CHAPTER - I

STRAIN VARIATION EQUATION IN RELATIVISTIC CONTINUUM MECHANICS

1.1 Introduction

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1. INTRODUCTION

(i) Continuum Mechanics :

Mechanics is the behaviour of systems under the action of forces. When we ignore the molecular nature of the system and treat it as a continuum, we have continuum mechanics. In fact continuum is a collection of particles such that the study of individual particles is glossed over. We concentrate our attention on an element of the continuum. The element is small enough to be homogeneous i.e., all the particles in the element can be considered to have the same velocity, the same density, the same energy, the same temperature, the same pressure etc. Such an approach to continuum mechanics encompasses, the special media like, the fields of gravitation, electromagnetism besides the ordinary materials like water, air, and iron.

(ii) Relativistic Continuum Mechanics :

General Relativistic Continuum mechanics pertains to the fast motion of ponderable gravitating matter at high pressure. An explanation of the adjectives fast, ponderable and high is necessary. The velocity v of an element of the continuum is considered fast when it is comparable to c, the velocity of light i.e.,

$$v \approx c$$

The magnitude of c is huge and is approximately 1,86000 miles per second, or $c = (2.997924562 + 0.000000011) \times 10^{10}$ cm/sec.

The ponderamotive nature of material of the continuum is expressed through the order relation

$$M \approx \frac{c^2 R}{G}$$

where M is the mass of an element, R is the radius of the gravitating matter, G is the universal constant of gravitation with magnitude $(6.673 + 0.003) \times 10^{-8}$ $cm^3/g\ sec^2$.

The pressure p is considered high when the order relation

$$p \approx \varrho^* c^2$$

where c^* is the energy density of the gravitating matter, is valid.

These order relations are realisable in the case of supermassive objects like the neutron stars of relativistic astrophysics.

For the earth, we have

$$M_E \approx 6 \times 10^{-10} \frac{c^2 R_E}{G}$$

and for the sun, we have

$$M_{\odot} \approx 2 \times 10^{-6} \frac{c^2 R_{\odot}}{G}$$

where E, $\odot$ these suffix stand for earth and sun respectively.

It follows that these small orders of magnitude exclude the investigation of the motion of the Earth and the Sun from the scope of relativistic continuum mechanics.

We conclude that strong and rapidly varying gravitational fields are essential for testing the relevance of relativistic continuum mechanics. Since $\frac{GM}{c^2R} \simeq 1$ can be achieved only for supermassive objects or dense collapsing objects (Narlikar 1983), the testing of the predictions of general relativistic continuum mechanics cannot be accomplished in the terrestrial laboratory.

The aim of this chapter is to derive the strain variation equation, starting from the Ricci identity satisfied by the time like flow vector of the continuum and then exploiting the Einstein field equations of gravitating matter with the energy balance equation. Consequence of contracted Bianchi identity providing transition from Kinematics to dynamics. The definition of the strain tensor field in Relativistic Continuum mechanics is developed in Section 2. Different nomenclatures for the strain field, in vogue with research workers are cited. Some special strain tensor fields in Cosmology are to be found in Sec.3. Sec.4 contains the derivation of the Lie derivative of material tensor field as a computational aid for following sections. The kinematical form of the strain variation equation and the Dynamical form of the strain variation equation are described in Section 5 and 6 respectively. Newtonian and Special Relativistic approximation for the strain variation are cited in the last section. These equations are applied to disordered radiation field in Chapter III.

2. THE DEFINITION OF STRAIN TENSOR FIELD IN RELATIVITY

(i) The matter flow vector :

If g_{ab} is the metric tensor of the space-time manifold and T is the proper time we have the relation

$$ds^2 = c^2 dT^2$$

or

$$g_{ab} dx^a dx^b = c^2 dT^2$$

$$a, b = 1, 2, 3, 4$$

where x^a are the co-ordinates.

In the relation

$$g_{ab} \frac{dx^a}{dT} \cdot \frac{dx^b}{dT} = c^2 ,$$

we introduce the notation

$$u^a = \frac{dx^a}{dT}$$

and infer

$$g_{ab} u^a u^b = c^2$$

or equivalently

$$u^a u_a = c^2 \quad \ldots \quad (2.1)$$

Thus in the relativistic continuum mechanics the velocity u^a of the element of the continuum satisfies the condition (2.1). The vector u^a is the tangent vector to the world line of a particle (in the element) of the continuum. Hence u^a is just the matter flow vector.

Note: The explicit occurrence of c in (2.1) is the harbinger of the spirit of relativity. (one of the principles of Special Relativity is devoted to the paramount role of c). Several authors (e.g., Wald 1984) take $c = 1$ and so $ds = d\tau$ in which case the flow vector is $u^a = \frac{dx^a}{ds}$, and then (2.1) becomes $u^a u_a = 1$. In this case u^a becomes the unit time like vector field representing the 4-velocity of the (fluid) element of the continuum.

(ii) The orthogonal projection of the flow gradient tensor :

The orthogonal projection operator γ_{ab} for the signature (--- +) of the metric tensor g_{ab} is

$$\gamma_{ab} = g_{ab} - \frac{1}{c^2} u_a u_b \qquad \ldots \qquad (2.2)$$

which acts as a positive definite metric tensor on the tangent subspace orthogonal to the flow vector. We can also express (2.2) in the form

$$g^{\perp}_{a}{}^{\perp}_{b} = g_{ab} - \frac{1}{c^2} u_a u_b \; .$$

In order to obtain an expression for the projection of the flow gradient tensor $u_{a;b}$ on the tangent subspace, we proceed as follows:

$$u_{a;b} = \delta^c_a \delta^d_b u_{c;d}$$

$$= (\gamma^c_a + \frac{1}{c^2} u^c u_a)(\gamma^d_b + \frac{1}{c^2} u^d u_b) u_{c;d}$$

by (2.2)

M

$$= u_{\overset{\perp}{a};\overset{\perp}{b}} + \frac{1}{c^2}\,\dot{u}_a u_b$$

since $\gamma^c_a \gamma^d_b u_{c;d} = u_{\overset{\perp}{a}\,\overset{\perp}{b}}$,

$$\dot{u}_a u^a = 0\ ,$$

$$u_{c;b} u^c = 0\ .$$

Therefore we have

$$u_{\overset{\perp}{a}\,\overset{\perp}{b}} = u_{a;b} - \frac{1}{c^2}\dot{u}_a u_b\ . \qquad \ldots \qquad (2.3)$$

Def : Strain tensor field is defined as the symmetric part of $u_{\overset{\perp}{a}\,\overset{\perp}{b}}$. It is denoted by Θ_{ab} .

From (2.3) we have

$$u_{\overset{\perp}{b};\overset{\perp}{a}} = u_{b;a} - \frac{1}{c^2}\dot{u}_b u_a$$

and hence

$$u_{\overset{\perp}{a};\overset{\perp}{b}} + u_{\overset{\perp}{b};\overset{\perp}{a}} = u_{a;b} + u_{b;a} - \frac{1}{c^2}(\dot{u}_a u_b + \dot{u}_b u_a)$$

so

$$\Theta_{ab} = \frac{1}{2}\,(u_{\overset{\perp}{a};\overset{\perp}{b}} + u_{\overset{\perp}{b};\overset{\perp}{a}})$$

or

$$\boxed{\Theta_{ab} = \frac{1}{2}\left[\, u_{a;b} + u_{b;a} - \frac{1}{c^2}\,(\dot{u}_a u_b + \dot{u}_b u_d\,)\,\right]} \qquad (2.4)$$

This is the explicit expression of the strain tensor field in terms of the flow gradient.

If w_{ab} is the skewsymmetric part of $u_{\overset{\perp}{a};\overset{\perp}{b}}$, we have

$$u_{\overset{\perp}{a};\overset{\perp}{b}} = \theta_{ab} + w_{ab}$$

with

$$\theta_{[ab]} = 0 \quad w_{(ab)} = 0 .$$

The tensor w_{ab} is referred as the rotation tensor field. Accordingly we have the following expression

$$\boxed{w_{ab} = \frac{1}{2}\left[u_{a;b} - u_{b;a} - \frac{1}{c^2}(\dot{u}_a u_b - u_a \dot{u}_b) \right]} \qquad \ldots (2.5)$$

(iii) <u>The decomposition of the strain tensor</u> :

From (2.4) we get the trace of θ_{ab} as

$$g^{ab}\theta_{ab} = \frac{1}{2}(u_{a;}{}^{a} + u_{b;}{}^{b}) ,$$

since $\dot{u}^a u_a = 0$

$$= u_{a;}{}^{a} .$$

The expansion scalar is defined by

$$\theta = u^a_{;a} .$$

By adding and subtracting $\frac{1}{3}\theta g_{ab}$ (note the trace of $g_{\overset{\perp}{a}\overset{\perp}{b}}$ is 3), we rearrange (2.4) as follows

$$\theta_{ab} = \frac{1}{2}(u_{a;b} + u_{b;a} - \frac{1}{c^2}[\dot{u}_a u_b + \dot{u}_b u_a] - $$
$$- \frac{1}{3}\theta g_{\overset{\perp}{a}\overset{\perp}{b}} + \frac{1}{3}\theta g_{\overset{\perp}{a}\overset{\perp}{b}}$$

$$= \sigma_{ab} + \frac{1}{3}\theta g_{\overset{\perp}{a}\overset{\perp}{b}} , \quad \text{say} \qquad \ldots (2.5)$$

The tensor field σ_{ab} is known as the shear field. Equation (2.5) gives the decomposition of the strain tensor in terms of the expansion scalar and the shear tensor. Thus σ_{ab} is the trace free part of the strain, since $\sigma_a{}^a = 0$. The trace of strain is just the expansion.

Note: $g_{a}^{\perp}{}_{b}^{\perp}$ is termed as relative strain tensor (Carter and Quintana 1977).

(iv) Synonyms of the strain tensor :

Due to lack of unanimity among research workers on mathematical notation and terminology in relativistic continuum mechanics, there exists a lot of confusion. In order to ameliorate this difficulty we herein give the synonyms of strain tensor used by different authors in relativistic continuum mechanics and continuum mechanics.

Relativistic Continuum Mechanics

Sr.No.	Author/s	Synonyms	Notation
1.	Straumann (1984)	Expansion tensor $\pounds_u g_{a\perp b\perp}$	θ_{ab}
2.	Choquet-Bruhat et.al. (1982)	Strain tensor	$\pounds_u g_{a}^{\perp}{}_{b}^{\perp} = \theta_{ab}$
3.	Radhakrishna (1977)	Deformation	θ_{ab}
4.	Carter and Quintana (1977)	Relative strain rate	θ_{ab}
5.	Ehlers (1973)	Rate of deformation	θ_{ab}
6.	Carter and Quintana (1972)	Strain-rate tensor	θ_{ab}
7.	Sachs (1971)	Deformation	θ_{ab}

Continuum Mechanics

Sr.No.	Author	Synonyms	Notation
1.	Paria (1983)	Strain	e_{ab}
2.	Harris (1977)	Deformation or Deformation rate or Strain tensor	γ_{ab}
3.	Walter (1975)	Strain	e_{ab}
4.	Sommerfield (1967)	Strain-tensor	ϵ_{ab}
5.	Brillouin (1964)	Strain	e_{ab}
6.	Fredricson (1964)	Strain rate	e^{*}_{ab}
		General strain rate	E^{*}_{ab}
7.	Aris (1962)	Deformation of strain tensor	e'_{ab}
8.	Eringen (1962)	Deformation field	d_{ab}
9.	Prager (1961)	Rate of deformation	Q_{ab}
10.	Bird, Stewart and Light Foot (1960)	Rate of strain	$\frac{1}{2}\Delta$
11.	Sokolniff (1956)	Deformation	e_{ab}
12.	Rivlin (1955)	Deformation	$\frac{1}{2}A_1$
13.	Truesdell (1952)	Deformation field	$d_{ij}, ij=1,2,3$
14.	Oldroyd (1950)	Rate of strain	$e^{(1)}_{ab}$

3. SPECIAL STRAIN TENSOR FIELDS

We give below a brief historical account of special strain tensor fields studied by research workers.

i) Shear-Free Fluids : $(\theta_{ab} = \frac{1}{3}\theta\gamma_{ab})$

(a) : Especially the shear-free case has attracted many investigators, since the space times which are suitable for the description of collapsing stars or expanding cosmologies are characterized by

$$\theta \neq 0$$

$$\text{and} \quad \sigma_{ab} = w_{ab} = 0 .$$

The famous Friedman Robertson Walker (1922) spacetimes in cosmology are described through

$$p = p(\rho^{*}) ,$$
$$\theta_{ij} = \frac{1}{3}\theta\gamma_{ij} ,$$
$$w_{ij} = 0 ,$$
$$\dot{u}_i = 0 ,$$

besides being spatially homogeneous and spherically symmetric, in stationary axisymmetric spacetimes (including static axisymmetric and static cylindrical symmetric cases).

(b) : Wyman's (1946) solutions of Einstein Field equation are compatible with,

$$\theta_{ij} = \frac{1}{3}\theta\gamma_{ij} ,$$
$$w_{ij} = 0 ,$$
$$\theta\dot{u}^a = 0 .$$

(c) : It is well-known (Collins, 1985) that for Newtonian theory, shear free fluids with barotropic equation of state $p = p(\rho^*)$ are, either, (i) non-rotating, or (ii) stationary, or (iii) static. The pressure and density are independent in Schwarzschild (1916) interior solution. Therefore it does not have a barotropic relation of the type $p = p(\rho^*)$.

(d) : Barnes (1984) was concerned with fluids satisfying,

$$\theta_{ij} = \frac{1}{3}\theta\gamma_{ij} ,$$

$$w_{ij} = 0 ,$$

but with no barotropic equation of state.

(e) : Pressure-free (dust) models were considered by Ellis (1967) who has proved that

$$\sigma = 0 \Longrightarrow \theta w = 0$$

(ii) <u>Born-rigid fluids</u> : ($\theta_{ab} = 0$)

There do not exist rigid bodies in relativistic continuum mechanics since there is an upper limit (namely the velocity of light) for the velocity the propagation of any interaction (Landau and Lifschitz 1975). In the absence of the concept of rigidity in relativity, Pirani (1962) introduced the concept of Born-rigidity through

$$\mathcal{L}_u\gamma_{ab} = 0$$

or

$$\theta_{ab} = 0 .$$

Thus Born-rigid flow of a continuum is identical with shear-free and expansion free flow, since,

$$\theta_{ab} = 0 \Leftrightarrow \theta = 0, \ \sigma_{ab} = 0 .$$

(a) A strange solution was reported by Godel in 1949. His solution for pressure-free fluid admits closed time-like lines with

$$\theta_{ij} = 0 ,$$
$$\dot{u}_i = 0 ,$$
$$w_{ij} \neq 0 ,$$

but with covariantly constant vorticity

$$w_{i;j} = 0 .$$

(b) In a series of papers Kransinski (1914, 1975, 1978) described fluids with $\theta_{ij} = 0$, $w_{ij} \neq 0$ together with a killing vector parallel to the vorticity vector.

(c) <u>Inertial Reference Frame</u> : (IRF)

$(\theta_{ab} = 0, \ w_{ab} = 0, \ \dot{u}^a = 0)$

When the particles of a continuous medium move rectilinearly with constant velocity, we have

$$u_{a;b} = 0$$

i.e., $\theta = \sigma_{ab} = w_{ab} = \dot{u}_a = 0$.

This case has been named as an inertial reference frame by Audretsch (1971). Accordingly a cloud of test particles which moves rigidly and without rotation represents an

Inertial Reference Frame. The role of such frames in rheometrodynamics has been studied by Radhakrishna and Shah (1985).

4. LIE DERIVATIVE OF MATERIAL TENSORS

The strain tensor θ_{ab} is a covariant material tensor i.e., u-orthogonal tensor :

$$\theta_{ab}u^a = 0 \ , \quad \theta_{ab}u^b = 0$$

For material tensors, we prove the following lemma (on their Lie derivatives) as a computational aid in deriving the strain variation equation.

Lemma : If M_{ab} is a material tensor then

$$£_u M_{ab} = M_{\overset{\perp}{a}\overset{\perp}{b};k}u^k + M_{ak}u^k{}_{;\overset{\perp}{b}} + M_{kb}u^{\overset{\perp}{k}}{}_{;\overset{\perp}{a}} \qquad \ldots (4.1)$$

From the definition of Lie derivative of tensors (Trautmann 1964) we have,

$$£_u M_{ab} = M_{ab;k}u^k + M_{ak}u^k{}_{;b} + M_{kb}u^k{}_{;a} \ .$$

$$= M_{ab;k}u^k + M_{ak}(u^{\overset{\perp}{k}}{}_{;\overset{\perp}{b}} + \frac{1}{c^2}\dot{u}^k u_b) + M_{bk}(u^{\overset{\perp}{k}}{}_{;\overset{\perp}{a}} + \frac{1}{c^2}\dot{u}^k u_a)$$

by (2.3)

$$£_u M_{ab} = \delta^p_a \delta^q_b M_{pq;k}u^k + M_{ak}(u^{\overset{\perp}{k}}{}_{;\overset{\perp}{b}} + \frac{1}{c^2}\dot{u}^k u_b) + $$
$$+ M_{bk}(u^{\overset{\perp}{k}}{}_{;\overset{\perp}{a}} + \frac{1}{c^2}u^k u_a)$$

$$= (\gamma_a{}^p + \frac{1}{c^2}u^p u_a)(\gamma_b{}^q + \frac{1}{c^2}u^q u_b)M_{pq;k}u^k +$$

$$+ M_{ak}(u^{k\perp}{}_{;b}{}^{\perp} + \frac{1}{c^2}\dot{u}^k u_b) + M_{bk}(u^{k\perp}{}_{;a}{}^{\perp} + \frac{1}{c^2}u^k u_a)$$

by def γ^a_b

$$= M_{\perp a \perp b;k}u^k + \frac{1}{c^2}\gamma^p_a u^q u_b \dot{M}_{pq} + \frac{1}{c^2}\gamma^q_b u^p u_a \dot{M}_{pq} -$$

$$- \frac{1}{c^4}u_a u_b u^q \dot{u}^p M_{pq} + M_{ak}(u^{k\perp}{}_{;b}{}^{\perp} + \dot{u}^k u_b) +$$

$$+ M_{bk}(u^{k\perp}{}_{;a}{}^{\perp} + \frac{1}{c^2}u^k u_b)$$

on expansion.

$$\pounds_u M_{ab} = M_{\perp a \perp b}u^k - \frac{1}{c^2}\gamma_a{}^p \dot{u}_b u^q M_{pq} - \frac{1}{c^2}\gamma_b{}^q u_a M_{pq}\dot{u}^p +$$

$$+ M_{ak}(u^k{}_{;b} + \frac{1}{c^2}u^k u_b) + M_{bk}(u^k{}_{;a} + \frac{1}{c^2}u^k u_a) \ ,$$

since $(M_{pq}u^q)^{\cdot} = 0$ implies

$$\dot{M}_{pq}u^q = - M_{pq}\dot{u}^q$$

and

$$u_a u_b (u^q M_{pq})\dot{u}^p = 0 \ .$$

$$\pounds_u M_{ab} = M_{\perp a \perp b;k}u^k - \frac{1}{c^2}u_b \dot{u}^q M_{aq} - \frac{1}{c^2}u_a \dot{u}^p \gamma^q_b M_{pq} +$$

$$+ M_{ak}(u^{k\perp}{}_{;b}{}^{\perp} + \frac{1}{c^2}\dot{u}^k u_b) + M_{bk}(u^{k\perp}{}_{;a}{}^{\perp} + \frac{1}{c^2}u^k u_a)$$

on rearrangement,

since $\gamma^p_a M_{pq} = M_{aq}$,

$$\pounds_u M_{ab} = M_{\underset{a}{\perp}\,\overset{\perp}{b};k} u^k + M_{ak} u^{k^\perp}{}_{;b} + M_{bk} u^{k^\perp}{}_{;\overset{\perp}{a}} \quad . \qquad \ldots (4.2)$$

on cancelling u^k term.

5. KINEMATICAL FORM OF STRAIN VARIATION EQUATION

We establish the strain variation equation by exploiting the contracted Ricci identity for the flow vector u^a and the Lie derivative of material tensor. The strain variation equation expresses the variation of the strain tensor along the flow in terms of the curvature tensor (which is absent in the Newtonian mechanics) and acceleration field $\dot{u}^a$ and spatial projection of the flow gradient viz. $u_{\overset{\perp}{a};\overset{\perp}{b}}$.

Claim :

$$\pounds_u \Theta_{ab} = \gamma_a{}^c \gamma_b{}^d \dot{u}_{(c;d)} + u^{\overset{\perp}{c}}{}_{;\overset{\perp}{a}} u_{\overset{\perp}{c};\overset{\perp}{b}} - \frac{1}{c^2} \dot{u}_a \dot{u}_b - u^c u^d R_{acbd} \; .$$

Proof : By (4.1) of Lemma, we have,

$$\pounds_u \Theta_{ab} = \gamma^c_a \gamma^d_b \Theta_{cd;k} u^k + \Theta_{ak} v^k{}_{.b} + \Theta_{kb} v^k{}_{.a} \; ,$$

on putting $v^k{}_{.b} \equiv u^{\overset{\perp}{k}}{}_{;\overset{\perp}{b}}$

$$= \frac{1}{2} \gamma^c_a \gamma^d_b \left[u_{c;d} + u_{d;c} - \frac{1}{c^2} \dot{u}_c \dot{u}_d - \frac{1}{c^2} u_c \dot{u}_d \right]_{;k} u^k +$$

$$+ \Theta_{ak} v^k{}_{.a} + \Theta_{kb} v^k{}_{.a} \; .$$

(by definition of Θ_{ab}).

$$\pounds_u \Theta_{ab} = \frac{1}{2}\gamma^c_a\gamma^d_b \,[u_{c;d;k}u^k + u_{d;c;k}u^k - \frac{1}{c^2}\dot{u}_c\dot{u}_d - \ldots$$

$$- \frac{1}{c^2}(\dot{u}_c)_{;k}u^k u_d - \frac{1}{c^2}(\dot{u}_d)_{;k}u^k u_c \,] + \Theta_{ak}v^k_{.b} +$$

$$+ \Theta_{kb}v^k_{.b} \quad \text{on expansion.} \qquad \ldots (5.1)$$

Using Ricci identity,

$$u_{c;d;k}u^k = u_{c;k;d}u^k + u^k u^q R_{qcdk}$$

and

$$\gamma^c_a u_c = 0 \; ,$$

Eq. (5.1) reduce to

$$\pounds_u \Theta_{ab} = \frac{1}{2}\gamma^c_a\gamma^d_b \,[\, u_{c;k;d}u^k + u_{d;k;c}u^k + u^k u^q \,(R_{qcdk} +$$

$$+ R_{qdck}) - \frac{2}{c^2}\dot{u}_c\dot{u}_d \,] + \Theta_{ak}v^k_{.b} + \Theta_{kb}v^k_{.a}$$

$$\pounds_u \Theta_{ab} = \frac{1}{2}\,\gamma^c_a\gamma^d_b \,[\, (\, u_{c;k;d}u^k + u_{c;k}u^k_{;d} \,) + (u_{d;k;c}u^k +$$

$$+ u_{d;k}u^k_{;c} \,) - u_{c;k}u^k_{;d} - u_{d;k}u^k_{;c} + u^k u^q (R_{qcdk} +$$

$$+ R_{qdck} \,) - \frac{2}{c^2}\,\dot{u}_c\dot{u}_d \,] + \Theta_{ak}v^k_{.b} + \Theta_{kb}v^k_{.a} \qquad \ldots (5.2)$$

by adding and subtracting $u_{c;k}u^k_{\;;d}$,

$$\pounds_u\theta_{ab} = \frac{1}{2}\gamma^c_a\gamma^d_b[\dot{u}_{c;d} + \dot{u}_{d;c}] + \frac{1}{2}\gamma^c_a\gamma^d_b[-\frac{2}{c^2}\dot{u}_c\dot{u}_d -$$

$$- u_{c;k}u^k_{;d} - u_{d;k}u^k_{;c} + 2u^ku^qR_{qcdk}] +$$

$$+ \theta_{ak}V^k_{.b} + \theta_{kb}V^k_{.b} \quad .$$

since

$$R_{qcdk}u^ku^q = R_{qdck}u^ku^q \quad .$$

$$\pounds_u\theta_{ab} = \gamma^c_a\gamma^d_b\dot{u}_{(c;d)} + \frac{1}{2}(\delta^c_a - \frac{1}{c^2}u_au^c)(\delta^d_b - \frac{1}{c^2}u_bu^d) \times$$

$$\times [-\frac{2}{c^2}\dot{u}_c\dot{u}_d - u_{c;k}u^k_{;d} - u_{d;k}u^k_{;c} + 2u^ku^qR_{qcdk}] +$$

$$+ \theta_{ak}v^k_{.b} + \theta_{kb}v^k_{.a} \text{ , by def. of } \gamma^a_b$$

$$\pounds_u\theta_{ab} = \gamma^c_a\gamma^d_b\dot{u}_{(c;d)} - \frac{1}{c^2}\dot{u}_a\dot{u}_b + \frac{1}{2}[-(v_{ak} + \frac{1}{c^2}\dot{u}_au_k) \times$$

$$\times (v^k_{.b} + \frac{1}{c^2}\dot{u}_ku_b) - (v_{bk} + \frac{1}{c^2}\dot{u}_bu_k)(v^k_{.a} + \frac{1}{c^2}\dot{u}^ku_a) +$$

$$+ 2u^qu^kR_{qabk} + \frac{1}{c^2}u_{a;k}u_bu^k + \frac{1}{c^2}u_{b;k}u_au^k -$$

$$- \frac{2}{c^2}u^ku^q(R_{qadk}u_bu^d + R_{qcbk}u_au^c] + \theta_{ak}v^k_{.b} + \theta_{kb}v^k_{.a}$$

Since $u^k u^q u^d R_{qadk} = 0$ identically.

$$\pounds_u \Theta_{ab} = \gamma^c_a \gamma^d_b \dot{u}_{(c;d)} - \frac{1}{c^2}\dot{u}_a \dot{u}_b + \frac{1}{2}\Big[- v_{ak} v^k_{.b} - \frac{1}{c^2} v_{ak}\dot{u}^k u_b - v_{bk} v^k_{.a} - \frac{1}{c^2} v_{bk}\dot{u}^k u_a + 2\, u^k u^q R_{qabk} + \frac{1}{c^2} u_{a;k} u_b \dot{u}^k + \frac{1}{c^2} u_{b;k} u_a \dot{u}^k\Big] + \Theta_{ak} v^k_{.b} + \Theta_{kb} v^k_{.a}$$ on expansion,

we have

$$v_{ak}\dot{u}^k u_b = u_{a;k}\dot{u}^k u_b - \frac{1}{c^2}(\dot{u}_a u_k)\dot{u}^k u_b,$$

$$v_{ak}\dot{u}^k u_b = u_{a;k}\dot{u}^k u_b .$$

For $u_k u^k = 0$.

$$\pounds_u \Theta_{ab} = \gamma^c_a \gamma^d_b \dot{u}_{(c;d)} - \frac{1}{c^2}\dot{u}_a \dot{u}_b + \frac{1}{2}\Big[- (\Theta_{ak} + w_{ak})(\Theta^k_{.b} + w^k_{.b}) - (\Theta_{bk} + w_{bk})(\Theta^k_{.a} + w^k_{.a})\Big] + \Theta_{ak}(\Theta^k_{.b} + w^k_{.b}) + \Theta_{kb}(\Theta^k_{.a} + w^k_{.a}) + u^k u^q R_{qabk} .$$

$$\pounds_u \Theta_{ab} = \gamma^c_a \gamma^d_b \dot{u}_{(c;d)} - \frac{1}{c^2}\dot{u}_a \dot{u}_b + w_{ak} w^k_{.b} + \Theta_{ak} w^k_{.b} + \Theta_{kb}\Theta^k_{.a} + w^k_{.a}\Theta_{kb} + u^k u^q R_{qabk} .$$

$$\pounds_u \Theta_{ab} = \gamma^c_a \gamma^d_b \dot{u}_{(c;d)} - \frac{1}{c^2}\dot{u}_a \dot{u}_b + v^k_{.a} v_{kb} + u^k u^q R_{qabk}$$

$$\boxed{\pounds_u \Theta_{ab} = \gamma^c_a \gamma^d_b \dot{u}_{(c;d)} - \frac{1}{c^2}\dot{u}_a \dot{u}_b + v^k_{.a} v_{kb} - u^c u^d R_{acbd}} \qquad \ldots(5.3)$$

This is the Kinematical form of the strain variation equation since only Kinematical variables like u^a, $\dot{u}^a$, v^{ab} and the curvature tensor are involved.

6. DYNAMICAL FORM OF STRAIN VARIATION EQUATION

In order to obtain the dynamical form of strain variation, we introduce the condition that medium has an energy momentum tensor of the form

$$T^{ab} = p^{ab} + \rho^* u^a u^b \qquad \ldots (6.1)$$

where ρ^* is the (time-like eigen value) density and p^{ab} are components of the pressure tensor which is symmetric and orthogonal to the flow vector u^a

i.e.,
$$p^{ab} = p^{ba}$$
$$p^{ab} u_b = 0 \; .$$

The trace of T^{ab} is

$$T^a_a = T = p = \rho^* c^2 \qquad \ldots (6.2)$$

since $u^a u_a = c^2$, $p = p^a_a$.

The Ricci tensor for such a medium is given by Einstein's field equations :

$$R_{ab} = \frac{-8\pi G}{c^4} \left(T_{ab} - \frac{1}{2} T g_{ab} \right)$$

or equivalently

$$R_{ab} = -\frac{8\pi G}{c^4}\left\{ p_{ab} - \frac{1}{2}\, p g_{ab} + \rho^*(u_a u_b - \frac{c^2}{2} g_{ab})\right\} \tag{6.3}$$

The dynamical equation of motion viz.

$$T^{ab}{}_{;b} = 0$$

implies that

$$p^{ab}{}_{;b} + \dot{\rho}^* u^a + \rho^* \dot{u}^a + \rho^* u^a u^b{}_{;b} = 0$$

$$\text{i.e., } \rho^* \dot{u}^a = -p^{ab}{}_{;b} - u^a(\dot{\rho}^* + \rho^* u^b{}_{;b}) \,. \tag{6.4}$$

On contraction with u_a, $\therefore$

$$u_a p^{ab}{}_{;b} = -c^2(\dot{\rho}^* + \rho^* u^b{}_{;b}) \tag{6.5}$$

since $\dot{u}^a u_a = 0$.

Covariantly differentiating $p^{ab}u_a = 0$ with respect x^b we have

$$u_a p^{ab}{}_{;b} = -p^{ab}(u_{a;b})$$

$$\text{i.e., } u_a p^{ab}{}_{;b} = -p^{ab}[\,\theta_{ab} + w_{ab} + \frac{1}{c^2}\dot{u}_a u_b\,]$$

Since p^{ab} is symmetric and w_{ab} is skew symmetric

$$p^{ab} w_{ab} = 0 \,.$$

So

$$u_a p^{ab}{}_{;b} = -p^{ab}\theta_{ab} \quad \text{since } p^{ab}u_b = 0 \,. \tag{6.6}$$

From equation 6.5

$$\dot{\rho}^* + \rho^* u^b{}_{;b} = -\frac{1}{c^2} u_a p^{ab}{}_{;b}$$

From 1.6.6 and $\Theta = u^b{}_{;b}$

$$(\dot{\varrho}^*) + \varrho^*\Theta = \frac{1}{c^2} p^{ab}\Theta_{ab} \qquad \ldots (6.7)$$

Consequently (6.4) reduces to

$$\varrho^*\dot{u}^a = - p^{ab}{}_{;b} - \frac{1}{c^2} u^a p^{cb}\Theta_{cb}$$

$$\dot{u}^a = - \frac{1}{\varrho^*} [p^{ab}{}_{;b} + \frac{1}{c^2} u^a p^{cb}\Theta_{cb}] \qquad \ldots (6.8)$$

This equation expresses the relation between the kinematical quantity $\dot{u}^a$ and the dynamical quantity p^{ab} .

The Kinematical equation of strain tensor viz. (5.3) can be written by using the relation characterizing

$$R_{abcd} = C_{abcd} - \frac{1}{2} (g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{cb}R_{ad}) - $$
$$- \frac{R}{6} (g_{ad}g_{cb} - g_{ac}g_{bd}) \qquad \ldots (6.8')$$

as

$$\pounds_u\Theta_{ab} = \gamma^c_a\gamma^d_b\dot{u}_{(c.d)} + v^c{}_{.a}v_{cb} - \frac{1}{c^2} u_a\dot{u}_b - u^c u^d C_{acbd} + $$
$$+ \frac{1}{2}\gamma_{ab}[R_{cd}u^c u^d - \frac{R}{3} c^2] + \frac{1}{2}\gamma^c_a\gamma^d_b R_{cd}c^2 \qquad \ldots (6.9)$$

we eliminate $\dot{u}^c$ by (6.8) and the following derived relations,

$$(\dot{u}_a)_{;b} = \frac{1}{\varrho^{*2}} (\varrho^*{}_{;b}f_a) - \frac{1}{\varrho^*} f_{a;b} + \frac{1}{\varrho^{*2}c^2} \varrho^*{}_{;b}u_a p^{km}\Theta_{km} - $$
$$- \frac{1}{\varrho^* c^2} u_{a;b}p^{km}\Theta_{km} - \frac{1}{\varrho^* c^2}u_a p^{km}{}_{;b}\Theta_{km} - $$
$$- \frac{1}{\varrho^* c^2} u_a p^{km}\Theta_{km;b} \qquad \ldots (6.10)$$

where we have put

$$f_a = p^k_{a;k} \ .$$

$$(\dot{u}_a)_{;b} = \frac{1}{\varrho^{*2}} [\varrho^*_{;b} f_a - \varrho^* f_{a;b}] + \frac{1}{\varrho^{*2} c^2} [(\varrho^*_{;b} u_a - \varrho^* \theta_{ab} -$$

$$- \varrho^* u_a u_b) p^{km} \theta_{km} - \varrho^* u_a (p^{km} \theta_{km})_{;b}] -$$

$$- \frac{1}{\varrho^* c^2} w_{ab} p^{km} \theta_{km}, \quad \text{by decomposition of } u_{a;b}$$

we note that

$$\gamma^c_a \gamma^d_b \dot{u}_{(c;d)} = \gamma_{c(a} \gamma_{b)}{}^d \dot{u}^c_{;b} \ . \qquad \ldots(6.11)$$

We obtain the dynamical equation for strain tensor

$$\mathcal{L}_u \theta_{ab} = \frac{1}{\varrho^{*2}} \gamma_{c(a} \gamma_{b)}{}^d (\varrho^*_{;d} f^c - \varrho^* f^c_{;d}) - C_{ab} - \frac{4\pi G}{3} \varrho^* \gamma_{ab} +$$

$$+ w_{ca} w^c_{.b} + 2 w^c_{.(a} \theta_{b)c} + g^{cd} \theta_{ac} \theta_{bd} - \frac{1}{c^2 \varrho^{*2}} [\gamma_{ac} \gamma_{bd} f^c f^d +$$

$$+ \varrho^* p^{cd} \theta_{cd} \theta_{ab}] - \frac{4\pi G}{c^2} [p_{ab} - \frac{2}{3} p^c_c \gamma_{ab}] \qquad \ldots(6.12)$$

Remark : The sign in the last two terms in (1.6.12) differs from that of Carter and Quintana (1972) due to the fact that our signature of the metric (---+) differs from theirs.

7. NEWTONIAN AND SPECIAL RELATIVISTIC APPROXIMATION

O'neill (1983,p.333) has given the distinction between the General Relativistic continuum mechanics and the Newtonian mechanics in the following table :

	Gravitation	Speeds
General relativity	Arbitrary	Arbitrary
Special relativity	Negligible	Arbitrary
Newtonian physics	Weak	Low

DISCUSSION :

(i) To obtain the Newtonian approximation of the Strain variation Equation in dynamic form viz. (6.12) we deal with the case of low speeds (i.e. $v << c$ or $\frac{v}{c} \to 0$). The strain variation equation becomes

$$\pounds_u \Theta_{ab} = \frac{1}{\rho^{*2}} \gamma_{c(a}\gamma_{b)}{}^{d} \left(\rho^*_{;d} f^c - \rho^* f^c_{;d} \right) - C_{ab} - \frac{4\pi G \rho^*}{3} \gamma_{ab} + w_{ca} w^c{}_{.b} + 2w^c{}_{(a}\Theta_{b)c} + g^{cd}\Theta_{ac}\Theta_{bd}$$

(ii) The special relativistic approximation is obtained by putting

$$c_{ab} = 0 \quad G = 0 \quad \text{in} \quad (6.12)$$

$$\pounds_u \Theta_{ab} = \frac{1}{\rho^*} \gamma_{c(a}\gamma_{b)}{}^{d} \left(\rho^*_{,d} f^c - \rho^* f^c_{;d} \right) + w_{ca} w^c{}_{.b} + 2\, w^c{}_{.(a}\Theta_{b)c} + g^{cd}\Theta_{ac}\Theta_{bd} - \frac{1}{c^2 \rho^{*2}} \left\{ \gamma_{ac}\gamma_{bd} f^c f^d + \rho^* p^{cd} \Theta_{cd} \Theta_{ab} \right\}$$