$0 \leq \delta \leq 1.$$

4. We call for the following Lemmas which are used in our discussion.

Lemma 4.4.1

If $f \in S(m, M)$ and $|z| \leq r < 1$, then

$$Q_1(r) = \frac{1 - ar}{1 + br} \leq \mathrm{Re} \ \frac{zf'(z)}{f(z)} \leq \frac{1 + ar}{1 - br} = Q_2(r)$$

where $a = (M^2 - m^2 + m)/M$, $b = (m-1)/M$, $(m,M) \in E$.

Equality occurs for the function $f(z) = z/(1 \pm bz)^{(a+b)/b}$

This lemma is due to Silverman[19].

Lemma 4.4.2

If $p \in P(\mu)$ and $|z| \leqslant r < 1$, then,

$$\operatorname{Re}\left\{\frac{z\,p'(z)}{p(z)}\right\} \leqslant \frac{2r\,(1-\mu)}{(1-r)\,[1+(1-2\mu)\,r]} = \theta_1(r,\mu)$$

Equality occurs only for the function,

$$p(z) = \frac{1+(1-2\mu)\,\epsilon z}{(1-\epsilon z)}, \quad |\epsilon| = 1$$

This lemma is due to Libera[13].

Lemma 4.4.3

If $g(z)/z \in P(\delta)$ and $|z| \leqslant r < 1$, then

$$\operatorname{Re}\left\{\frac{z\,g'(z)}{g(z)}\right\} \leqslant \frac{1-2(2\delta-1)\,r+(2\delta-1)\,r^2}{(1-r)[1-(2\delta-1)\,r]} = \theta_2(r,\delta)$$

Equality occurs for the function $g(z) = z\left\{1+(2\delta-1)z\right\}/(1+z)$

This lemma is due to Libera[13]

Lemma 4.4.4

If $g \in V(\delta)$, then $|z| = r$, $0 \leqslant r < 1$,

$$\operatorname{Re}\left\{\frac{z\,g'(z)}{g(z)}\right\} \leqslant \frac{1+2(1-2\delta)r+\delta(2\delta-1)r^2}{(1-r)\,(1-\delta r)} = \eta(r,\delta).$$

..65

Equality occurs for the function $g(z)=z\,(1-\delta z)/(1-z)^{3-2\delta}$ $\lambda/0$ $\lambda/0$

This lemma is due to Singh and Goel [20] $\lambda/0$

5. Statements and Proofs of Theorems :

Theorem : 4.5.1

Let $F \in S(\alpha,\beta,\varsigma)$, g, $h \in S(m, M)$, $P \in \rho(\mu)$ $a, b, c > 0$, then the function f defined by

$$F(z)^a = \frac{a+1}{z}\int_0^z f(t)^a \left[\frac{g(t)}{h(t)}\right]^b p(t)^c\,dt,$$

is starlike, the radius of which is given by the polynomial

$$P(r) = 0$$

$$P(r) \equiv \frac{b(1-ar)}{1+br} - \frac{b(1+ar)}{1-br} - \frac{2cr(1-\mu)}{(1-r)[1+(1-2\mu)r]}$$

$$+ \frac{a\beta(1-2\varsigma\alpha) - \beta(2\varsigma-1)}{a[1+\beta(1-2\varsigma\alpha)r] + [1-\beta(2\varsigma-1)r]}$$

$$+ \frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)r} + \frac{a[1-\beta(1-2\varsigma\alpha)r]}{1+\beta(2\varsigma-1)r} = 0.$$

Proof :

We have from the integral operator,

$$F(z)^a = \frac{a+1}{z}\int_0^z [f(t)]^a \left[\frac{g(t)}{h(t)}\right]^b p(t)^c\,dt$$

after logarithmic differentiation and by use of lemma Which

$$f(z)^a \geqslant \frac{F(z)^a h(z)^b}{g(z)^b p(z)^c}\left[\frac{a}{a+1}\cdot\frac{1+\beta(1-2\varsigma\alpha)|z|}{1-\beta(2\varsigma-1)|z|} + \frac{1}{a+1}\right]$$

$$\therefore \frac{a\,z\,f'(z)}{f(z)} \geqslant \frac{az\,F'(z)}{f(z)} + \frac{bz\,h'(z)}{h(z)} - \frac{b\,z\,g'(z)}{g(z)} - \frac{cz\,P'(z)}{p(z)}$$

$$+ \frac{a\beta\,(1-2\varsigma\alpha) - \beta\,(2\varsigma-1)}{a\left[1+\beta(1-2\varsigma\alpha)\,|z|\right] + \left[1-\beta(2\varsigma-1)\,|z|\right]} +$$

$$+ \frac{\beta\,(2\varsigma-1)}{1-\beta(2\varsigma-1)\,|z|} .$$

$$\therefore \frac{az\,f'(z)}{f(z)} \geqslant \frac{b\,z\,h'(z)}{h(z)} - \frac{b\,z\,g'(z)}{g(z)} - \frac{cz\,p'(z)}{p(z)} +$$

$$+ \frac{a\,\beta\,(1-2\varsigma\alpha) - \beta\,(2\varsigma-1)}{a\left[1+\beta(1-2\varsigma\alpha)\,|z|\right] + \left[1-\beta(2\varsigma-1)\,|z|\right]} +$$

$$+ \frac{\beta\,(2\varsigma-1)}{1-\beta\,(2\varsigma-1)\,|z|} + \frac{1-\beta\,(1-2\varsigma\alpha)\,|z|}{1+\beta\,(2\varsigma-1)\,|z|}$$

$\therefore$ Using the lemmas stated and taking the real parts we obtain.

$$a\,\mathrm{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\} \geqslant \frac{b\,(1-ar)}{1+br} - \frac{b(1+ar)}{1-br} - \frac{C\,2\,r\,(1-\mu)}{(1-r)\,[1+(1-2\mu)r]}$$

$$+ \frac{a\beta\,(1-2\varsigma\alpha) - \beta\,(2\varsigma-1)}{a\left[1+\beta(1-2\varsigma\alpha)|z|\right] + \left[1-\beta\,(2\varsigma-1)\,|z|\right]}$$

$$+ \frac{\beta\,(2\varsigma-1)}{1-\beta\,(2\varsigma-1)\,|z|} + \frac{a\left[1-\beta\,(1-2\varsigma\alpha)\,|z|\right]}{1+\beta\,(2\varsigma-1)\,|z|} \geqslant 0$$

Thus f(z) is starlike and the radius of starlikeness isgiven by the equation.

$$\frac{b(1-ar)}{1+br}-\frac{b(1+ar)}{1-br}-\frac{2cr(1-\mu)}{(1-r)[1+(1-2\mu)r]}$$

$$+\frac{a\beta(1-2\varsigma\alpha)-\beta(2\varsigma-1)}{a[1+\beta(1-2\varsigma\alpha)\gamma]+[1-\beta(2\varsigma-1)r]}$$

$$+\frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)r}+\frac{a[1-\beta(1-2\varsigma\alpha)r]}{1+\beta(2\varsigma-1)r}=0.$$

Replacing $\beta = \varsigma = 1$, we get the class of starlike functions.

<u>Corollary</u> :

Let $F \in S^*(\alpha)$, $g, h \in S(m, M)$, $P \in p(\mu)$, $a,b,c > 0$, then the function f defined by

$$F(z)^a = \frac{a+1}{z}\int_0^z (f(t))^a\left[\frac{g(t)}{h(t)}\right]^b p(t)^c\, dt$$

is starlike, the radius of which is given by the equation

$$\frac{b(1-ar)}{(1+br)}-\frac{b(1+ar)}{1-br}-\frac{2cr(1-\mu)}{(1-r)[1+(1-2\mu)r]}$$

$$+\frac{a(1-2\alpha)-1}{a[1+(1-2\alpha)r]+[1-r]}$$

$$+\frac{1}{(1-r)}+\frac{a(1-(1-2\alpha)r)}{1+r}=0$$

This result is new one.

Also putting $\alpha = 0$, $\varsigma = \frac{\alpha + 1}{2}$ we get the class studied by Lakshminarsimhan [12], denoted by $S\left(0, \beta, \frac{\alpha+1}{2}\right)$

Corollary :

Let $F \in S\left(0, \beta, \frac{\alpha+1}{2}\right)$, $g, h, \in S(m, M)$ $p \in P(\mu)$ $a, b, c > 0$, then they function f defined by

$$F(z)^a = \frac{a+1}{z} \int_0^z f(t)^a \left[\frac{g(t)}{h(t)} \right]^b p(t)^c \, dt$$

is starlike, the radius of which is given by the equation,

$$\frac{b\,(1 - ar)}{1 + br} - \frac{b(1 + ar)}{1 - br} - \frac{2cr\,(1 - \mu)}{(1 - r)[1 + (1-2\mu)\,r]}$$

$$+ \frac{a\beta\,(1) - \beta\,(\alpha)}{a\,[1 + \beta\gamma] + [1 - \beta\,(\alpha)\,\gamma]} + \frac{\beta\alpha}{1 - \beta\alpha\gamma} + \frac{a\,[1 - \beta\gamma]}{1 + \beta\alpha\gamma} \geqslant 0$$

This result is also new and not found in literature.

Lastly replacihg β by 1 and ς by β, we get the class studied by Juneja - Mogra [9] .

Corollary :

Let $f \in S(\alpha, 1, \beta)$, $g, h \in S(m, M)$, $p \in \mathcal{P}(\mu)$ and for $\delta = 1/2$, we have this

$$\frac{b(1-ar)}{1+br} - \frac{br}{(1-r)\log(1-r)} - \frac{2cr\ (1-\mu)}{(1-r)[1+(1-2\mu)r]}$$

$$+ \frac{a\ (1-2\alpha\beta) - (2\beta-1)}{a\ [1+(1-2\alpha\ \beta)\ |z|\ ] + [1-(2\beta-1)|z|]}$$

$$+ \frac{(2\beta-1)}{1-(2\beta-1)r} + \frac{a\ [\ 1-(1-2\alpha\ \beta)r\ ]}{1+(2\beta-1)\ r} \geqslant 0 \quad \delta = 1/2$$

Theorem 4.5.2 :

If $F \in S(\alpha, \beta, \varsigma)$, $g \in V(\delta)$, $h \in S(m, M)$ $p \in P(\mu)$, $a, b, c > 0$, then the function f defined by $F(z)^a = \int_0^z f(t)^a.$ $[g(t)/h(t)]^b$ $p(t)^c\,dt$ is univalent and starlike, the radius of starlikeness of which is given by the equation,

$$\frac{b(1-ar)}{1+br} - b\,\frac{1+2(1-2\delta)r+\delta(2\delta-1)r^2}{(1-r)(1-\delta r)}$$

$$\frac{-2\,cr\,(1-\mu)}{(1-r)\,[\,1+(1-2\mu)r\,]} + \frac{a\,\beta\,(1-2\varsigma\alpha) - \beta\,(2\varsigma-1)}{a[1+\beta(1-2\varsigma\alpha)\,r\,] + [1-\beta\,(2\varsigma-1)r\,]}$$

$$+ \frac{\beta\,(2\varsigma-1)}{1-\beta(2\varsigma-1)\,r} + \frac{a\,[\,1-\beta\,(1-2\varsigma\alpha)r\,]}{1+\beta\,(2\varsigma-1)\,r} = 0$$

Proof :- The routine calculations and applications of the appropriate lemmas yield,

$$a\,\mathrm{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\} \geqslant b\,\mathrm{Re}\left\{\frac{z\,h'(z)}{h(z)}\right\} - b\,\mathrm{Re}\left\{\frac{z\,g'(z)}{g(z)}\right\}$$

$$- c\,\mathrm{Re}\left\{\frac{z\,p'(z)}{p(z)}\right\} + \frac{a\,\beta\,(1-2\varsigma\alpha) - \beta\,(2\varsigma-1)}{a\,(1+\beta\,(1-2\varsigma\alpha)|z|)[1-\beta(2\varsigma-1)\,|z|\,]}$$

$$+\frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)|z|}+\frac{a\left[1-\beta(1-2\varsigma\alpha)|z|\right]}{1+\beta(2\varsigma-1)|z|}$$

$$\geqslant \frac{b(1-ar)}{1+br}-b\,\frac{1+2(1-2\delta)r+\delta(2\delta-1)r^2}{(1-r)(1-\delta r)}$$

$$\frac{-2cr(1-\mu)}{(1-r)\left[1+(1-2\mu)r\right]}+\frac{a\beta(1-2\varsigma\alpha)-\beta(2\varsigma-1)}{a\left[1+\beta(1-2\varsigma\alpha)|z|\right]+\left[1-\beta(2\varsigma-1)|z|\right]}$$

$$+\frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)|z|}+\frac{a\left[1-\beta(1-2\varsigma\alpha)|z|\right]}{1+\beta(2\varsigma-1)|z|}\geqslant 0$$

Particular Cases :

For $\beta = \varsigma = 1$, our class $S(\alpha,\beta,\varsigma)$ yields the family of Starlike functions of order α, $S^*(\alpha)$, Then we have the following interesting particular case.

Corollary 1.:

Let $F \in S^*(\alpha)$, $g \in V(\delta)$, $h \in S\ (m, M)$ $p \in P(\mu)$, $a, b, c > 0$. then the function f defined by

$$F(z)^a = \int_0^z f(t)^a \left[\frac{g(t)}{h(t)}\right]^b p(t)^c\,dt$$ is univalent

and starlike, the radius of which is given by the equation

$$= \frac{b(1 - ar)}{1+br} - b\,\frac{1 + 2(1-2\delta)r + \delta(2\delta - 1)r^2}{(1-r)(1-\delta r)}$$

$$- \frac{2Cr\,(1-\mu)}{(1-r)\,[1 + (1-2\mu)r]} + \frac{a(1-2\alpha) - 1}{a[1+(1-2\alpha)|z|]+[1-|z|]}$$

$$+ \frac{1}{(1-|z|)} + \frac{a\,[1 - (1-2\alpha)\,|z|]}{1+|z|}$$

Next we have a corresponding result for these univalent holomorphic functions investigated by Lakshminarsimhan [12]

Corollary :

If $F \in S\left(\alpha\ \beta, \frac{\alpha+1}{2}\right)$, $g \in V(\delta)$, $h \in S\,(m, M)$ $P \in P\,(\mu)$, a, b, c > 0, then the function F defined by

$$F(z)^a = \int_0^z f(t)^a \left[\frac{g(t)}{h(t)}\right]^b p(t)^c\, dt$$

is starlike, the radius of which is given by the equation

$$\frac{b(1-ar)}{1+br} - b\,\frac{1+2(1-2\delta)r + \delta(2\delta - 1)r^2}{(1-r)(1-\delta r)}$$

$$\frac{-2cr\,(1-\mu)}{(1-r)\,[1+(1-2\mu)r]} + \frac{a\ \ \beta - \alpha\beta}{a\,[1+\beta\,|z|] + [1-\alpha\beta\,|z|]}$$

$$+ \frac{\alpha\beta}{1-\alpha\beta|z|} + \frac{a\,[1-\beta\,|z|]}{1+\alpha\beta\,|z|} \geqslant 0$$

With the usual substitutions, namely $\beta = 1$ and ς replaced by β, we obtain a result on these lines for those univalent holomorphic functions studied by Juneja - Mogra [9] .

Corollary :

If $F \in S(\alpha, 1, \beta)$, $g \in V(\delta)$, $h \in S(m, M)$ $p \in P(\mu)$, $a, b, c, > 0$, then the function f defined by

$$F(z)^a = \int_0^z f(t)^a \left[\frac{g(t)}{h(t)} \right]^b p(t)^c \, dt$$ is starlike

and univalent, the radius of which is given by the equation.

$$\frac{b(1-ar)}{1+br} + b \frac{1+2(1-2\delta)r + \delta(2\delta-1)r^2}{(1-r)(1-\delta r)} - \frac{2cr(1-\mu)}{(1-r)[1+(1-2\mu)r]} + \frac{a(1-2\alpha\beta)-(2\beta-1)}{a(1+(1-2\alpha\beta)|z|)+[1-(2\beta-1)]}$$

$$+ \frac{(2\beta-1)}{1-(2\beta-1)|z|} + \frac{a[1-(1-2\alpha\beta)|z|]}{1+(2\beta-1)|z|} \geq 0$$

All the above results are novel results and not found in the literature.

Theorem 4.5.3

Let $F \in S(\alpha, \beta, \varsigma)$, $\frac{g(z)}{z} \in P(\delta)$, $h \in S(m, M)$, $p \in P(\mu)$, $a, b, c > 0$. Then the function f defined by

$$F(z)^a = \frac{a+1}{z} \int_0^z f(t)^a \left[\frac{g(t)}{h(t)} \right]^b \cdot p(t)^c \, dt$$

is starlike and the radius of starlikeness is given by the equation.

$$\frac{b(1-ar)}{1+br} - \frac{b\left[1-2(2\delta-1)r+(2\delta-1)r^2\right]}{(1-r)\left[1-(2\delta-1)r\right]} - \frac{c\ 2r\ (1-\mu)}{(1-r)\left[1+(1-2\mu)r\right]}$$

$$+ \frac{a\beta(1-2\varsigma\alpha) - \beta(2\varsigma-1)}{a\left[1+\beta(1-2\varsigma\alpha)r\right] + \left(1-\beta(2\varsigma-1)r\right)}$$

$$+ \frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)r} + \frac{a\left[1-\beta(1-2\varsigma\alpha)r\right]}{1+\beta(2\varsigma-1)r}$$

Where is the equation?

Proof : By the customary calculations and using the lemmas, we arrive at

$$a\,\mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant \frac{b(1-ar)}{1+br} - \frac{b\left[1-2(2\delta-1)r+(2\delta-1)r^2\right]}{(1-r)\left\{1-(2\delta-1)r\right\}}$$

$$- \frac{c\ 2r\ (1-\mu)}{(1-r)\left[1+(1-2\mu)r\right]} + \frac{a\beta(1-2\varsigma\alpha) - \beta(2\varsigma-1)}{a\left[1+\beta(1-2\varsigma\alpha)|z|\right] + \left[1-\beta(2\varsigma-1)|z|\right]}$$

$$+ \frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)|z|} + \frac{a\left[1-\beta(1-2\varsigma\alpha)|z|\right]}{1+\beta(2\varsigma-1)|z|}$$

NO

We note the following known cases -

Corollary :

Let $F \in S^*(\alpha)$, $g(z)/_z \in P(\delta)$, $h \in S(m, M)$

$p \in P(\mu)$, $a, b, c > 0$, Then the function f defined by

$$F(z)^a = \frac{a+1}{z} \int_0^z f(t)^a \left[\frac{g(t)}{h(t)}\right]^b \cdot p(t)^c\, dt$$

is starlike and the radius of starlikeness is given by the equation,

$$\frac{b(1-ar)}{1+br}-\frac{b\left[1-2(2\delta-1)r+(2\delta-1)r^2\right]}{(1-r)\left\{1-(2\delta-1)r\right\}}$$

$$\frac{-2cr(1-\mu)}{(1-r)\left[1+(1-2\mu)r\right]}+\frac{a(1-2\alpha)-1}{a\left[1+(1-2\alpha)|z|\right]+\left[1-|z|\right]}$$

$$+\frac{1}{(1-|z|)}+\frac{a\left[1-(1-2\alpha)|z|\right]}{1+|z|}=0.$$

Suppose that F is holomorphic in E, then $F \in S(0, \beta, \frac{1+\alpha}{2})$, a class studied by Lakshminarsimhan [12] we have

<u>corollary</u> :

Let $F \in S(0, \beta, \frac{1+\alpha}{2})$, $\frac{g(z)}{z} \in P(\delta)$, $h \in S(m, M)$ $p \in P(\mu)$, a, b, c > 0, Then the function f defined by

$$F(z)^a = \frac{a+1}{z}\int_0^z f(t)^a\left[\frac{g(t)}{h(t)}\right]^b p(t)^c\,dt$$

is starlike and the radius of starlikeness is given by the equation.

$$\frac{b(1-ar)}{1+br}-\frac{b\left[1-2(2\delta-1)r+(2\delta-1)r^2\right]}{(1-r)\left\{1-(2\delta-1)r\right\}}$$

$$\frac{-2cr(1-\mu)}{(1-r)\left[1+(1-2\mu)r\right]}+\frac{a(\beta)-\alpha\beta}{a\left[1+\beta|z|\right]+\left[1-\beta\alpha|z|\right]}+$$

$$+\frac{\alpha\beta}{1-\alpha\beta|z|}+\frac{a\left[1-\beta\alpha|z|\right]}{1+\alpha\beta|z|}=0.$$

Lastly, we have the following particular case for the univalent functions considered by Juneja-Mogra [9].

Corollary :

Let $F \in S(\alpha, 1, \beta)$, $g(z)/z \in p\ (\delta)$, $h \in S(m, M)$, $p \in P\ (\mu)$, $a, b, c > 0$. Then the function f defined by

$$F(z)^a = \frac{a+1}{z} \int_0^z f(t)^a \left[\frac{g(t)}{h(t)}\right]^b p\ (t)^c\ dt$$

is starlike and the radius of starlikeness is given by the equation :

where is the equation?

$$\frac{b(1-ar)}{1+br} - \frac{b[1-2(2\delta-1)r+(2\delta-1)r^2]}{(1-r)\{1-(2\delta-1)r\}} - \frac{2cr(1-\mu)}{(1-r)[1+(1-2\mu)r]} + \frac{a(1-2\alpha\beta)-(2\beta-1)}{a[1+(1-2\alpha\beta)|z|]+[1-(2\beta-1)|z|]} + \frac{(2\beta-1)}{1-(2\beta-1)|z|} + \frac{a[1-(1-2\alpha\beta)|z|]}{1+(2\beta-1)|z|}$$

$a, b, c > 0$, then the function f defined by

$$F(z)^a = \frac{a+1}{z} \int_0^z f(t)^a \left[\frac{g(t)}{h(t)}\right]^b p(t)^c\ dt \text{ is}$$

Starlike, the radius of which is given by

$$\frac{b(1-ar)}{1+br} - \frac{b(1+ar)}{1-br} - \frac{2cr(1-\mu)}{(1-r)(1+(1-2\mu)r)} + \frac{a(1-2\alpha\beta)-(2\beta-1)}{a[1+(1-2\alpha\beta)|z|]+[1-(2\beta-1)|z|]} + \frac{(2\beta-1)}{1-(2\beta-1)|z|} + \frac{a[1-(1-2\alpha\beta)|z|]}{1+(2\beta-1)|z|} \geqslant 0 \ \cancel{\ } \ 0$$

This result is not found in the literature.

Theorem 4.5.4

Let $F \in S(\alpha, \beta, \varsigma)$, $g \in K(\delta)$, $h \in S(m, M)$

$P \in p(\mu)$, a,b, c > 0, Then the function f defined by

$$F(z)^{a} = \frac{a+1}{z} \int_{0}^{z} f(t)^{a} \left[\frac{g(t)}{h(t)} \right]^{b} (p(t))^{c}\, dt$$

is univalent and starlike, where radius of starlikeness is given by, the equation -

Where are the equations?

$$\frac{b(1-ar)}{1+br} - \frac{b(2\delta-1)r}{(1-r)^{2(1-\delta)}[1-(1-r)^{2\delta-1}]} - C\frac{2r(1-\mu)}{(1-r)[1+(1-2u)r]}$$
$$+ \frac{a\beta(1-2\varsigma\alpha) - \beta(2\varsigma-1)}{a[1+\beta(1-2\varsigma\alpha)r] + [1-\beta(2\varsigma-1)r]}$$
$$+ \frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)r} + \frac{a[1-\beta(1-2\varsigma\alpha)r]}{1+\beta(2\varsigma-1)r} \quad \text{for } \delta \neq 1/2$$

$$\frac{b(1-ar)}{1+br} - \frac{br}{(1-r)\log(1-r)} - \frac{C2r(1-\mu)}{(1-r)[1+(1-2\mu)r]}$$
$$+ \frac{a\beta(1-2\varsigma\alpha) - \beta(2\varsigma-1)}{a[1+\beta(1-2\varsigma\alpha)r] + [1-\beta(2\varsigma-1)r]} + \frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)r}$$
$$+ \frac{a[1-\beta(1-2\varsigma\alpha)r]}{1+\beta(2\varsigma-1)r} \quad \text{for } \delta = 1/2$$

X/6

Proof :

Making use of Lemmas stated above and proceeding exactly on the same lines as in above theorem, we can write down.

$$a\,\mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant b\,\mathrm{Re}\left\{\frac{Z h'(z)}{h(z)}\right\} - b\,\mathrm{Re}\left\{\frac{z g'(z)}{g(z)}\right\}$$
$$- C\,\mathrm{Re}\left\{\frac{z p'(z)}{p(z)}\right\} + \frac{+a\beta(1-2\varsigma\alpha) - \beta(2\varsigma-1)}{a[1+\beta(1-2\varsigma\alpha)|z|] + [1-\beta(2\varsigma-1)|z|]}.$$
$$+ \frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)|z|} + a\left[\frac{1-\beta(1-2\varsigma\alpha)|z|}{1+\beta(2\varsigma-1)|z|}\right]$$

..

$$\geqslant \frac{b(1-ar)}{1+br} - \frac{b(2\delta-1)r}{(1-r)^{2(1-\delta)}\left[1-(1-r)^{2\delta-1}\right]}$$

$$- C\frac{2r(1-\mu)}{(1-r)[1+(1-2\mu)r]}$$

$$+ \frac{a\beta(1-2\varsigma\alpha) - \beta(2\varsigma-1)}{a[1+\beta(1-2\varsigma\alpha)r] + [1-\beta(2\varsigma-1)r]}$$

$$+ \frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)r} + \frac{a[1-\beta(1-2\varsigma\alpha)r]}{1+\beta(2\varsigma-1)r} \geqslant 0$$

$$(\delta \neq 1/2)$$

for $\delta = 1/2$, we can have, this,

$$a \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{b(1-ar)}{1+br} - \frac{br}{(1-r)\log(1-r)}$$

$$- C.\frac{2r(1-\mu)}{(1-r)[1+(1-2\mu)r]} + \frac{a\beta(1-2\varsigma\alpha) - \beta(2\varsigma-1)}{a[1+\beta(1-2\varsigma\alpha)r] + [1-\beta(2\varsigma-1)r]}$$

$$+ \frac{\beta(2\varsigma-1)}{1-\beta(2\varsigma-1)r} + \frac{a[1-\beta(1-2\varsigma\alpha)r]}{1+\beta(2\varsigma-1)r} \geqslant 0 \quad (\text{for } \delta = 1/2)$$

Thus f(z) is starlike and the radius of Starlikeness is given by the root of the equation $P(r) = 0$. We list the particular cases of the above result,

For $\beta = \varsigma = 1$, we shall obtain a result which seems to be new.

...

Corollary 1 :

Suppose that $F \in S^*(\alpha)$, $0 \leq \alpha < 1$, $g \in K(\delta)$, $h \in S(m, M)$, $P \in p(\mu)$, $a,b,c > 0$ Then the function f defined by

$$F(z)^a = \frac{a+1}{z} \int_0^z f(t)^a \left[\frac{g(t)}{h(t)}\right]^b p(t)^c \, dt$$

is univalent and starlike whose radius of starlikeness is given by the equation

$$\frac{b(1-ar)}{1+br} - \frac{br}{(1-r)\log(1-r)} - \frac{2Cr(1-\mu)}{(1-r)[1+(1-2\mu)r]} + \frac{a(1-2\alpha)-2}{a[1+(1-2\alpha)r]+[1-r]} \cdot$$

$$+ \frac{1}{1-r} + \frac{a[1-(1-2\alpha)r]}{1+r} \geqslant 0, \quad \text{for } \delta = 1/2$$

for $\delta \neq 1/2$, the radius is given by the equation

$$\frac{b(1-ar)}{1+br} - \frac{b(2\delta-1)r}{(1-r)^{2(1-\delta)}[1-(1-r)^{2\delta-1}]}$$

$$- \frac{2cr(1-\mu)}{(1-r)[1+(1-2\mu)r]} + \frac{a(1-2\alpha)-1}{a[1+(1-2\alpha)r]+[1-r]}$$

$$+ \frac{1}{1-r} + \frac{a[1-(1-2\alpha)r]}{1+r} \qquad (\delta \neq 1/2)$$

Where is the equation

••

For $\alpha = 0$, $\zeta = \frac{1+\alpha}{2}$ and β unchanged we shall get the region of starlikeness, that involve univalent holomorphic functions studied by Lakshminarsimhan[12]; needless to say that the result appears to be new and it would be given,

Corollary 2 -

Suppose $f \in S(o, \beta, \frac{1+\alpha}{2})$, $g \in K(\delta)$,

$h \in S(m, M)$, $P \in \rho(\mu)$, a, b, $c > 0$, Then the function f defined by

$$F(z)^a = \frac{a+1}{z} \int_0^z f(t)^a \left[\frac{g(t)}{h(t)}\right]^b (p(t))^c dt$$

is univalent and starlike, where radius of starlikeness is given by the equation,

$$\frac{b(1-ar)}{1+br} - \frac{b(2\delta-1)r}{(1-r)^{2(1-\delta)}[1-(1-r)^{2\delta-1}]} - \frac{2Cr(1-\mu)}{(1-r)[1+(1-2\mu)r]}$$

$$+\frac{a\beta-\alpha\beta}{a[1+\beta r]+[1-\beta\alpha r]} + \frac{\alpha\beta}{1-\alpha\beta r} + \frac{a[1-\beta r]}{1+\alpha\beta r} \geqslant 0 \quad \delta \neq 1/2$$

and $$\frac{b(1-ar)}{1+br} - \frac{br}{(1-r)\log(1-r)} - \frac{2cr(1-\mu)}{(1-r)[1+(1-2\mu)r]}$$

$$+ \frac{a\beta - \alpha\beta}{a[1+\beta r\ ] + [1-\alpha\beta r\ ]}$$

$$+ \frac{\alpha\beta}{1-\alpha\beta r} + \frac{a\ [1-\beta r]}{1+\alpha\beta r} \geqslant 0 \quad \text{for } \delta = 1/2$$

hand

In the same manner for α unchanged, β replaced by 1 and ς replaced by β, we have the region of Starlikeness; studied by Juneja - Mogra[9]

Corollary 3 :

Suppose the $F \in S(\alpha, 1, \beta)$, $g \in K\ (\delta)$ $h \in S\ (m, M)$, $p \in p\ (\mu)$ $a, b, c > 0$, Then the function f defined by

$$F(z)^a = \left(\frac{a+1}{z}\right) \int_0^z \dot{f}\ (t)^a \left[\frac{g(t)}{h(t)}\right]^b p\ (t)^c\ dt$$

is univalent and starlike, where radius of starlikeness is given by

hand

$$\frac{b\ (1-ar)}{1+br} - \frac{b\ (2\delta-1)\ r}{(1-r)^{2(1-\delta)}\left[1-(1-r)^{2\delta-1}\right]}$$

$$- \frac{2\ Cr\ (1-\mu)}{(1-r)\ [1+(1-2\mu)r]} + \frac{a\ (1-2\alpha\beta) - (2\beta-1)}{a[1+(1-2\alpha\beta)r\ ] + [1-(2\beta-1)r\ ]}$$

$$+ \frac{(2\beta-1)}{1-(2\beta-1)r} + \frac{a\ [1-(1-2\alpha\beta)\ r\ ]}{1+(2\beta-1)\ r} \geqslant 0 \quad \delta \neq 1/2 .$$

REFERENCES

No uniformity in listing the references.

1. Ahuja, O.P.

   Certain generalisation of the Robertson functions, Yokohama Mathematical Journal, Vol - 31, 1983, pp 5-11

2. Barr, A.F.

   Raddii of univalence of certain Classes of analytic functions, Ph.D. dissertation, Uhiv. of Missisipi (1971)

3. Bhargav, G.P. and Pandey R.K.

   On the radius of Starlikeness of Classes of regular functions, Bulletin of the Institute of Mathematics Academia Sinica Vol 13, No. 1, March 1985 pp 13-21.

4. Caplinger, T.R.

   On certain classes of analytic functions, Ph.D. Dissertation, University of Missisipi, (1972)

5. Causey, W.M. and Merkes, E.P.

   Radii of Starlikeness of certain Classes of analytic functions, Jour. Math Anal. Appl. 31, 579-586 (1970).

6. Dutton, C.E.

   Geometric properties of some classes of Univalent starlike functions, Ph.D. Dissertation, Temple University 1978.

7. Gupta, V.P., and Iqbal Ahmed :

   On Starlike functions, Bull Austr. Math Soc No. 22, (1980) 241- 247.

8. Gupta, V.P. and Jain, P.K.

   Certain Classes of Univalent Function with negative Coefficients, Ball Austr. Math Soc. 15(1976). 467-473.

..2

9. Juneja, O.P. and Mogra, M-L.

Raddi of convexity for certain classes of Univalent analytic functions, Pacific Jour, Math. 78, (1978), 359 - 368.

10. Kulkarni, S.R.

Some Problems connected with Univalent functions, Ph.D. Thesis, (unpublished) Shivaji University, Kolhapur 1981.

11. Kulkarni, V.N. and Swamy, S.R.

On the Univalence of Some analytic functions, Indian J. Pure, Appl. Math 9, No. 5 (1978) pp 467-480.

12. Lakshminarsimhan, T.V.

On subclasses of functions starlike in the disc, Jour Indian Math. Soc. 41(1977), 233-243,

13 Libera R.J.

Some classes of regular univalent functions, Proc. Amer, Math Soc 19(1965), 755-758.

14. Mac Gregor, T.H.

The radius of univalence of certain analytic functions, Proc. Amer Math Soc 14 (1963), 514-520.

15. Ratti, J.S.

The radius of univalence of certain analytic functions, Math Z 107, (1968), 241-248

16. Robertson, M.S.

On the theory of Univalent functions, Ann.Math 37 (1936), 374-408.

17. Shah, G.M.

On the Univalence of some analytic functions, Pacific J.Math 43, 239-250 (1972).

18. Soni, Anilkumar : Studies in Univalent Functions Ph.D.Dissertation, Bowling Green State University, 1981.

19. Silverman. H.

Univalent functions with negative coefficients, Proc. Amer Math Soc. 51, (1975), 109-116

20. Singh, V and Goel, R.M.

On the radii of converxity and Starlikeness of some classes of functions, J. Math Soc Japan, 23(1971) 323- 339.

££££££££