CHAPTER - III

STRUCTURE THEOREM FOR STIELTJES TRANSFORMABLE GENERALIZED FUNCTIONS

3.1 Introduction :

One of the most interesting and important problem in the theory of generalized functions is the problem of finding the representation of generalized functions expressing them in terms of differential operators acting on functions.

Gel'fand and Shilov [1], Koh [3], Pandey [5], Pathak [6], Prasad [7], Sonavane [8], Joshi [2] and Malgonde [4] have investigated the representation of different kinds of generalized functions. In the present chapter we obtain the structure theorem for Stieltjes transformable generalized functions.

The transform defined by the equation

$$S[j(\tau),\ \tau \to y\ ] = J(y) = \Gamma(\rho) y^{m\rho-1} \int_0^\infty \frac{j(\tau)\ d\tau}{(y^m + \tau^m)^\rho} \ ,(m,\rho > 0) \qquad \ldots (3.1.1)$$

(whenever the integral on the right-hand side converges for a complex y with $Re\, y > o$) is a simple generalization of the Stieltjes transform of a function $f(\tau) \in L\,(o, \infty)$, which is defined by the qequation

$$F(y) = \int_{o}^{\infty} \frac{f(\tau)\; d\tau}{(y+\tau)} \qquad \ldots (3.1.2)$$

Through out this chapter D(I) will denote the space of all smooth functions with compact supports on I : (o, ∞).

For a fixed real number a let $\lambda_a(\tau)$ be a smooth function defined on I : (o,∞) such that $\lambda_a(\tau) > o$ and

$$\lambda_a(\tau) = \begin{cases} \tau^a & \text{if } o < \tau < 1 \\ \tau^b & \text{if } 2 < \tau < \infty \end{cases} \qquad \ldots (3.1.3)$$

Let α be a fixed real number. Let $J_{a,m,\alpha}(I)$ denote the space of all smooth functions $\psi(\tau)$ on I : (o, ∞) such that

$$\gamma_k(\psi) \triangleq \gamma_{a,m,\alpha,k}(\psi) \triangleq$$

$$= \sup_{o<\tau<\infty} \lambda_a(\tau)(1+\tau^m)^{\alpha} \left| \tau^{mk}(\tau^{1-m} D_\tau)^k \psi(\tau) \right| < \infty \qquad \ldots (3.1.4)$$

for k = 0, 1, 2, ... and $D_\tau = \frac{d}{d\tau}$

We assign to $J_{a,m,\alpha}(I)$ the topology generated by the collection of seminorms $\left\{\gamma_k\right\}_{k=o}^{\infty}$. $J_{a,m,\alpha}(I)$ is sequentially complete, Housdorff, locally convex topological linear space. It is complete and therefore a Frechet space [11]. $J'_{a,m,\alpha}(I)$

denote the dual of $J_{a,m,\alpha}(I)$. $J'_{a,m,\alpha}(I)$ is also sequentially complete. The members of $J'_{a,m,\alpha}(I)$ are the generalized functions on which our Stieltjes transformation will be defined. A function $\Gamma(\varrho)y^{m\varrho-1}(y^m + \tau^m)^{-\varrho}$ belongs to $J_{a,m,\alpha}(I)$ for some real numbers a and α such that $\alpha \leqslant \varrho$ and $a > m(\varrho - \alpha)$. We shall call the generalized function j, a Stieltjes transformable if $j \in J'_{a,m,\alpha}(I)$. We define Stieltjes transform

$$S[j(\tau), \tau \to y] = J(y) = \left\langle j(\tau), \Gamma(\varrho)y^{m\varrho-1}(y^m + \tau^m)^{-\varrho} \right\rangle \quad \ldots (3.1.5)$$

of j as an application of j belonging to $J'_{a,m,\alpha}(I)$ to the function $\Gamma(\varrho)y^{m\varrho-1}(y^m + \tau^m)^{-\varrho}$, which is an element of $J_{a,m,\alpha}(I)$ for $\alpha \leqslant \varrho$ and $a > m(\varrho - \alpha)$.

<u>Lemma 3.1.1</u> : Let $J_{a,m,\alpha}(I)$ be the space of all smooth functions ψ defined on $I : (o,\infty)$ satisfying (3.1.4), then for $\psi \in D(I)$ and $j \in J'_{a,m,\alpha}(I)$ there exists a positive constant c and a non-negative integer q such that

$$\left|\langle j,\psi\rangle\right| \leqslant c.\max_{1\leqslant k\leqslant q+1} \int_o^\infty \left| \lambda_a(\tau)(1+\tau^m)^{\alpha}\, \tau^{k-1} D_\tau^k\, \psi(\tau)\right| d\tau \quad \ldots (3.1.6)$$

where $\lambda_a(\tau)$ is given by (3.1.3)

Proof : For each $j \in J'_{a,m,\alpha}(I)$ and in view of boundedness property of generalized functions, we have a constant c_1 and a non-negative integer q such that

$$\left|\langle j,\psi\rangle\right| \leqslant C_1 \max_{0\leqslant r\leqslant q} \gamma_r(\psi)$$

$$\leqslant C_1 \max_{0\leqslant r\leqslant q} \sup_{0<\tau<\infty} \left|\lambda_a(\tau)(1+\tau^m)^{\alpha}\tau^{mr}(\tau^{1-m}D_\tau)^r\psi(\tau)\right|$$

$$\leqslant C_1 \max_{0\leqslant r\leqslant q} \sup_{0<\tau<\infty} \left|\sum_{k=0}^{r} C_k\lambda_a(\tau)(1+\tau^m)^{\alpha}\tau^k D_\tau^k\psi(\tau)\right|$$

$$\therefore \left|\langle j,\psi\rangle\right| \leqslant C_2 \max_{0\leqslant k\leqslant q} \sup_{0<\tau<\infty} \left|\lambda_a(\tau)(1+\tau^m)^{\alpha}\tau^k D_\tau^k\psi(\tau)\right| \quad \ldots(3.1.7)$$

where C_k are integral constants depending on k, C_2 is a suitable constant and C_2 and q depend on j and not on ψ.

Further from elementary calculus, we have

$$\lambda_a(\tau)(1+\tau^m)^{\alpha}\tau^k D_\tau^k\psi(\tau) = \int_0^{\tau} D_t[\lambda_a(t)(1+t^m)^{\alpha}t^k D_t^k\psi(t)]dt. \quad \ldots(3.1.8)$$

Combining (3.1.7) and (3.1.8) we obtain

$$\left|\langle j,\psi\rangle\right| \leqslant C_2 \max_{0\leqslant k\leqslant q} \sup_{0<\tau<\infty} \left|\int_0^{\tau} D_t[\lambda_a(t)(1+t^m)^{\alpha}t^k D_t^k\psi(t)]dt\right|$$

$$\leqslant C_2 \cdot \max_{0 \leqslant k \leqslant q} \sup_{0 < \tau < \infty} \left| \int_0^{\tau} D_t[\lambda_a(t)(1+t^m)^{\alpha} t^k] D_t^k \psi(t)\,dt + \right.$$

$$\left. + \int_0^{\tau} \lambda_a(t)(1+t^m)^{\alpha}\, t^k D_t^{k+1} \psi(t)\,dt \right|$$

$$\leqslant C_2 \cdot \max_{0 \leqslant k \leqslant q} \sup_{0 < \tau < \infty} \left[\left| \int_0^{\tau} D_t[\lambda_a(t)(1+t^m)^{\alpha} t^k] D_t^k \psi(t)\,dt \right| \right.$$

$$\left. + \left| \int_0^{\tau} \lambda_a(t)(1+t^m)^{\alpha} t^k\, D_t^{k+1} \psi(t)\,dt \right| \right]$$

Now consider

$$\left| \int_0^{\tau} D_t[\lambda_a(t)(1+t^m)^{\alpha} t^k]\, D_t^k \psi(t)\,dt \right|$$

$$= \left| \lambda_a(\tau)(1+\tau^m)^{\alpha}\, \tau^k\, D_\tau^k \psi(\tau) - \int_0^{\tau} \lambda_a(t)(1+t^m)^{\alpha} t^k D_t^{k+1} \psi(t)\,dt \right|$$

$$\leqslant \left| \lambda_a(\tau)(1+\tau^m)^{\alpha}\, \tau^k\, D_\tau^k\, \psi(\tau) \right| +$$

$$+ \left| \int_0^{\tau} \lambda_a(t)(1+t^m)^{\alpha}\, t^k\, D_t^{k+1} \psi(t)\,dt \right|$$

Therefore,

$$\left| \langle J, \psi \rangle \right| \leqslant C_2 \max_{0 \leqslant k \leqslant q} \sup_{0 < \tau < \infty} \left[\left| \lambda_a(\tau)(1+\tau^m)^{\alpha}\, \tau^k\, D_\tau^k \psi(\tau) \right| \right.$$

$$+ \left| \int_0^{\tau} \lambda_a(t)(1+t^m)^{\alpha} t^k D_t^{k+1} \psi(t)\,dt \right| +$$

$$+\left|\int_0^{\tau} \lambda_a(t)\ (1+t^m)^{\alpha}\ t^k.\ D_t^{k+1}\psi(t)dt\right|\Bigg]$$

$$\leqslant C_2 \max_{0\leqslant k\leqslant q} \sup_{0<\tau<\infty}\Bigg[\left|\lambda_a(\tau)(1+\tau^m)^{\alpha}\tau^k D_\tau^k\psi(\tau)\right| + $$

$$+\ 2\left|\int_0^{\tau} \lambda_a(t)(1+t^m)^{\alpha}.t^k.D_t^{k+1}\psi(t)dt\right|\Bigg]$$

$$\leqslant C_2. \max_{0\leqslant k\leqslant q}\Bigg[\sup_{0<\tau<\infty}\left|\lambda_a(\tau)(1+\tau^m)^{\alpha}\ \tau^k D_\tau^k\ \psi(\tau)\right|$$

$$+\ 2\sup_{0<\tau<\infty}\int_0^{\tau}\left|\lambda_a(t)(1+t^m)^{\alpha}.t^k D_t^{k+1}\psi(t)\right|dt\Bigg]$$

But for $\psi \in D(I) \subset J_{a,m,\alpha}$, we have

$$\sup_{0<\tau<\infty}\left|\lambda_a(\tau)(1+\tau^m)^{\alpha}\ \tau^k D_\tau^k\psi(\tau)\right| < \infty\ (= N \text{ say})$$

$$\therefore \left|\langle J,\psi\rangle\right| \leqslant C_2. \max_{0\leqslant k\leqslant q}\left[N + 2\sup_{0<\tau<\infty}\int_0^{\tau}\left|\lambda_a(t)(1+t^m)^{\alpha}t^k D_t^{k+1}\psi(t)\right|dt\right]$$

$$\leqslant C_2. \max_{0\leqslant k\leqslant q}\left[N + 2\int_0^{\infty}\left|\lambda_a(\tau)(1+\tau^m)^{\alpha}\ \tau^k D_\tau^{k+1}\psi(\tau)\right|\ d\tau\right]$$

$$\leqslant C_2. \max_{0\leqslant k\leqslant q}\left[(P+2)\int_0^{\infty}\left|\lambda_a(\tau)(1+\tau^m)^{\alpha}\ \tau^k\ D_\tau^{k+1}\psi(\tau)\right|dt\right]$$

where $N \leq P \int_0^{\infty}\left|\lambda_a(\tau)(1+\tau^m)^{\alpha}\ \tau^k D_\tau^{k+1}\psi(\tau)\right|\ dt$.

$$\therefore \left|\langle J,\psi\rangle\right| \leqslant C. \max_{0\leqslant k\leqslant q}\int_0^{\infty}\left|\lambda_a(\tau)(1+\tau^m)^{\alpha}\ \tau^k\ D_\tau^{k+1}\psi(\tau)\right|d\tau$$

where $C = C_2.\ (P+2)$

$$\therefore \left|\langle j,\psi\rangle\right| \leqslant C. \max_{1\leqslant k\leqslant q+1} \int_0^\infty \left| \lambda_a(\tau)(1+\tau^m)^\alpha \tau^{k-1} D_\tau^k \psi(\tau) \right| d\tau$$

This completes the proof.

Now we can prove the main structure theorem.

<u>Theorem 3.1.1</u> : Let $j \in J'_{a,m,\alpha}(I)$ and $\psi \in D(I)$ then there exists $(q+1)$ L_∞ functions $g_k(\tau), 1 \leqslant k \leqslant q+1$, defined on $(\dot{L}_\infty(o,\infty))^{q+1}$ such that

$$\langle j, \psi\rangle = \sum_{k=1}^{q+1} \left\langle (-1)^k D_\tau^{k+1} \int_0^\tau \lambda_a(t)(1+t^m)^\alpha t^{k-1} g_k(t)dt, \psi\right\rangle$$

...(3.1.9)

<u>Proof</u> : On account of Lemma 3.1.1, we have

$$\left|\langle j,\psi\rangle\right| \leqslant C. \max_{1\leqslant k\leqslant q+1} \int_0^\infty \left| \lambda_a(\tau)(1+\tau^m)^\alpha \tau^{k-1} D_\tau^k \psi(\tau) \right| d\tau$$

$$\therefore \left|\langle j,\psi\rangle\right| \leqslant C. \max_{1\leqslant k\leqslant q+1} \left\| \lambda_a(\tau)(1+\tau^m)^\alpha \tau^{k-1} D_\tau^k \psi(\tau) \right\|_{L_1(o,\infty)}$$

... (3.1.10)

where $L_1(o,\infty)$ is a space of all equivalence classes of Lebesque integrable functions on (o,∞) whose topology is defined through the norm

$$\left\| \psi(\tau) \right\|_{L_1(o,\infty)} = \int_0^\infty \left| \psi(\tau) \right| d\tau < \infty, \; \psi \in L_1(o,\infty) \quad \text{... (3.1.11)}$$

we consider the product space

$$L_1(o,\infty) \times L_1(o,\infty) \times \overset{q+1}{\cdots\cdot} \times L_1(o,\infty) = (L_1(o,\infty))^{q+1}$$ and

the injection $H : \psi \dashrightarrow (\lambda_a(\tau)(1+\tau^m)^\alpha \tau^{k-1} D_\tau^k \psi(\tau)), 1 \le k \le q+1$ of $D(I)$ into $(L_1(o,\infty))^{q+1}$.

Estimate (3.1.10) can be read as saying that the linear functional $H(\psi) \to \langle j,\psi\rangle$ is continuous on $H(D(I))$ for the topology induced by $(L_1(o,\infty))^{q+1}$ [11, Lemma 1.10.1, p.26]. Therefore by the Hahn-Banach Theorem [9, p.184], it can be extended as a continuous linear functional in the whole of $(L_1(o,\infty))^{q+1}$. But the dual of $(L_1(o,\infty))^{q+1}$ is canonically isomorphic with $(L_\infty(o,\infty))^{q+1}$ [9, Theorem 20.3 and p.259]. Therefore there exists (q+1) L_∞ functions $g_k(\tau)$, $1 \le k \le q+1$ such that

$$\langle j, \psi\rangle = \sum_{k=1}^{q+1} \left\langle g_k(\tau), \; \lambda_a(\tau)(1+\tau^m)^\alpha \tau^{k-1} D_\tau^k \psi(\tau)\right\rangle$$

$$= \sum_{k=1}^{q+1} \left\langle D_\tau \int_0^\tau \lambda_a(t)(1+t^m)^\alpha t^{k-1} g_k(t)\,dt, \; D_\tau^k \psi(\tau)\right\rangle$$

Since $\psi \in D(I)$ so also $D_\tau^k \psi(\tau)$. Now we know that $\lambda_a(\tau)(1+\tau^m)^\alpha \tau^{k-1} g_k(\tau)$ is locally integrable function on I .

$$\therefore \quad D_{\tau} \int_{0}^{\tau} \lambda_a(t)(1+t^m)^{\alpha} . t^{k-1} g_k(t)dt =$$

$$= \lambda_a(\tau)(1+\tau^m)^{\alpha} . \tau^{k-1} g_k(\tau) \quad \text{belongs to}$$

$D'(I)$ [10, p.54]. Further this function is locally integrable on I. So it generates a regular distribution in $D'(I)$ i.e.

$D_{\tau} \int_{0}^{\tau} \lambda_a(t)(1+t^m)^{\alpha} t^{k-1} g_k(t)dt$ is a generalized function.

Therefore on applying generalized differentiation [10,p.47] k-times, we have

$$\langle j, \psi \rangle = \sum_{k=1}^{q+1} \left\langle (-1)^k D_{\tau}^{k+1} \int_{0}^{\tau} \lambda_a(t)(1+t^m)^{\alpha} . t^{k-1} g_k(t)dt, \psi \right\rangle$$

or

$$j = \sum_{k=1}^{q+1} (-1)^k D_{\tau}^{k+1} \int_{0}^{\tau} \lambda_a(t)(1+t^m)^{\alpha} t^{k-1} g_k(t)dt.$$

This completes the proof.

REFERENCES

[1] Gel'fand and Shilov : Generalized Functions, Academic Press, New York, London (1964).

[2] Joshi,V.G. : Some contribution to the Theory of Generalized Integral Transformations, Ph.D.Thesis, Shivaji University, (1983).

[3] Koh,E.L. : A Representation Theorem of Hankel Transformable Generalized Functions, SIAM Journal of Mathematical Analysis, Vol.1,No.1(Feb.1970) pp.33-36.

[4] Malgonde,S.P. : A study of some Integral Transforms and Their Applications, Ph.D.Thesis, Shivaji University, (1983).

[5] Pandey,J.N. : A Representation Theorem for a class of Convolution Transformable Generalized Functions. SIAM J.Math.anal. Vol.II,No.2 (1971),pp.286-289.

[6] Pathak,R.S. : A Representation Theorem for a Class of Stieltjes Transformable Generalized Functions, J.Indian Math.Soc.38 (1974) pp.339-344.

[7] Prasad,R.D. : Kontorvich-Lebdev-Transformation, Ph.D.Thesis, Jabalpur University (1974).

[8] Sonavane,K.R. : Some Properties of Generalized Functions and Generalized Integral Transforms, Ph.D.Thesis, Marathwada University (1975).

[9] Treves,F. : Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, London, (1967).

[10] Zemanian,A.H. : Distribution Theory and Transform Analysis, McGrow Hill Book Co. Inc.New York, (1965).

[11] ___________ : Generalized Integral Transformations, Interscience Publisher, New York (1968).

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