CHAPTER - 5

CHAPTER - V

Pathak type Mapping

5.0.0 Introduction :

The well-known Banach contraction principle states that every contraction mapping of a complete metric space X into itself has a unique fixed point. This principle has been generalised in various ways by many authors.

This principle has not found much attention in terms of a rational expression. some authors who have made an attempt to generalise this principle through such an expression are as follows :

In 1975, Dass and Gupta [12] have tried to generalise Banach contraction principle for the mapping $T : X \to X$ satisfying

$$d(Tx,Ty) \leqslant \alpha d(y,Ty) \frac{[1+d(x,Tx)]}{1 + d(x,y)} + \beta d(x,y) \quad \ldots \quad (i)$$

for all x,y in X and $\alpha, \beta > 0$ and $\alpha + \beta < 1$. They have showed that T has a unique fixed point in X.

In April 1979, H.Chatterjee [9] has obtained fixed point theorems for mapping of Dass and Gupta, by using inequality (i), in arbitrary topological space. Using the same mapping of Dass and Gupta,, he has obtained in [10] and [11] respectively, a unique fixed point theorem for a pair of continuous self-mappings of a metric space and a fixed point theorem of a continuous mapping of a compact metric space.

In 1980, Jaggi and Dass [20] have proved the following fixed point theorem through a rational expression which comes out to be an extension of the well-known Banach's contraction mapping theorem.

<u>Theorem</u> :

Let F be a self map defined on a metric space (x,d) satisfying the following:

(i) for some $\alpha,\beta \in (0,1)$ with $\alpha+\beta < 1$

$$d[F(x),F(y)] \leq \frac{\alpha d[x,F(x)] \,.\, d[y,F(y)]}{d[x,F(y)] + d[y,F(x)] + d(x,y)} + \beta\, d(x,y),$$

for all $x,y \in X$ and $x \neq y$

(ii) there exists $x_o \in X$ such that

$$\{F^n(x_o)\} \supset \{F^{n_k}(x_o)\} \text{ with } \lim_{k\to\infty} F^{n_k}(x_o) \in X$$

Then F has a unique fixed point

$$u = \lim_{k\to\infty} F^{n_k}(x_o)$$

In 1982, Sharma and Yuel [44] proved the following fixed point theorem for mapping in a normed space as follows :

Theorem :

Let T be a mapping of a normed space X into itself satisfying

$$d(Tx,Ty) \leq q \max\left\{d(x,y), \frac{d(y,Ty) + [1+d(x,Tx)]}{1 + d(x,Tx)}, \frac{1}{2}\frac{d(x,Ty)\ [1+d(x,Tx)+ d(y,Ty)]}{1 + d(x,y)}\right\}$$

for all x,y in X and $0 < q < 1$ where the sequence $\{x_n\}$ is given by

$$x_{n+1} = (1-C_n)\, x_n + C_n Tx_n \quad \text{for} \quad n \geq 0 \quad \text{where}$$

C_n satisfy $C_o = 1$, $0 < C_n < 1$ for $n > 0$ and

$\sum C_n$ diverges and $\lim_n C_n = h > 0$

If $\{x_n\}$ converges in X, then it converges to a fixed point of T.

In 1984, Sharma and Yuel [45] have obtained the following unique fixed point theorem.

Theorem :

Let T be a mapping of a complete metric

space X into itself such that

$$d(Tx,Ty) \leq \alpha \left\{\frac{d(x,Tx)\ d(y,Ty)}{d(x,y)}\right\} + \beta\left\{d(x,Tx) + d(y,Ty)\right\} + \nu\left\{d(x,Ty) + d(y,Tx)\right\} + \delta\, d(x,y)$$

for all x,y in X where

$0 \leq \frac{\beta + \nu + \delta}{1-\alpha-\beta-\nu} < 1$, $\beta + \nu < 1$, $2\nu + \delta < 1$,

$\nu \geq 0$. Then T has a unique fixed point.

In the same year 1984, Bajaj [1] has obtained the following theorem for pair of mappings :

<u>Theorem</u> :

Let S and T be a pair of self mappings of a complete metric space (X,d) and satisfy the inequality

$$d(Sx,Ty) \leq \alpha\, \frac{d(x,Sx).d(x,Ty) + [d(x,y)]^2 + d(x,Sx).d(x,y)}{d(x,Sx) + d(x,y) + d(x,Ty)}$$

for all x,y in X with $x \neq y$, $0 < \alpha < 1$

and $d(x,Sx) + d(x,y) + d(x,Ty) \neq 0$

Then S and T have common fixed point. Further if $d(x,Sx) + d(x,y) + d(x,Ty) = 0$ implies $d(Sx,Ty) = o$, then S and T have a unique common fixed point.

Naik [32] had obtained a unique fixed point theorem as follows :

Theorem :

Let T be a non-empty, bounded, closed and convex subset of a reflexive Banach space and let K have normal structure. Let $T : K \to K$ be a continuous map of K satisfy

$$||Tx-Ty|| \leq \frac{||x-Tx||\ ||x-Ty|| + ||y-Ty||\ ||y-Tx||}{||x-Ty|| + ||y-Tx||}$$

$$\text{if } ||x-Ty||\ ||y-Tx|| > 0$$

$$= 0 \qquad\qquad \text{otherwise}$$

Then T has a unique fixed point in K.

In 1988, H.K. Pathak [35] have defined the following mappingin a normed space using more general rational inequality .

Definition :

Let X be a normed space then T, a self mapping of X called a 'generalised contraction mapping' if

$$||Tx-Ty|| \leq q \max \{ ||x-y||\frac{||x-Tx|| \left[1-||x-Ty||\right]}{1 + ||x - Tx||} ,$$

$$\frac{||x-Ty|| \left[1-||x-Tx||\right]}{1 + ||x-Ty||}, \quad \frac{||Tx-y|| \left[1-||y-Ty||\right]}{1 + ||Tx-y||},$$

$$\frac{||y-Ty|| \left[1-||Tx-y||\right]}{1 + ||y-Ty||} \}$$

for all x,y in X where $0 < q < 1$.
He proved the following fixed point theorem.

Let X be a closed, convex subset of a normed space N and T be a 'generalised contraction mapping' of X and T be continuous on X. Let $\{x_n\}$ be the sequence of Mann iterates associated with T defined by

$x_{n+1} = (1-C_n)\, x_n + C_n T x_n$ for $n \geq 0$ where
C_n satisfy $C_0 = 1$, $0 < C_n < 1$ for $n > 0$
and $\lim_{n \to \infty} C_n = h > 0$. If $\{x_n\}$ converges in X.,
thenit converges to a fixed point of T.

He has extended the above theorem 1for a pair of mappings T_1 and T_2 and obtained common fixed point for them.

Purpose of This Chapter :

From the above all results it is clear that the Banach contraction principle has undergone many generalisations through rational expressions

in metric spaces as well as in normed spaces. But this principle has not found any attention in terms of rational expressions in Hilbert spaces.
In this chapter, we have made an attempt to generalise this principle through such rational expressions in Hilbert spaces.

Now we define a Pathak type mapping in Hilbert space as follows :

5.0.1 Definition : A self mapping $T : C \to C$, where C is a closed convex subset of a Hilbert space H, called Pathak type mapping if it satisfies the condition,

$$||Tx-Ty||^2 \leq q \max \left\{ ||x-y||^2, \frac{||x-Tx||^2 \left[1-||x-Ty||^2\right]}{1+||x-Tx||^2}, \frac{||x-Ty||^2 \left[1-||x-Tx||^2\right]}{1+||x-Ty||^2}, \frac{||Tx-y||^2 \left[1-||y-Ty||^2\right]}{1+||Tx-y||^2}, \frac{||y-Ty||^2 \left[1-||Tx-y||^2\right]}{1+||y-Ty||^2} \right\}$$

for all x,y in C where $0<q<1$

Main Result :

We hare establish two fixed point theorems using the technique as appeared in Rhoades [37] and Yuel and Sharma [44] for mappings satisfying 5.0.1.

Our first result runs as follows :

5.0.2 Theorem :

Let C be a closed, convex subset of a real Hilbert space H and T be a mapping satisfying 5.0.1 and T be continuous on C, $\{x_n\}$ be sequence of Mann iterates associated with mapping T given by $x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T x_n$ for $n \geq 0$ where $\{\alpha_n\}$ satisfies

(i) $\alpha_0 = 1$ (ii) $0<\alpha_n<1$ for $n > 0$

(iii) $\lim_{n\to\infty} \alpha_n = h \in (0,1)$,

If $\{x_n\}$ converges in C, then it converges to a fixed point of T.

Proof :

Let $z \in C$ such that $\lim_{n\to\infty} x_n = z$

Now we show that z is the fixed point of T consider,

$$||z-Tz||^2 = ||z-x_{n+1} + x_{n+1} - Tz||^2$$

$$\leqslant \|z-x_{n+1}\|^2 + \|x_{n+1}-T_2\|^2 + 2\,\|z-x_{n+1}\| \times \|x_{n+1}-T_2\|$$

$$= \|z-x_{n+1}\|^2 + \|(1-\alpha_n)x_n + \alpha_n Tx_n - Tz\|^2 + 2\|z-x_{n+1}\| \times \|x_{n+1}-Tz\|$$

$$= \|z-x_{n+1}\|^2 + \|(1-\alpha_n)x_n - (1-\alpha_n)\,Tz + \alpha_n Tx_n - \alpha_n Tz\|^2 + 2\|z-x_{n+1}\| \times \|x_{n+1}-Tz\|$$

Therefore,

$$\|z-Tz\|^2 \leqslant \|z-x_{n+1}\|^2 + (1-\alpha_n)^2\,\|x_n-Tz\|^2 + \alpha_n^2\,\|Tx_n-Tz\|^2 + 2\,\alpha_n^2(1-\alpha_n)^2\|x_n-Tz\| \times \|Tx_n-Tz\| + 2\|z-x_{n+1}\| \times \|x_{n+1}-Tz\|$$

$$\leqslant \|z-x_{n+1}\|^2 + (1-\alpha_n)^2\,\|x_n-Tz\|^2 + \alpha_n^2\, q \max\left\{\|x_n-z\|^2, \frac{\|x_n-Tx_n\|^2\left[1-\|x_n-Tz\|^2\right]}{1+\|x_n-Tx_n\|^2}, \frac{\|x_n-Tz\|^2\left[1-\|x_n-Tx_n\|^2\right]}{1+\|x_n-Tz\|^2}, \frac{\|Tx_n-z\|^2\left[1-\|z-Tz\|^2\right]}{1+\|Tx_n-z\|^2}, \frac{\|z-Tz\|^2\left[1-\|Tx_n-z\|^2\right]}{1+\|z-Tz\|^2}\right\}$$

$$+ 2\alpha_n^2 (1-\alpha_n)^2 ||x_n - Tz|| \times ||Tx_n - Tz|| +$$

$$2 ||z - x_{n+1}|| \times ||x_{n+1} - Tz||$$

we observe that

$$||x_n - Tx_n||^2 = ||x_n - x_{n+1}||^2 / \alpha_n^2$$

Thus the abve inequality reduces to

$$||z-Tz||^2 \leq ||z - x_{n+1}||^2 + (1-\alpha_n)^2 ||x_n - Tz||^2$$

$$+ \alpha_n^2 q \max\{ ||x_n - z||^2 \quad \frac{||x_n - x_{n+1}||^2 / \alpha_n^2 \left[1 - ||x_n - Tz||^2\right]}{1 + ||x_n - x_{n+1}||^2 / \alpha_n^2}$$

$$\frac{||x_n - Tz||^2 \left[1 - ||x_n - x_{n+1}||^2 / \alpha_n^2\right]}{1 + ||x_n - Tz||^2},$$

$$\frac{||Tx_n - z||^2 \left[1 - ||z-Tz||^2\right]}{1 + ||Tx_n - z||^2}, \frac{||z-Tz||^2 \left[1 - ||Tx_n - z||^2\right]}{1 + ||z-Tz||^2} \}$$

$$+ 2\alpha_n^2 (1-\alpha_n)^2 ||x_n - Tz|| \times ||Tx_n - Tz|| + 2||z - x_{n+1}|| \times ||x_{n+1} - Tz||$$

Now taking the limit as $n \to \infty$ and
Using (iii) and continuty of T we have

$$||z-Tz||^2 \leq (1-h)^2 ||z-Tz||^2 + h^2 q \max\{ 0,$$

$$0, , \frac{||z-Tz||^2}{1 + ||z-Tz||^2} \quad \frac{||z-Tz||^2 \left[1 - ||z-Tz||^2\right]}{1 + ||z-Tz||^2},$$

$$\frac{||z-Tz||^2 \left[1-||z-Tz||^2\right]}{1+||z-Tz||^2}\}$$

$$\Rightarrow ||z-Tz||^2 \leqslant (1-h) \, ||z-Tz||^2 + hq \max\{0,\ 0,$$

$$\frac{||z-Tz||^2}{1+||z-Tz||^2},\ \frac{||z-Tz|| \left[1-||z-Tz||^2\right]}{1+||z-Tz||^2}$$

$$\frac{||z-Tz||^2 \left[1-||z-Tz||^2\right]}{1+||z-Tz||^2}\}$$

$$\Rightarrow ||z-Tz||^2 \leqslant (1-h) \, ||z-Tz||^2 + \frac{hq\, ||z-Tz||^2}{1+||z-Tz||^2}$$

$$\Rightarrow (1-1+h) \, ||z-Tz||^2 \leqslant \frac{hq\, ||z-Tz||^2}{1+||z-Tz||^2}$$

$$\Rightarrow ||z-Tz||^2 \leqslant q \, \frac{||z-Tz||^2}{1+||z-Tz||^2} \qquad \because h > 0$$

$$\Rightarrow ||z-Tz||^2 + ||z-Tz||^4 \leqslant q \, ||z-Tz||^2$$

$$\Rightarrow ||z-Tz||^4 \leqslant -(1-q) \, ||z-Tz||^2$$

suppose that $z \neq Tz$, then let $||z-Tz||^2 = \delta$

then we have $\delta < -(1-q)$

which is a contradiction since $||.|| \nless 0$

Hence $Tz = z$ i.e. z is the fixed point of T

Now we give an example to verify the above theorem.

Example :

Let H = R, the set of real numbers regarded as a Hilbert space.

Let $T : [0,1] \to [0,1]$ be a mapping such that $Tx = x/2$

Then,

$$||Tx-Ty||^2 = \frac{1}{4}||x-y||^2$$

setting $\frac{1}{4} \leq q < 1$ we see that all the conditions of above theorem are satisfied. Here 0 is the fixed point of T.

Now, we extended the above theorem 5.0.2 for a pair of mappings T_1 and T_2.

5.0.3 Theorem :

Let C be a closed, convex subset of a Hilbert space H and let T_1 and T_2 be pair of mappings satisfying.

$$||T_1x-T_2y||^2 \leq q \max \{||x-y||^2, \frac{||x-T_1x||^2 \left[1-||x-T_2y||^2\right]}{1 + ||x-T_1x||^2}$$

$$\frac{||x-T_2y||^2\left[1-||x-T_1x||^2\right]}{1+||x-T_2y||^2}, \frac{||T_1x-y||^2\left[1-||y-T_2||^2\right]}{1+||T_1x-y||^2}$$

$$\frac{||y-T_2y||^2\left[1-||T_1x-y||^2\right]}{1+||y-T_2y||^2}\}$$

for all x,y in C with $0<q<1$ and

T_1 and T_2 are continuous on C.

The sequence $\{x_n\}$ of Mann iterates associated with T_1 and T_2 are given by

for $x_o \varepsilon$ C, set $x_{2n+1} = (1-\alpha_n)x_{2n} + \alpha_n T_1 x_{2n}$

anx $x_{2n+2} = (1-\alpha_n)\, x_{2n+1} + \alpha_n T_2 x_{2n+1}$

for $n \geqslant 0$ where $\{\alpha_n\}$ satisfy conditions

(i) to (iii) of theorem 5.0.2. Here we further assume that $h \neq \frac{1}{2}$. If $\{x_n\}$ converges to z in C and if z is a fixed point of either T_1 or T_2 then z is the common fixed point of T_1 and T_2

Proof :

Let $z \varepsilon C$ such that $\lim_{n\to\infty} x_n = z$

and let $T_1 z = z$

Now we show that z is the common fixed point of T_1 and T_2

Consider,

$$\|z-T_2z\|^2 = \|z-x_{2n+1} + x_{2n+1} - T_2z\|^2$$

$$\leqslant \|z-x_{2n+1}\|^2 + \|(1-\alpha_n)x_{2n} + \alpha_n T_1x_{2n} - T_2z\|^2$$

$$+ 2\|z-x_{2n+1}\| \times \|x_{2n+1} - T_2z\|$$

$$\leqslant \|z-x_{2n+1}\|^2 + \|(1-\alpha_n)x_{2n} - (1-\alpha_n) T_2z +$$

$$\alpha_n T_1x_{2n} - \alpha_n T_2z\|^2 + 2\|z-x_{2n+1}\| \times \|x_{2n+1} - T_2z\|$$

$$\Rightarrow \|z-T_2z\|^2 \leqslant \|z-x_{2n+1}\|^2 + (1-\alpha_n)^2\|x_{2n}-T_2z\|^2 +$$

$$\alpha_n^2\|T_1x_{2n}-T_2z\|^2 + 2\,\alpha_n^2(1-\alpha_n)^2\|x_{2n}-T_2z\| \times$$

$$\|T_1x_{2n}-T_2z\| + 2\|z-x_{2n+1}\| \times \|x_{2n+1} - T_2z\|$$

$$\Rightarrow \|z-T_2z\|^2 \leqslant \|z-x_{2n+1}\|^2 + (1-\alpha_n)^2 \|x_{2n} - T_2z\|^2$$

$$+ \alpha_n^2\, q \max\Big\{\|x_{2n}-z\|^2, \frac{\|x_{2n}-T_1x_{2n}\|^2\left[1-\|x_{2n}-T_2z\|^2\right]}{1 + \|x_{2n} - T_1x_{2n}\|^2},$$

$$\frac{\|x_{2n} - T_2z\|^2 \left[1 - \|x_{2n} - T_1x_{2n}\|^2\right]}{1 + \|x_{2n} - T_2z\|^2},$$

$$\frac{\|T_1x_{2n}-z\|^2\left[1-\|z-T_2z\|^2\right]}{1 + \|T_1x_{2n}-z\|^2}, \frac{\|z-T_2z\|^2\left[1-\|T_1x_{2n}-z\|^2\right]}{1 + \|z-T_2z\|^2}\Big\}$$

$$+ 2\, \alpha_n^2 (1-\alpha_n)^2 ||x_{2n} - T_2 z|| \times ||T_1 x_{2n} - T_2 z||$$

$$+ 2 ||z - x_{n+1}|| \times ||x_{2n+1} - T_2 z||$$

we observe that

$$||x_{2n} - T_1 x_{2n}||^2 = ||x_{2n} - x_{2n+1}||^2 / \alpha_n^2$$

Hence the above inequality reduces to

$$||z - T_2 z||^2 \leq ||z - x_{2n+1}||^2 + (1-\alpha_n)^2 ||x_{2n} - T_2 z||^2$$

$$+ \alpha_n^2\, q \max\Big\{ ||x_{2n} - z||^2, \frac{||x_{2n} - x_{2n+1}||^2 / \alpha_n^2 \left[1 - ||x_{2n} - T_2 z||^2\right]}{1 + ||x_{2n} - x_{2x+1}||^2 / \alpha_n^2}$$

$$\frac{||x_{2n} - T_2 z||^2 \left[1 - ||x_{2n} - x_{2n+1}||^2 / \alpha_n^2\right]}{1 + ||x_{2n} - T_2 z||^2},$$

$$\frac{||T_1 x_{2n} - z||^2 \left[1 - ||z - T_2 z||^2\right]}{1 + ||T_1 x_{2n} - z||^2}$$

$$\frac{||z - T_2 z||^2 \left[1 - ||T_1 x_{2n} - z||^2\right]}{1 + ||z - T_2 z||^2}$$

$$+ 2\, \alpha_n^2 (1-\alpha_n)^2 ||x_{2n} - T_2 z|| \times ||T_1 x_{2n} - T_2 z||$$

$$+ 2 ||z - x_{n+1}|| \times ||x_{2n+1} - T_2 z||$$

Now taking the limit as $n \to \infty$ and using (iii)

and continuity of T_1

we obtain for $T_1 z = z$

$$||z-T_2z||^2 \leq (1-h)^2 ||z- T_2z||^2 + h^2 q \max\{0, o, \frac{||z - T_2z||^2}{1 + ||z- T_2z||^2}, 0, \frac{||z-T_2z||^2}{1 + ||z- T_2z||^2}\}$$

$$+ 2h^2 (1-h)^2 ||z - T_2z||^2 + 0$$

$$\Rightarrow ||z- T_2z||^2 \leq (1-h) ||z - T_2z||^2 + hq \frac{||z- T_2z||^2}{1+||z-T_2z||^2} + 2h (1-h) ||z- T_2z||^2$$

$$\Rightarrow h(2h-1) ||z-T_2z||^2 \leq \frac{h q ||z- T_2z||^2}{1 + ||z- T_2z||^2}$$

$$\Rightarrow (2h-1) ||z-T_2z||^2 + (2h-1)||z-T_2z||^4 \leq q||z-T_2z||^2$$

$$\Rightarrow (2h-1)||z- T_2z||^4 \leq \left[q -(2h-1)\right] ||z-T_2z||^2$$

$$\Rightarrow ||z - T_2z||^4 \leq - \left[1 - \frac{q}{2h-1}\right] ||z-T_2z||^2$$

For $0 < h \neq \frac{1}{2} < 1$ we have $\frac{q}{2h - 1} = P < 1$

Suppose $z \neq T_2z$. Let $||z - T_2z||^2 = \delta$,

we have $\delta < - [1-P]$

which is a contradiction.

Hence $z = T_2z$

Similarly we can prove that if $T_2 z = z$, then

$T_1 z = z$ i.e. z is the common fixed point of T_1 and T_2.

This completes the proof.

Note :

Here we conclude this chapter by indicating an open question for further research work.

(i) If T satisfies contractive definition 5.0.1 and the continuity of mapping T is removed, does T have a fixed point ?