ARTICLE TWO

Self-reciprocal Functions

The Hankel transformation of a function f(x) defined by

$$F(y) = h_{\mu}(f) = \int_{o}^{\infty} (xy)^{1/2} J_{\mu}(xy)\, f(x)\, dx \qquad (2.1)$$

where $o < y < \infty$, μ is any real numbers and $J_{\mu}(z)$ is the Bessel function of first kind of order μ. The simple generalization of the transform h_{μ} is given by [9, VIII] for $\lambda > o$,

$$F(y) = h_{\mu,\lambda}(f) = \lambda \int_{o}^{\infty} (xy)^{\lambda-1/2} J_{\mu}(x^{\lambda} y^{\lambda})\, f(x) dx. \qquad (2.2)$$

when $\lambda = 1$, (2.2) reduce to (2.1).

A function f is called a self-reciprocal function $R_{\mu,\lambda}$, if it is a solution of the integral equation $h_{\mu,\lambda}(f)=f$. Firstly, we have discussed the solution of an integral equation (2.2) and solution of another type of integral equations. Secondly, we have established a few results on self-reciprocal functions. The general solution of the equation

$$f(x) = \lambda \int_{o}^{\infty} (xy)^{\lambda-1/2} J_{\mu}(x^{\lambda} y^{\lambda})\, f(y)\, dy. \qquad (2.3)$$

is of the form

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2\lambda}\, \Gamma(\mu/2+s/2+1/2-1/4\lambda)\psi(s)\, x^{-s}\, ds, \qquad (2.4)$$

where $\psi(s) = \psi(1-s)$.

For, in view of Mellin transform [23, p.7] to (2.2), we have

$$M(s) = \lambda \int_0^\infty y^{\lambda-1/2} f(y)dy \int_0^\infty x^{s-1+\lambda-1/2} J_\mu(x^\lambda y^\lambda)\, dx$$

$$= \int_0^\infty y^{\lambda-1/2} f(y)\, dy \int_0^\infty (uy^{-\lambda})^{\frac{2s-1}{2\lambda}} J_\mu(u) y^{-\lambda}\, du$$

$$= \int_0^\infty y^{-s} f(y)\, dy \int_0^\infty u^{\frac{2s-1}{2\lambda}} J_\mu(u)\, du ,$$

on changing the order of integration and putting $u(x) = x^\lambda y^\lambda$.

Now using the known result [23, p. 182] we get

$$M(s) = 2^{\frac{2s-1}{2\lambda}} \frac{\Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)}{\Gamma(\mu/2-s/2\lambda+1/2+1/4\lambda)} \int_0^\infty f(y)y^{-s}\, dy$$

$$= 2^{\frac{2s-1}{2\lambda}} \frac{\Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)}{\Gamma(\mu/2-s/2\lambda+1/2+1/4\lambda)} M(1-s)$$

Putting $M(s) = 2^{s/2\lambda}\, \Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)\, \psi(s)$,
where $\psi(s) = \psi(1-s)$, we obtain (2.4).

$f(x)$ is of the same form as given in Titchmarsh [23, p.247] with μ replaced by ν and $\lambda = 1$. Thus, $f(x)$ given by (2.3) is a self-reciprocal function $R_{\mu,\lambda}$.

We have the following theorem, which is a generalization of the result given in Titchmarsh [23, p. 268].

Theorem - 2.1 : If f is a self-reciprocal function $R_{\mu,\lambda}$, $\lambda > 0$ and

$$K(x^{\lambda}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s/\lambda}\,\Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)\,\Gamma(\nu/2+s/2\lambda+1/2-1/4\lambda)\,\chi(s)\,x^{-s}\,ds \qquad \ldots \qquad (2.5)$$

Where $\chi(s) = \chi(1.s)$

then

$$g(x) = \int_{0}^{\infty} f(y)\,K(x^{\lambda}y^{\lambda})\,dy \qquad \ldots \qquad (2.6)$$

is a self-reciprocal function $R_{\nu,\lambda}$.

Proof : Since f is a self-reciprocal function $R_{\mu,\lambda}$, we have

$$f(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s/2\lambda}\,\Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)\psi(s)y^{-s}\,ds, \qquad (2.7)$$

where $\psi(s) = \psi(1-s)$. ... ... (2.8)

Let $K(s) = 2^{s/\lambda}\,\Gamma(\nu/2+s/2\lambda+1/2-1/4\lambda)\,\Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)\,\chi(s)x^{-s}$

and $F(s) = 2^{s/2\lambda}\,\Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)\,\psi(s)$.

By (2.5) and (2.7), K(s) and F(s) are the Mellin transforms of the functions $K(x^{\lambda}y^{\lambda})$ and f(y) respectively.

By Titchmarsh [23, p. 51; (2.1.12)], we have

$$g(x) = \int_0^\infty f(y)\, k(x^\lambda y^\lambda)\, dy = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)\, F(1-s)\, ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s/\lambda}\,\Gamma(\nu/2+s/2\lambda+1/2-1/4\lambda)\,\Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)\,\chi(s)\,x^{-s}\,\left(2^{1/2-s/2\lambda}\,\Gamma(\mu/2-s/2\lambda+1/2+1/4\lambda)\,\psi(1-s)\right) ds.$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s/2\lambda}\,\Gamma(\nu/2+s/2\lambda+1/2-1/4\lambda)\,\psi_1(s)\,x^{-s}\,ds,$$

where $\psi_1(s) = 2^{1/2\lambda}\,\Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)\,\Gamma(\mu/2-s/2\lambda+1/2+1/4\lambda)\,\psi(1-s)\,\chi(s).$

Since $\psi_1(s) = \psi_1(1-s)$.

Therefore, $g(x)$ is of the same form as (2.4) with μ replaced by ν. Thus, $g(x)$ is a self-reciprocal function $R_{\nu,\lambda}$.

In perticular if we put

$$\chi(s) = \frac{2^{\nu/2-\mu/2-1/2\lambda}}{\Gamma(\mu/2+s/2\lambda+1/2-1/4\lambda)\,\Gamma(\mu/2-s/2\lambda+1/2+1/4\lambda)}$$

in theorem 2.1, then we have

$$K(x^\lambda) = x^{\lambda(\nu/2-\mu/2+1)-1/2}\, J_{\frac{\nu+\mu}{2}}(x^\lambda).$$

Therefore, if we define the integral transform $h_{\nu,\mu,\lambda}$ by

$$h_{\nu,\mu,\lambda}(f(x)) = \int_0^\infty (xy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(x^\lambda y^\lambda) f(y)\, dy, \quad (2.9)$$

then by theorem 2.1, we have the following

Theorem - 2.2 : If $f(x)$ is a self-reciprocal function $R_{\mu,\lambda}$, $\lambda > 0$, then $h_{\nu,\mu,\lambda}(f)$ is a self-reciprocal function $R_{\nu,\lambda}$. That is, $h_{\mu,\lambda}(f) = f$ implies that $h_{\nu,\lambda}\,(h_{\nu,\mu,\lambda}(f)) = h_{\nu,\mu,\lambda}(f)$.

Proof : To prove the theorem, it is sufficient to show that the solution of the integral equation (2.9) is of the form of (2.7) with the condition (2.8). By applying Mellin transform to (2.9), we have

$$M(h_{\nu,\mu,\lambda}(f(x)),s) = M(s) = \lambda \int_0^\infty x^{s-1}\, dx \int_0^\infty \Big\{ f(y)\, (xy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(x^\lambda y^\lambda) \Big\}\, dy$$

Changing the order of integration and putting $u(x) = x^\lambda y^\lambda$, we obtain

$$M(s) = \int_0^\infty u^{(\nu/2-\mu/2+1)-1/2\lambda} J_{\frac{\nu+\mu}{2}}(u)\, du \int_0^\infty f(y)\, y^{-s}\, dy$$

Again using the known result [23, p.182] we get

$$M(s) = 2^{s/\lambda+\nu/2-\mu/2-1/2\lambda} \frac{\Gamma(\nu/2+s/2\lambda+1/2-1/4\lambda)}{\Gamma(\mu/2-s/2\lambda+1/2+1/4\lambda)} \int_0^\infty f(y)y^{-s}\,dy$$

$$= 2^{s/\lambda+\nu/2-\mu/2-1/2\lambda} \frac{\Gamma(\nu/2+s/2\lambda+1/2-1/4\lambda)}{\Gamma(\mu/2-s/2\lambda+1/2+1/4\lambda)} \quad M(1-s)$$

Putting $M(s) = 2^{s/2}\,\Gamma(\nu/2+s/2\lambda+1/2-1/4\lambda)\ \chi(s)$, we obtain a general solution of (2.9)

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2\lambda}\,\Gamma(\nu/2+s/2\lambda+1/2-1/4\lambda)\,\chi(s)\ x^{-s}\,ds,$$

where $\chi(s)$ is again satisfies the condition $\chi(s) = \chi(1-s)$.

$f(x)$ is of the same form as (2.4), with μ replaced by ν. Thus, $f(x)$ is a self-reciprocal function $R_{\nu,\lambda}$.

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