CHAPTER - 1

CHAPTER - I

INTRODUCTION

This chapter is introductory in nature which contains two parts. Part first includes the basic concepts and fixed point theorems which are needed for our investigations. Part two deals with a historical development of the fixed point theorey.

Part - I

1.1 SOME BASIC CONCEPTS

1.1.1 METRIC SPACES

Definition :

Let X be a non-empty set and d be a function from X x X into R^+ such that for all x,y and z in X we have

(i) $d(x,y) \geq 0$

(ii) $d(x,y) = 0$ if and only if $x = y$

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,z) \leq d(x,y) + d(y,z)$

Then d is called a metric or a distance function and the pair (x,d) is called a metric space. The space (x,d) is also denoted by X if the metric d is understood.

d(x,y) is called the distance between x and y.

Definition :

Let (x,d) be a given metric space.
Let $x_o \varepsilon X$ and a real number $r > 0$ be given.
Then the sets

(i) $B(x_o;r) = \{ x \varepsilon X / d(x,x_o) < r \}$ is called an open sphere.

(ii) $\bar{B}(x_o;r) = \{ x \varepsilon X / d(x,x_o) \leq r \}$ is called a closed sphere.

(iii) $S(x_o;r) = \{ x \varepsilon X / d(x,x_o) = r \}$ is called a sphere with the centre at x_o and radius r.

Convergent Sequence

Definition

Let (X,d) be a metric space and $\{ x_n \}$ is said to be convergent if there exists a point x in X such that for each $\varepsilon > 0$ there exists a positive integer N such that for all $n \geq N$

$$d(x_n\ x) < \varepsilon$$

i.e. $d(x_n x) \to 0$ as $n \to \infty$

The point x is called the limit of the sequence $\{x_n\}$ and we write $x_n \to x$ in the form

$$\lim_{n \to \infty} x_n = x$$

we say that $\{x_n\}$ is a convergent sequence with limit x.

Definition :

Let (X.d) be a metric space and $\{x_n\}$ be a sequence in it. The sequence $\{x_n\}$ is said to be a cauchy sequence if for every $\epsilon > 0$, there exists a positive integer N such that

$$d(x_m, x_n) < \epsilon \text{ for all } m, n \geq N$$

Theory :

Every convergent sequence in a metric space is a cauchy sequence but not conversely.

Complete Metric Space

Definition :

A metric space X is said to be complete if every cauchy sequence in X converges to a point in X.

Subsequence

Definition :

A subsequence of a sequence $\{x_n\}$ is a sequence whose terms are chosen from the terms of the sequence $\{x_n\}$ and arranged in the same order as their relative order in $\{x_n\}$. A subsequence of $\{x_n\}$ is often designated as $\{x_{n_k}\}$ with terms x_{n_1}, x_{n_2} ..

Note : If a sequence $\{x_n\}$ converges to x then any subsequence of $\{x_n\}$ also converges to x.

Bounded Sequence

Definition :

A sequence $\{x_n\}$ is said to be bounded if there exists numbers m_1, m_2 such that

$$m_1 \leq \{x_n\} \leq m_2 \quad \text{for } n \in N$$

$\{x_n\} X$

Monotonic Sequence

Definition :

A sequence $\{x_n\}$ is said to be strictly monotonicaly increasing if

$$x_{n+1} > x_n \text{ , for all } n$$

and strictly monotonically decreasing if

$$x_{n+1} < x_n \text{ for all } n$$

A sequence which is either monotonically increasing or decreasing is called monotonic sequence.

Theorem :

A monotonic sequence is convergent if and only if it is bounded.

Compact Metric Space

Definition :

A metric space X is said to be compact if every sequence in X has a convergent subsequence.

Open Set

Definition :

Let (X,d) be a metric space. A subset M of X is said to be open if and only if to each $x \epsilon M$, there exists $r > o$, such $S(x,r) \subset M$.

Closed Set

Definition :

A subset M of a metric space (X,d) is said to be closed if the complement of M in X is open.

1.1.2 NORMED LINEAR SPACES

Let X be a real or complex vector space or linear space of finite or infinite dimension. Let K be the field of complex numbers C, or real numbers R.

Norm of a Space

Definition :

Let X be a linear space on K. A norm on X is a real valued function $||.|| \ X \to R^+$ defined on X such that for any $x, y \ \epsilon X$ and for $\lambda \ \epsilon \ K$, we have

(i) $||x|| \geqslant 0$

(ii) $||x||=0$ if and only if $x=0$

(iii) $||\lambda x|| = |\lambda| \ ||x||$

(iv) $||x+y|| \leqslant ||x|| + ||y||$

Definition :

A normed linear space is a vector with a norm

$(X, ||\cdot||)$

and it is denoted by $(x, ||.||)$

Result :

What is X?

A norm on X defines a metric d on X which is given by

$$d(x,y) = ||x-y|| \text{ for all } x,y \in X$$

and is called the metric induced by the norm. Thus every normed linear space X is a metric space with this metric defined on X.

Theorem :

In a normed linear space, norm is a continuous function

$\{x_n\} \to x$ in X.

i.e If $x_n \to x$ then $||x_n|| \to ||x||$

Strong Conversion

Definition :

A sequence $\{x_n\}$ in a normed linear space (or normed space) X is said to be strongly convergent (or convergent in the norm) if there exists an $x \in X$ such that

$$\lim_{n\to\infty} ||x_n - x|| = 0 \text{ i.e } \lim_{n\to\infty} x_n = x \text{ or } x_n \to x$$

we say that $\{x_n\}$ converges strongly to x and x is the strong limit of $\{x_n\}$

1.1.3 Banch Spaces

A normed linear space X is said to be complete if every cauchy sequence in X converges to an element of X. A complete normed linear space is called a Banch Space.

Convex Set

Definition :

Let B be an arbitrary Banch Space. A convex set in B is a non-empty sub-set S with the property that for all $x, y \in X$, $z = tx + (1-t)y \in S$ for every real number t such that $0 \leq t \leq 1$.

Note :

(i) The empty set and the set containing one point are convex.

(ii) Every subspace of a vector space is convex. In particular every vector space is convex.

1.1.4 HILBERT SPACE

Inner Product Space

Definition :

Let X be a linear space over the scalar field K (real or complex). A function $\langle \, , \, \rangle : X \times X \to K$ is called an innear product on X if for all $x, y, z \in X$ and $\lambda \in K$ we have

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ where bar denotes complex conjugates.

(iii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$

(iv) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

An inner product space or pre Hilbert space is a linear space X with an inner product on it.

If K=R, then $\langle x, y \rangle$ is a real number and if

K=C, then it is a complex number.

Properties of inner Product Space :

(i) Every inner product space is a normed space but not all normed spaces are inner product spaces.

(ii) In an inner product space, the inner product is continuous.

(iii) If x and y are in an inner product space, then

$$||x+y||^2 + ||x-y||^2 = 2\left(||x||^2 + ||y||^2\right)$$

which is known as Parallelogram Law.

(iv) If x, y are in an inner product space X then $|\langle x, y\rangle| \leq ||x|| \cdot ||y||$ which is Cauchy-Schwarz inequality

(v) Let X be a normed linear space in which the Parallelogram Law holds. Then X can be made into an inner product space by defining the given norm as $\langle x, x\rangle = ||x||^2$

Hilbert Space

Definition :

A complete inner product space is called a Hilbert Space.

Result :

Every Hilbert space is a Banch space but converse is not true.

Fixed Point

Definition :

Let X be a set and $T : X \to X$ be a self map. A fixed point of T is a point $x \in X$ such that $Tx = x$ i.e. the image Tx coincides with x.

Example,

(i) A mapping $x \to x^3$ of R into itself has three fixed points (0, -1, 1)

(ii) A translation has no fixed point. ?

(iii) A rotation of the plane has a single fixed point i.e. the centre of rotation.

(iv) A mapping $Tx = x^2-6$ defined on R has $x = -2$, $x = 3$ are fixed points.

The following definitons in Hilbert Space are due to Browder and Petryshyn [7]

Let C be a convex subset of a real Hilbert Space H and T be a nonlinear (possibly) mapping ? from C into H, then we have,

Definition :

T is said to be strictly contractive if there exists a constant K with $0 < K < 1$ such that

$$||Tx-Ty|| < K\, ||x-y|| \quad \text{for all } x, y \in C$$

Definition :

T is said to be contractive (or if for all $x, y \in C$,

$$||Tx - Ty|| < ||x - y||$$

Definition :

T is said to be strictly pseudocontractive if there exists a constant $0 < K < 1$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + K||(I-T)x - (I-T)y||^2$$

for al $x, y, \in C$

Definition :

T is said to be pseudocontractive if for all $x, y \in C$,

$$||Tx - Ty||^2 \leq ||x - y||^2 + ||(I-T)x - (I-T)y||^2$$

These mappings admit iterative methods for the construction of their fixed points.

Identity Mapping

Definition :

The identify mapping $I : X \to X$ is defined by $I(x) = x$

Continuous Mapping

Definition

A mapping T of a metric space X into a metric space Y is said to be continuous at $x \in X$, If $x_n \to x$ in X then

$Tx_n \to Tx$ in Y

Definition :

Let H be a Hilbert Space and C be a convex subset of H. T be a mapping from C in to H. A mapping T is said to be monotone [1] if

a Re $\langle Tx-Ty, \; x-y \rangle \geq 0$ for all x,y in C

Now We mention here some fixed point theorems.

1.1.5 FUNDAMENTAL FIXED POINT THEOREMS

Brouwer's [8] and Schayder's [41] fixed point theorems are fundamental theorems in the field of fixed point theory and its applications. Though Brouwer obtained his result in 1912, Poincare proved a slightly different version of it in 1986 which was subsequently rediscovered by Bohl P. in 1904.

Brower's Fixed Point Theorem

Every continuous map of the closed unit ball $S = \{x / ||x|| < 1\}$ in R^n, the n-dimentional Euclidean space to itself has a fixed point.

Birkhoff and Kellog [3] were the first to prove fixed point theorems in infinite dimensional spaces. They considered continuous self-maps defined on compact subsets of C[0,1] and $L^2[0,1]$ and established the existence of fixed points for them Schauder[41] generalised these results.

Schauder's Fixed Point Theorem

Let C be a non-empty convex compact subset of a normed linear space X. Then every continuous self map of C has a fixed point.

Many author's have extended Schauder's theorem in different spaces. Tychonoff [43] extended Brouwer's result to a compact convex subset of a locally convex linear topological space.

1.1.6 ITERATIVE METHODS

Mann Iteration Process :

Mann [30] gave the following iteration process. For a self-mapping T of a closed bounded interval of the real line having a unique fixed point, the iteration process

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T x_n \quad \ldots\ldots \quad (1.1.7)$$

with

$\alpha_n = \dfrac{1}{n+1}$, converges to the fixed

point of T as $n \to \infty$

Ishikawa Iteration Process

In 1974, Ishikawa [19] itroduced the following iterative procedure.

Let C be a non-empty convex subset of a Hilbert space H and T be a self map on C. Then the iteration scheme $\{x_n\}_{n=0}^{\infty}$ introduced by Ishikawa is as follows.

For any $x_o \varepsilon$ C,

$$\left.\begin{aligned} y_n &= (1-\beta_n)\, x_n + \beta_n\, Tx_n \quad , \; n \geqslant 0 \\ x_{n+1} &= (1-\alpha_n) x_n + \alpha_n Ty_n \quad , \; n \geqslant 0 \end{aligned}\right] \quad ..(1.1.8)$$

Where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positve numbers which satisfy the following three conditions.

$$\left.\begin{aligned} &\text{(i)} && 0 \leqslant \alpha_n \leqslant \beta_n < 1 \\ &\text{(ii)} && \lim_{n \to \infty} \beta_n = 0 \\ &\text{(iii)} && \sum_{n=0}^{\infty} \alpha_n \beta_n = \infty \end{aligned}\right] \quad ...\ (1.1.9)$$

Ishikawa Identity

For any x,y,z in a Hilbert space H and a real number t,

$$||tx+(1-t)y-Z||^2 = t||x-z||^2 + (1-t)||y-z||^2 - t(1-t)\,||x-y||^2 \quad ...\ (1.1.10)$$

Part - II

1.2 Historical Developments of a Fixed Point Theory in Hilbert Spaces

The theory of Hilbert spaces is originated in the year 1912 with the work 'Grundzuge einer allegemeinon Theorie der linearen Integralqleichungen' of the great German mathematician D.Hilbert [18]. However, several years elapsed before an axiomatic basis was provided by the famous mathematician J.Von Neumann [34]. The modern developments in Hilbert spaces are concerned largely with the theory of operator on the spaces.

Browder initiated the study of fixed point theory of non-expansive mappings in Hilbert spaces without compactness conditions. In 1965, Browder [5] proved the following theorem

Theorem :

Let Br be a closed ball of radius $r > 0$ in a real or complex Hilbert space H. ∂Br be the boundary of Brand S be a nonlinear contraction map of Br into H such that $Sx - \lambda x \neq 0$ for all x in Br and any $\lambda \neq 1$. Using the theory of monotone operators developed in [31, 4]. Browder [5] showed that T has atleast one fixed point in Br.

Petryshyn [36] studied an iteration method for the actual construction of fixed points of a nonlinear contraction map S under the additional assumption that S is demicompact. He has proved his main result in the following way.

Theorem :

Let S be a demicompact contraction of Br into H such that $Sx-\lambda x \neq 0$ for all $x \in \partial Br$ and $\lambda > 1$, then the set of fixed points Fr of S lying in Br is a nonempty convex set and for any $x_o \in Br$ and any $\beta > 0$ such that $0 < \beta < 1$ the sequence $\{x_{n+1}\}$ determined by the process

$$x_{n+1} = \beta r_n S x_n + (1-\beta)\, x_n , \quad n= 0,1,2,..$$

where the real numbers r_n, n= 0,1,2,.... are given by

$$r_n = \begin{cases} 1 & \text{if } ||Sx_n|| \leq r \\ \dfrac{r}{||Sx_n||} & \text{if } ||Sx_n|| \geq r \end{cases}$$

converges to a fixed point $\bar{z} \in F_r \subset B_r$ of S

Browder and Petryshyn [7] introduced the four classes of nonlinear mappings(strictly contractive, contractive, stricly Pseudocontractive and Pseudocontractive) which admit iterative methods for the construction of their fixed points. They established the following basic existence result.

Theorem :

Let C be a closed bounded convex subset of the Hilbert space H, T be a contractive mapping of c into C. Then T has at least one fixed point in C.

Based upon this theorem a number of theorems have been proved by theauthors. We give here a few of them

Theorem :

If T is contractive (non-expansive) mapping of C into C, where C is a closed convex subset of a Hilbert space H and the set F(T) of fixed points of T in C is non-empty, then the mapping defined by $T_\lambda = \lambda I + (1-\lambda) T$ for any given λ with $0 < \lambda < 1$ is a reasonable wanderer from C into C with the same fixed points as T.

Corollary :

If T is contractive (non-expansive) mapping of C into C with non-empty set F(T) of fixed points of T in C and if the mapping defined by

$T_\lambda = \lambda I + (1-\lambda) T$ for a given λ with $0 < \lambda < 1$, then T_λ maps with C into C, T_λ has the same fixed points as T and T_λ is a symptotically regular.

<u>Theorem</u> :

Let T be a self-map of a bounded closed convex subset C of a Hilbert space H. Suppose T is contractive and demicompact. Then the set F(T) of fixed points of T in C is a non-empty convex set and for any given $x_o \varepsilon$ C and any fixed $\lambda \geq 0$ with $0 < \lambda < 1$, the sequence $\{x_n\} = \{T_\lambda^n x_o\}$ determined by the process

$$x_n = \lambda T x_{n-1} + (1-\lambda) x_{n-1} \quad ; \quad n = 1,2,3,...$$

converges strongly to a fixed point of T in C.

Hicks and Huffman [17] generalised theorem (1.1.4) and (1.2.6) in generalised Hilbert space (see theorem 6,7 of [17]).

Ishkawa [19] has introduced a new itereation scheme, calledd as Ishikawa iteration scheme and proved that a sequence of Ishikawa iterates for a Lipschitzian Pseudocontractive maping in a convex compact subset of a Hilbert space converges strongly to a fixed point of this mapping.

Das and Debata [13] have extended and generalised the result of Ishikawa [19] by taking simultaneously a more generalised iteration scheme involving a family of maps and secondly by taking less restrictive hemicontractive mappings. Their result states as follows :

Theorem :

Let $\{T_j\}$, $j = 1,2,....K$, $K \geq 2$ be a family of hemicontractive maps defined on a convex, compact subset C of a Hilbert space H and have at least one common fixedd point in C. Let the family of maps $\{T_j\}$ satisfy

$$||T_i x - T_j y|| \leq M \; ||x-y||$$

for all $x,y \in C$ and all pairs (i,j), M being a positive constant. Then the sequence $\{x_n\}$ converges to a common fixed point of the family of maps $\{T_j\}$ where x_n is defined iteratively for each positive integer n by $x_1 \in C$

$$x_{n+1} = (1-\alpha_n)\; x_n + \alpha_n T_k u_{k-1}(n)$$

Where,

$$u_o(n) = x_n, \; u_j(n) = (1-\beta_n)\; x_n + \beta_n \; T_j \; u_{j-1}(n)$$

for $j= 1,2,...K$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequencces in [0,1] such that

(i) $0 \leq \alpha_n \leq \beta_n \quad 1$ for $n = 1,2,\ldots$

(ii) $\lim_{n \to \infty} \beta_n = 0$

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n^{k-1} = \infty$ for each $k \geq 2$

The authors [13] claimed that for $K = 2$, $T_1 = T_2$, the above theorem includes the result of Ishikawa [19] as a corollary. They further claim that the Ishikawa iteration can be extended to Lipschitzian hemicontractive mappings.

In 1976 and 1983, Rhoades [38] and Naipally and Singh [33] studied the Ishikawa iteration scheme, respectively, and put forth the following questions :

Can the Ishikawa iteration procedure be extended to quasi-contractive and hemicontractive mappings ?

Liu Qihou [27,28] studied the above questions and proved the convergence theorem of the sequence of Ishikawa iterates for quasicontractive mappings and Lipschitzian hemicontractive mappings. After this, Liu Qihou [29] continue to studied the above questions and proved the following two theorems.

Theorem :

Let C be a convex compact subset of a Hilbert space and $T : C \to C$, a continuous hemicontractive mapping. Suppose that the number of the fixed points of T is finite. Then, for each $x_o \in C$, the sequence of Ishikawa iterates $\{x_n\}_{n=0}^{\infty}$ must converge to a fixed point of T.

Theorem :

Let C be a convex compact subset of a Hilbert space and $T : C \to C$, a continuous generalized contractive mapping. Then, for each $x_o \in C$, the sequence of Ishikawa iterates $\{x_n\}_{n=0}^{\infty}$ must converge to a fixed point of T.

Here we complete the brief survey of the development of the fixed point theory in Hilbert Space.