CHAPTER - 4

CHAPTER - IV

Generalised contraction mapping in Hilbert space

4.0.0 Introduction

The well-known Banach [2] contraction principle has been extended by a number of research workers working in the field of fixed point theory in several directions to different spaces which can be stated as follows

Let X be a banach space and C be a closed convex subset of X, then a contraction mapping T of C into itself satisfying.

$$||Tx-Ty|| \leqslant ||x-y||$$

for some $\alpha \in (0,1)$ and for all x,y in C has a unique point $P \in C$ such that TP=P.

The definition of contraction mapping has undergone successive generalisations [39] in complete metric space by R.Kannan [21], Reich [40], Hardy and Rogers [16] proved some fixed point theorem by considering the following general form of contraction mapping.

Let X be a complete metric space, then a contraction mapping T of C into itself satisfying.

$$d(Tx,Ty) \leq a_1 d(x,y) + a_2 d(x,Ty) + a_3 d(y,Tx) + a_4 d(x,Tx) + a_5 d(y,Ty)$$

where $a_i \geq 0$ and $\sum_{i=1}^{5} a_i < 1$

Khan and Imdad [22] considered the above generalised contraction in Banach space in the following form :

T be a self map of closed convex subset of a Banach space X satisfying

$$||Tx-Ty|| \leq a||x-y|| + b\left[||x-Tx||+||y-Ty||\right] + C\left[||x-Ty||+||y-Tx||\right]$$

for every x and y in C, $a,b,c \geq 0$ and $0 \leq a+4b+4c < 2$

Naimpally and Singh [33] used the two contraction conditions and proved some fixed point theorems.

Ganguly [14] in his recent paper defined a generalised non-expansive mapping in the following way :

A self map T of a subset of a normed linear space X is said to be generalised non-expansive if

$$||Tx-Ty|| < \max \{||x-y||, ||x-Tx||, ||y-Ty||, ||x-Ty||, ||y-Tx||\}$$

The Purpose of This Chapter :

By considering the above generalisations of contraction mapping in different spaces, we have introduced the following new definition of generalised contraction mapping in Hilbert space.

Our definition runs as follows

4.0.1 Generalised contraction mapping

Definition :

Let C be a closed convex subset of a Hilbert space H. A mapping $T : C \to C$ is said to be generalised contraction if for all $x, y \in C$

$$||Tx-Ty||^2 \leqslant a_1||x-y||^2 + a_2||x-Tx||^2 + a_3||y-Ty||^2 + a_4||x-Ty||^2 + a_5||y-Tx||^2 + a_6||(I-T)x-(I-T)y||^2 \quad \ldots 4.0.2$$

Where $a_i \geqslant 0$ and $\sum_{i=1}^{6} a_i \leqslant 1$

...4.0.3

We observe that

(i) If $a_2 = a_3 = a_4 = a_5 = a_6 = 0$, $0 \leq \sqrt{a_1} = K < 1$ we get T as strictly contractive mapping.

(ii) If we put $\sqrt{a_1} = 1$, $a_2 = a_3 = a_4 = a_5 = a_6 = 0$ we obtain T as non-expansive mapping

(iii) If we put $a_1 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$ and $a_6 < 1$, we obtain T as stricly pseudocontractive.

(iv) If we put $a_1 = a_6 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$ we obtain T as pseudo-contractive mapping.

(v) If we put $a_2 = a_3$ and $a_1 = a_4 = a_5 = a_6 = 0$ T becomes a Kannan type mapping which we have studied in Chapter-III.

Our first result runs as follows :

4.0.3 Theorem :

Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to C$ such that it satisfies 4.0.2 and 4.0.3 with $a_4 \neq a_6$ and $0 \leq a_3 + a_4 + a_6 < 1$ Further we assume that T is monotone. Suppose x_o is any point in C and the sequence $\{x_n\}$ associated with T is defined by Ishikawa scheme I-1.1.8 and

I-1.1.9. Suppose $\underline{\lim}\ \alpha_n = \alpha > 0$. If the sequence $\{x_n\}$ converges to P, then P is a fixed point of T

Proof :

From equation I-1.1.8 we have

$$x_{n+1} - x_n = \alpha_n (Ty_n - x_n)$$

suppose $x_n \to P$, then $||x_{n+1} - x_n||^2 \to 0$ amd since $\{\alpha_n\}$ is bounded away from zero,

$$||Ty_n - x_n||^2 \to 0 \qquad \text{.... (A)}$$

Using triangle inequality , we have;

$$||Ty_n - P||^2 \leqslant \Big| ||Ty_n - x_n|| + ||x_n - x_{n+1}|| \Big|^2$$

$$\to 0 \quad \text{as} \quad n \to \infty$$

Using I-1.1.8 and I-1.1.10, where t stands for β_n we obtain the following :

$$||y_n - x_n||^2 = ||\beta_n Tx_n + (1-\beta_n)x_n - x_n||^2$$

$$= \beta_n ||Tx_n - x_n||^2 - \beta_n(1-\beta_n)||Tx_n - x_n||^2$$

$$= \beta_n^2 ||Tx_n - x_n||^2$$

$$\leqslant ||Tx_n - X_n||^2$$

$$\leqslant \{ ||Tx_n - Ty_n|| + ||Ty_n - x_n|| \}^2$$

Using Triangle inequality,

$$\leq ||Tx_n-Ty_n||^2+ ||Ty_n-x_n||^2$$

$$+ 2||Tx_n-Ty_n|| \times ||Ty_n-x_n||$$

$$\ldots\ldots (4.0.4)$$

Now,

$$||y_n-Tx_n||^2 = ||\beta_n Tx_n+(1- \beta_n) x_n- Tx_n||^2$$

$$= (1- \beta_n)||x_n-Tx_n||^2- \beta_n(1-\beta_n)||Tx_n-x_n||^2$$

$$= (1- \beta_n)^2 ||Tx_n-x_n||^2$$

$$\leq ||Tx_n- x_n||^2$$

$$\leq \left| ||Tx_n-Ty_n||+||Ty_n-x_n||^2 \right|$$

Using T. inequality

$$\leq ||Tx_n-Ty_n||^2+||Ty_n-x_n||^2+2||Ty_n-Tx_n|| \times$$

$$||Ty_n- x_n|| \quad \ldots.(4.0.5)$$

Since T satisfies 4.0.2 , we have

$$||Tx_n-Ty_n||^2 \leq a_1||x_n-y_n||^2+a_2||x_n-Tx_n||^2 +$$

$$a_3||y_n-Ty_n||^2+ a_4||x_n-Ty_n||^2+$$

$$a_5||y_n-Tx_n||^2+ a_6 \{ ||x_n-y_n||^2$$

$$||Tx_n-Ty_n||^2- 2 \langle x_n-y_n , Tx_n-Ty_n \rangle \}$$

$$\leqslant (a_1+a_6)||x_n-y_n||^2 + a_2||x_n-Tx_n||^2 + a_3||y_n-Ty_n||^2 + a_4||x_n-Ty_n||^2 + a_5||y_n-Tx_n||^2 + a_6||Tx_n-Ty_n||^2 \quad \ldots\ldots(4.0.6)$$

Since H is a real Hilbert space and T is monotone

Using relations 4.0.4 and 4.0.5 in 4.0.6. we get

$$||Tx_n-Ty_n||^2 \leqslant (a_1+a_6)\left[||Tx_n-Ty_n||^2 + ||Ty_n-x_n||^2 + 2||Tx_n-Ty_n|| \times ||Ty_n-x_n||\right]$$
$$+ a_2\left[||x_n-Ty_n||^2 + ||Tx_n-Ty_n||^2 + 2||x_n-Ty_n|| \times ||Tx_n-Ty_n||\right] + a_3\left[||x_n-y_n||^2+||x_n-Ty_n||^2 + 2||x_n-y_n|| x ||x_n-Ty_n||\right] + a_4||x_n-Ty_n||^2$$
$$+ a_5\left[||Tx_n-Ty_n||^2+||Ty_n-x_n||^2+2||Ty_n-Tx_n|| \times ||Ty_n-x_n||\right] + a_6||Tx_n-Ty_n||^2$$

$$\Longrightarrow 1-(a_1+a_2+a_3+a_5+2a_6)\ ||Tx_n-Ty_n||^2$$
$$\leqslant (a_1+a_2+ 2a_3+a_4+a_5+a_6)\ ||Ty_n-x_n||^2$$
$$+ 2(a_1+a_2+ 2a_3+a_5+a_6)\ ||Ty_n-Tx_n|| \times ||Ty_n-x_n||$$

$$\Rightarrow \quad (a_4-a_6)\,||Tx_n-Ty_n||^2 \leqslant (1+a_3)\,||Ty_n-x_n||^2$$

$$+\, 2(a_1+a_2+2a_3+a_5+a_6)\,||Tx_n-Ty_n||$$

$$\times\, ||Ty_n-x_n||$$

$$\text{since} \quad \sum_{i=1}^{6} a_i \leqslant 1$$

Taking limit of above as $n \to \infty$

we have,

$$||Tx_n-Ty_n||^2 \to 0 \qquad \therefore\ a_4 \neq a_6 \text{ and by (A)} \qquad \ldots (4.0.7)$$

Using triangle inequality, we have

$$||x_n-Tx_n||^2 \leqslant \Big[\, ||x_n-Ty_n|| + ||Tx_n-Ty_n|| \,\Big]^2$$

$$\to 0 \text{ by (A) and 4.0.7 as } n \to \infty$$

$$\ldots\ldots (4.0.8)$$

And

$$||P-Tx_n||^2 \leqslant \Big|\, ||P-x_n|| + ||x_n-Tx_n|| \,\Big|^2$$

$$\to 0 \text{ as } x_n \to P \text{ and } 4.0.8$$

$$\ldots\ldots (4.0.9)$$

Now we show that P is a fixedpoint of T.

Since T satisfies 4.0.2 , we have

$$||Tx_n - TP||^2 \leq a_1||x_n - P||^2 + a_2||x_n - Tx_n||^2$$

$$a_3||P-TP||^2 + a_4||X_n - TP||^2 +$$

$$a_5||P - Tx_n||^2 + a_6 \{ ||x_n - P||^2 + ||Tx_n - TP||^2$$

$$- 2 < x_n - P, \quad Tx_n - TP > \}$$

$$\longrightarrow \frac{a_3 + a_4}{1 - a_6} ||P-TP||^2$$ Since by data and 4.0.8 and 4.0.9

..... 4.1.0

Now, using triangle inequality

$$\Rightarrow \quad ||P-TP||^2 \leq \{ ||P-Tx_n|| + ||Tx_n - TP|| \}^2$$

$$\Rightarrow \quad ||P-TP||^2 \leq \frac{a_3 + a_4}{1 - a_6} ||P-TP||^2$$ by 4.0.9 and 4.1.0

$$\Rightarrow \left[1 - \frac{a_3 + a_4}{1 - a_6} \right] ||P-TP||^2 \leq 0$$

$$\Rightarrow \left[1 - (a_3 + a_4 + a_6) \right] ||P-TP||^2 \leq 0$$

$$\Rightarrow \quad ||P-TP||^2 \leq 0$$ by data

$$||P-TP|| = 0$$ $||.|| \nless 0$

$$\Rightarrow \quad TP = P$$

i.e. P is a fixed point of T

This proves the theorem.

Now we have generalised the theorem 4.0.3 as follows

4.2.0 Theorem:

Let C be a closed convex subset of a real Hilbert space H. Let T_1 and T_2 be two self maps satisfying 4.0.2 and 4.0.3 with $a_4 \neq a_6$ and $0 \leqslant a_3 + a_4 + a_6 < 1$. Further we assume that T is monotone suppose x_o is any point in C and the sequence $\{x_n\}$ associated with T_1 and T_2 is defined by Ishikawa scheme I-1.1.8 and I-1.1.9. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences bounded away from zero. If the sequence $\{x_n\}$ converges to P, then P is a fixed point of both T_1 and T_2

Proof :

Exactly on the same lines, we have proved this theorem as in Chapter [III, see Theorem 3.2.0].