CHAPTER - 3

CHAPTER - III

Kannan Type Mapping in H. Space

In this chapter we have used Ishikawa iteration for process to obtain a fixed point for a Kannan type mappin Hilbert space. Further, it is shown that the teorem can be extended for two different mappints T_1 and T_2 by the process of Ishikawa [19].

3.0 Kannan mapping in Hilbert space [23]

Definition :

Let C be a closed subset of a Hilbert space H and T : C → C be a self map satisfy the condition,

$$||Tx-Ty|| \leqslant \left[||X-Tx|| + ||y-Ty|| \right]$$

for all $x, y \in C$ where $0 < \alpha < \frac{1}{2}$. Then T is called Kannan mapping in Hilbert space.

3.0.1 In 1991 Kannan type mapping in Hilbert [24] space was defined by Koparde and Waghmode and proved the following theorem by using the Picard's iteration process.

Definition :

A mapping T : C → C, where C is a subset of a Hilbert space H, is called a Kannan type mapping if

$$||Tx-Ty||^2 \leqslant \alpha \left[||x-Tx||^2 + ||y-Ty||^2 \right]$$

for all $x, y \in C$ and $0 < \alpha < \frac{1}{2}$

3.0.2 Theorem : [24] Let C be a closed subset of a Hilbert space H. Let T be a self mapping on C satisfying

$$||Tx-Ty||^2 < \alpha\left[||x-Tx||^2 + ||y-Ty||^2\right]$$

for all $x, y \in C$ and $0 < \alpha < \frac{1}{2}$, Then T has a unique fixed point in C.

For our work we need the definition (3.0.1) and the theorem (3.0.2) and Ishikawa iteration process (I-1.1.8 and I- 1.1.9)

Our result runs as follows :

3.0.3 Theorem :

Let C be a closed convex subset of a Hilbert space H. Let T be a self map on C satisfying (3.0.1) with $\alpha(1+\beta_n^2) < 1$. Suppose x_o is any point in C and the sequence $\{x_n\}$ associated with T is defined by Ishikawa scheme I -1.1.8 and I - 1.1.9. Suppose that $\{\alpha_n\}$ is bounded away from zero.
i.e. $\underline{\lim}\ \alpha_n = \alpha > 0$. If the sequence $\{x_n\}$ converges to P, then P is a unquie fixed point of T.

Proof : Equation I-1.1.8 implies that

$$x_{n+1} - x_n = \alpha_n (Ty_n - x_n)$$

suppose $x_n \to P$, then $||x_{n+1} - x_n||^2 \to 0$ and since $\{\alpha_n\}$ is bounded away from zero we have

$$||Ty_n - x_n||^2 \to 0 \qquad \ldots\ldots. \ (3.0.4)$$

Using trianle inequality, we have

$$||Ty_n-P||^2 \leqslant \{ ||Ty_n-x_n|| + ||x_n-x_{n+1}|| \}^2$$

$$\to 0 \quad \text{as } n \to \infty$$

i.e. $||Ty_n-P||^2 \to 0$

Using I-1.1.8 and I-1.1.10, where t stand for β_n we obtain the following

inequality :

$$||y_n-Ty_n||^2 = ||\beta_n Tx_n+(1-\beta_n)x_n-Ty_n||^2$$

$$= \beta_n||Tx_n-Ty_n||^2+(1-\beta_n)||x_n-Ty_n||^2$$

$$- \beta_n(1-\beta_n)||Tx_n-x_n||^2$$

$$= \beta_n||Tx_n-Ty_n||^2 - \beta_n(1-\beta_n)||Tx_n-x_n||^2$$

by 3.0.4

..... 3.0.6

Since T satisfies that

$$||Tx-Ty||^2 \leqslant \alpha\{ ||x-Tx||^2 + ||y-Ty||^2\}$$

we have,

$$||Tx_n-Ty_n||^2 \leqslant \alpha\{ ||x_n-Tx_n||^2 + ||y_n-Ty_n||^2\}$$

Using 3.0.6, the above inequality becomes

$$||Tx_n-Ty_n||^2 \leqslant \alpha\{ ||x-Tx_n||^2 + \beta_n||Tx_n-Ty_n||^2$$

$$- \beta_n(1-\beta_n)||Tx_n-x_n||^2$$

$$\Longrightarrow ||Tx_n-Ty_n||^2 \leq \alpha\{ [1-\beta_n(1-\beta_n)] \; ||x_n-Tx_n||^2 + \beta_n ||Tx_n-Ty_n||^2$$

$$\Longrightarrow (1-\alpha\beta_n)||Tx_n-Ty_n||^2 \leq \alpha(1-\beta_n+\beta_n^2)||x_n-Tx_n||^2$$

$$\Longrightarrow ||Tx_n-Ty_n||^2 \leq \frac{\alpha(1-\beta_n+\beta_n^2)}{1-\alpha\beta_n} ||x_n-Tx_n||^2 \qquad \ldots\ldots (3.0.7)$$

Now we use triangle inequality to get

$$||x_n-Tx_n||^2 \leq \{ ||Tx_n-Ty_n||+||Ty_n-x_n|| \}^2$$

Therefore 3.0.7 becomes

$$||Tx_n-Ty_n||^2 \leq \frac{\alpha(1-\beta_n+\beta_n^2)}{1-\alpha\beta_n}\Big[||Tx_n-Ty_n||^2+||Ty_n-x_n||^2 + 2||Tx_n-Ty_n||.||Ty_n-x_n|| \Big]$$

$$= \frac{\alpha(1-\beta_n+\beta_n^2)}{1-\alpha\beta_n} ||Tx_n-Ty_n||^2$$

by 3.0.4

$$\Longrightarrow \left[1-\frac{\alpha(1-\beta_n+\beta_n^2)}{1-\alpha\beta_n}\right] ||Tx_n-Ty_n||^2 \leq 0$$

$$\Longrightarrow \left[1-\alpha(1+\beta_n^2)\right] ||Tx_n-Ty_n||^2 \leq 0$$

Since $(1+\beta_n^2) < 1$ for $0 < \alpha < \frac{1}{2}$ and $||.|| \nless 0$; we have

$$||Tx_n - Ty_n||^2 = 0 \quad \text{i.e. } ||Tx_n - Ty_n|| \to 0 \qquad \text{..... (3.0.8)}$$

Hence,

$$||x_n - Tx_n||^2 \leq \{ ||x_n - Ty_n|| + ||Tx_n - Ty_n|| \}^2 \to 0 \qquad \text{.....(3.0.9)}$$

and

$$||P - Tx_n||^2 \leq \{ ||P - x_n|| + ||x_n - Tx_n|| \}^2 \to 0 \qquad \text{......(3.1.0)}$$

Now we show that P is a fixed point of T.

As T satisfies the inequality in the statement we have,

$$||Tx_n - TP||^2 \leq \alpha \; ||x_n - Tx_n||^2 + ||P - TP||^2$$

$$\to \alpha \; ||P - TP||^2 \qquad \text{by } 3.0.9 \qquad \text{(..... 3.1.1)}$$

Next, using tringle inequality

$$||P - TP||^2 \leq \{ ||P - Tx_n|| + ||Tx_n - TP|| \}^2$$

$$\leq \alpha \, ||P - TP||^2 \quad \text{by (3.1.0 and 3.1.1)}$$

$$\Longrightarrow (1-\alpha) \; ||P - TP||^2 \leq 0$$

Since $0 < \alpha < \frac{1}{2}$ and $||.|| \nless 0$, we have

$||P-TP||^2 = 0$ i.e. $||P-TP|| = 0$

$\therefore$ TP=P i.e. P is a fixed point of T

Let if possible P and q be two fixed points of T, then

$$||P-q||^2 = ||TP-Tq||^2$$

$$\leq \alpha \{ ||P-TP||^2 + ||q-Tq||^2 \}$$

$$0$$

$$||P-q|| = 0 \qquad P = q$$

Therefore a mapping T : $C \to C$ has a unique fixed point in C.

We verify the above theorem by the following example,

Example :

Let T : $[0,1] \to [0,1]$ be a mapping defined by

$$Tx = \frac{x}{4} \text{ , for all x in } [0,1]$$

Then 0 is the only fixed point of T

Now,

$$||Tx-Ty||^2 = ||\frac{x}{4} - \frac{y}{4}||^2$$

$$= \frac{1}{16} ||x-y||^2$$

And

$$\alpha\left[||x-Tx||^2 + ||y-Ty^2\right]$$

$$= \alpha\left[||x-\frac{x}{4}||^2 + ||y-\frac{y}{4}||^2\right]$$

$$= \frac{9\alpha}{16}\left[\ ||x||^2 + ||y||^2\ \right]$$

Therefor for $\frac{1}{9} \leqslant \alpha < \frac{1}{2}$ we have

$$||Tx-Ty||^2 \leqslant \alpha \left|\, ||x-Tx||^2 + ||y-Ty||^2\right.$$

Also for any $x_o \in [0,1]$

$$x_n = \frac{1}{4^n}\ x_o$$

$$\to 0 \quad \text{as} \to n$$

By the statement of the theorem this limit 0 is the unique fixed point of T. This verifies the theorem.

3.2.0 <u>Theorem</u> :

Let C be a closed convex subset of a Hilbert space H. Let T_1 and T_2 be two self mappings on C satisfying

$$||T_1x-T_2y||^2 \leqslant \alpha \left[||x-T_1x||^2+||y-T_2||^2\right]$$

$\forall$ x,y in C with $0< \alpha < \frac{1}{2}$ and $\alpha + \beta_n^2 < 1$

Suppose x_o is any point in C and the sequence $\{x_n\}$ associated with T_1 and T_2 defined by Ishikawa scheme I-1.1.8 and I-1.1.9. Suppose further that $\{\alpha_n\}$ is bounded away from zero. If the sequence $\{x_n\}$ converges to P, then P is a unique common fixed point of T_1 and T_2

Proof :

From Chapter I-1.1.8 we have,

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T_2 y_n$$

$$\Longrightarrow x_{n+1} - x_n = \alpha_n\, (T_2 y_n - x_n)$$

Suppose $x_n \to P$, then $||x_{n+1} - x_n||^2 \to 0$ and $\{\alpha_n\}$ is a sequence bounded away from zero we have

$$||T_2 y_n - x_n||^2 \to 0 \qquad \text{....(3.2.1)}$$

Using triangle inequality it follows that

$$||T_2 y_n - P||^2 \leq \{\, ||T_2 y_n - x_n|| + ||x_n - x_{n+1}||\, \}^2$$

$$\to 0 \quad \text{as} \quad n \to \infty$$

i.e. $||T_2 y_n - P||^2 \to 0$ ....(3.2.2)

Using (I-1.1.8 and 1.1.10), where t stands for β_n we obtain the following inequality

$$||y_n - T_2 y_n||^2 = ||\beta_n T_1 x_n + (1-\beta_n) x_n - T_2 y_n||^2$$

$$= \beta_n ||T_1x_n - T_2y_n||^2 + (1-\beta_n)||x_n - T_2y_n||^2$$
$$- \beta_n(1-\beta_n)\ ||T_1x_n - x_n||^2$$

.....(3.2.3)

Using the inequality of the statement we have,

$$||T_1x_n - T_2y_n||^2 \leq \alpha\left[||x_n - T_1x_n||^2 + ||y_n - T_2y_n||^2\right]$$
$$\leq \alpha\left[||x_n - T_1x_n||^2 + \beta_n||T_1x_n - T_2y_n||^2 + (1-\beta_n)||x_n - T_2y_n||^2 - \beta_n(1-\beta_n)\ ||T_1x_n - x_n||^2\right]$$

by ... 3.2.3

$$\Longrightarrow ||T_1x_n - T_2y_n||^2 \leq \alpha\beta_n||T_1x_n - T_2y_n||^2 + \left[\alpha - \alpha\,\beta_n(1-\beta_n)\right]\ ||x_n - T_1x_n||^2$$

$$\Longrightarrow (1-\alpha\,\beta_n)||T_1x_n - T_2y_n||^2 \leq \left[\alpha - \alpha\beta_n(1-\beta_n)\right]||x_n - T_1x_n||^2$$

$$\Longrightarrow ||T_1x_n - T_2y_n||^2 \leq \frac{\alpha - \alpha\,\beta_n + \alpha\,\beta_n^2}{1 - \alpha\,\beta_n}\ ||x_n - T_1x_n||^2$$

.... (3.2.4)

Now,

$$||x_n - T_1x_n||^2 \leq \{\ ||T_1x_n - T_2y_n|| + ||T_2y_n - x_n||\ \}^2$$

Therefore 3.2.4 becomes

$$||T_1x_n-T_2y_n||^2 \leqslant \frac{\alpha-\alpha\beta_n+\alpha\beta_n^2}{1-\alpha\beta_n}\left[||T_1x_n-T_2y_n||^2 + ||T_2y_n-x_n||^2 + 2\,||T_1x_n-T_2y_n|| \times ||T_2y_n-x_n||\right]$$

Using Triangle inequality

$$\Rightarrow \left[1-\frac{\alpha-\alpha\beta_n+\alpha\beta_n^2}{1-\alpha\beta_n}\right]||T_1x_n-T_2y_n||^2 \leqslant 0$$

by 3.2.1

$$\Rightarrow \left[1-\alpha\,(1+\beta_n^2)\right]||T_1x_n-T_2y_n||^2 \leqslant 0$$

Since $0 \leqslant \alpha+\alpha\beta_n^2 < 1$ for $0<\alpha<\frac{1}{2}$,

we have $||T_1x_n-T_2y_n||^2 = 0$

.... (3.2.5)

Next, using triangle inequality, we have

$$||x_n-T_1x_n||^2 \leqslant \{||x_n-T_2y_n|| + ||T_1x_n-T_2y_n||\}^2 \qquad ..(3.2.6)$$

$\to 0$ by 3.2.1 and 3.2.5

and

$$||P-T_1x_n||^2 \leqslant \{||P-x_n||+||x_n-T_1x_n||\}^2$$

$\to 0$ as $x_n \to P$ and 3.2.6

Now we try to show that P is a fixed point of both T_1 and T_2 :

consider,

$$||T_1x_n-T_2P||^2 \leqslant \alpha\left[||x_n-T_1x_n||^2+||P-T_2P||^2 \right] \quad .....(2.2.7)$$

$$\rightarrow \alpha\, ||P-T_2\ ||^2 \quad \text{by } 3.2.6$$

Using triangle inequality we have

$$||P-T_2P||^2 \leqslant \{\, ||P-T_1x_n|| + ||T_1x_n-T_2P||\, \}^2$$

$$\leqslant \alpha\, ||P-T_2P||^2 \quad \text{by } 3.2.7$$

$$\Rightarrow (1-\alpha)\, ||P-T_2P||^2 \leqslant 0$$

Since $0 < \alpha < \frac{1}{2}$ and $||.|| \nless 0$ we have,

$$||P-T_2P|| = 0 \qquad T_2P = P$$

Similarly we can show that P is also a fixed point of T_1 i.e. $T_1P = P$. Thus P is a common fixed point of T_1 and T_2.

Let if possible P and q be two common fixed points of T_1 and T_2, then

$$||P-q||^2 = ||T_1P - T_2q||^2$$

$$\leqslant \alpha\left[||P-T_1P||^2+||q-T_2q||^2 \right]$$

$$= 0$$

$$\Rightarrow ||P-q|| = 0$$

$$\Rightarrow P = q$$

Hence T_1 and T_2 have unique common fixed point in C.