CHAPTER - 2

CHAPTER - II

Fixed point theorem & identity mapping

2.1 Let B denote a Banach space with the norm $||.||$ and C be a closed subset of B. The transformation $F : C \to C$ is called contraction if there exists a constant K with $0 < K < 1$ such that $||Fx-Fy|| \leq K \, ||x-y||$. If K =1 it is called non-expansive. Banach contraction principle states that a contraction mapping C into C has a unique fixed point. This conclusion is also true for $||Fx-Fy||^2 \leq K \, ||x-y||^2$ but it is no longer true for K = 1. However, Browder [5] has proved that every non-expansive mapping of a closed, bounded and convex subset of a uniformly, convex Banach space has at least one fixed point.

In 1971, Goebel and Zlotkiewicz [15] have proved the following theorem :

2.1.1 Theorem :

If C is a closed and convex subset of B and $F : C \to C$ satisfies

(i) $F^2 = I$, I is identity mapping

(ii) $||Fx-Fy|| \leq K \, ||x-y||$ where $0 \leq K < 2$,

then F has at least one fixed point.

In 1991, Sharma and Sahu [42] have obtained the most generalised theorem from the result of Goebel and Zlotkiewicz [15]

2.1.2 Theorem :

Let F be a mapping of a Banach space X into itself and satisfy

(i) $F^2 = I$, Where I : Identity mapping

(ii) $$||Fx-Fy|| \leq \frac{a||x-Fx||\ ||y-Fy||}{||x-Fy|| + ||y-Fy|| + ||x-y||} + \frac{b||x-Fx||\ ||y-Fy||}{||x-y||} + c\{||x-Fx|| + ||y-Fy||\} + d\{||x-Fy|| + ||y-Fx||\} + e\ ||x-y||$$

For every $x,y \in X$ and $x \neq y$, $a,b,c,d,e \geq 0$ $a + 4b + 4c + 4d + e < 2$, $2d + e < 1$. Then F has a unique Fixed Point.

2.1.3 Semi-generalised ν-contraction mapping [25]

Definition :

A mappint T from a closed subset C into C of a Hilbert Space H satisfying

$$||Tx-Ty||^2 \leq \alpha ||x-Tx||^2 + \beta ||y-Ty|| + \nu\ ||x-y||^2$$

is called semi-generalised ν -contraction with $0 < \alpha + \beta + \nu < 1$ and $\alpha\ \beta \cdot \nu > 0$

2.1.4 Parallelogram Law [26] P.130

Let H be a Hilbert space and $x, y \in H$ Then

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + ||y||^2 \qquad \text{...... (i)}$$

From (i) we have.,

$$||x+y||^2 \leqslant 2\left[||x||^2 + ||y||^2\right] \qquad \text{..... (ii)}$$

$$||x-y||^2 \leqslant 2\left[||x||^2 + ||y||^2\right]$$

Now we recall the definition of Identity mapping

2.1.5 A mapping $I : X \to X$ is called Identity mapping if $I(x) = x$ for all x in X

Using (2.1.4 and 2.1.5) and the equation(ii) we prove the following Theorem

2.1.6 Theorem :

Let F be a mapping of a Hilbert space H into itself satisfying

(i) $F^2 = I$; I is the identity mapping

(ii) $||Fx-Fy||^2 < \alpha||x-Fx||^2 + \beta||y-Fy||^2 + \nu||x-y||^2$

For all x,y in H where $\alpha, \beta, \nu \geqslant 0$ and $0 < \alpha + \beta + \nu < 1$

Proof : Let x be a fixed point of H.

Set $y = \frac{1}{2}(F + I)x, \quad z = F(y)$

and $u = 2y - z$

Then by using (i) and (ii) we get

$$||z-x||^2 = ||F(y)-x||^2 = ||F(y) - F^2(x)||^2$$

$$\leq \alpha\, ||F(x)-F^2(x)||^2 + \beta\, ||y-F(y)||^2 + \nu\, ||y-F(x)||^2$$

$$\leq \alpha\, ||F(x)-x||^2 + \beta\, ||y-Fy||^2 + \nu \Big|\, 2||y-F(y)||^2 + 2||F(y)-F(x)||^2$$

by 2.1.4

Next,

$$||F(y)-F(x)||^2 \leq 2||F(y)-x||^2 + 2||x-F(x)||^2$$

$$\leq 4||x-F(x)||^2 \quad \text{proved}$$

$$||F(y)-x|| \leq ||x-F(x)||$$

$$\therefore \quad ||z-x||^2 \leq (\alpha + 8\nu)||x-F(x)||^2 + (\beta + 2\nu)\, ||y-F(y)||^2$$

Now,

$$||u-x||^2 = ||2y-F(y)-x||^2$$

$$= ||(F+I)\,x-F(y)-x||^2$$

$$= ||F(y) - F(x)||^2$$

$$\leqslant \alpha \; ||x-F(x)||^2 + \beta \; ||y-F(y)||^2$$

$$+ \nu \; ||x-y||^2$$

$$< \alpha \; ||x-Fx||^2 + \beta \; ||y-Fy||^2 +$$

$$\nu \left[2||x-Fy||^2 + 2||y-Fy||^2 \right]$$

But

$$||x-Fy||^2 \leqslant 2||x-Fx||^2 + 2||Fx-Fy||^2$$

$$\leqslant 2||x-Fx||^2 + 8||x-Fx||^2$$

$$= 10||x-F(x)||^2$$

$\therefore$

$$||u-x||^2 \leqslant \alpha||x-Fx||^2 + \beta||y-Fy||^2$$

$$+ \nu\left[20||x-Fx||^2 + 2||y-Fy||^2\right]$$

$$\therefore \; ||u-x||^2 \leqslant (\alpha+20\nu) \; ||x-Fx||^2 + (\beta+2\nu) \; ||y-Fy||^2$$

Hence,

$$||z-u||^2 \leqslant 2||z-x||^2 + 2||u-x||^2$$

$$\leqslant 2(\alpha + 8\nu) \; ||x-Fx||^2 + 2 \; (\beta+2\nu)||y-Fy||^2$$

$$+ 2(\alpha+20\nu)||x-Fx||^2 + 2(\beta+2\nu)||y-Fy||^2$$

$$\therefore \; ||z-u||^2 \leqslant 4(\alpha+20\nu)||x-Fx||^2 + 4(\alpha+2\nu||y-Fy||^2$$

$$\therefore \; \nu < 1$$

We have,

$$||z-u||^2 = ||u-z||^2 = ||2y-z-z||^2$$
$$=4||y-z||^2$$
$$=4||y-F(y)||^2$$

Hence,

$$4||y-Fy||^2 = ||z-u||^2$$
$$\leq 4(+20\nu)||x-Fx||^2 + 4(\beta+2\nu)||y-Fy||^2$$

$$\therefore\ ||y-Fy||^2 \leq (\alpha+20\nu)||x-Fx||^2 + (\beta+2\nu)||y-Fy||^2$$

$$\left|1-(\beta+2\nu)\right| ||y-Fy||^2 \leq (\alpha+20\nu)\ ||x-Fx||^2$$

$$||y-Fy||^2\ \frac{\alpha+20\nu}{1-(\beta+2\nu)}||x-Fx||^2$$

Let $G = \frac{1}{2}(F+I)$, then for all $x \in H$

$$||G^2x-Gx||^2 = ||Gy-y||^2$$
$$= ||\tfrac{1}{2}(F+I)y-y||^2$$
$$=\tfrac{1}{4}||y-Fy||^2$$
$$\leq \frac{1}{4}\,\frac{\alpha+20\nu}{[1-(\beta+2\nu)]}\ ||x-Fx||^2$$
$$= \frac{1}{4}\,\frac{\alpha+20\nu}{1-(\beta+2\nu)}\ ||x-(2Gx-x)||^2$$
$$= \frac{\alpha+20\nu}{1-(\beta+2\nu)}\ ||Gx-x||^2$$

$$\therefore\ ||G^2x-Gx|| \leq ||Gx-x|| \qquad \because\ \frac{\alpha+20\nu}{1-(\beta+2\nu)} < 1$$

Therefore the sequence $\{x_n\}$ defined by $x_n = G^n(x)$ is a cauchy seeunce in H.

Since H is complete

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} G^n x = x_o$$

Consider,

$$||x_o - Gx_o||^2 = ||x_o - x_{n+1} + x_{n+1} - Gx_o||^2$$

$$\leq 2||x_o - x_{n+1}||^2 + 2||x_{n+1} - Gx_o||^2$$

$$= 2||x_o - x_{n+1}||^2 + 2||Gx_n - Gx_o||^2$$

$$= 2||x_o - x_{n+1}||^2 + 2||\tfrac{1}{2}(F+I)x_n - \tfrac{1}{2}(F+I)x_o||^2$$

$$= 2||x_o - x_{n+1}||^2 + \tfrac{1}{2}||(F+I)x_n - (F+I)x_o||^2$$

$$\leq 2||x_o - x_{n+1}||^2 + ||x_n - x_o||^2 + ||Fx_n - Fx_o||^2$$

$$\leq 2||x_o - x_{n+1}||^2 + ||x_n - x_o||^2 + \alpha\, ||x_n - Fx_n||^2$$

$$+ \beta\, ||x_o - Fx_o||^2 + \nu\, ||x_n - x_o||^2$$

$$= 2||x_o - x_{n+1}||^2 + (1+\nu)||x_n - x_o||^2 +$$

$$\alpha\, ||x_{n+1} - x_n|| + \beta\, ||Gx_o - x_o||^2$$

As $n \to \infty$, we get

$$||x_o - Gx_o||^2 \leq \beta\, ||Gx_o - X_o||^2$$

$\Longrightarrow$ $Gx_o = x_o$ $\qquad\qquad \because \beta < 1$

Now,

$$X_o = Gx_o$$

$$= \frac{1}{2}(F+I)\, x_o$$

$$2x_o = F(x_o) + Ix_o$$

$$2x_o = F(x_o) + x_o$$

$\Longrightarrow \quad F(x_o) = x_o$

$\Longrightarrow \quad x_o$ is a fixed point of F

<u>Uniqueness</u> :

Let if possible x_o and y_o are two distinct fixed points of F in H

Then,

$$||x_o - y_o||^2 = ||Fx_o - Fy_o||^2$$

$$\leq \alpha||x_o - Fx_o||^2 + \beta||y_o - Fy_o||^2 + \nu||x_o - y_o||^2$$

$$\therefore \; ||x_o - y_o||^2 \leq \nu\,||x_o - y_o||^2$$

$$\Longrightarrow (1-\nu)\;||x_o - y_o||^2 \leq 0$$

Since $\nu < 1$ and $||.|| \nless 0$

We have $||x_o - y_o||^2 = 0$

$$\Longrightarrow ||x_o - y_o|| = 0$$

$$\Longrightarrow x_o = y_o$$

Hence x_o is a unque fixed point of F.