C H A P T E R - III

EINSTEIN COLLINEATION
IN THE SPACE-TIME
OF
INTINITELY CONDUCTING FERROFLUID.

1. INTRODUCTION :

The famous Einstein field equations form the bridge between geometry and dynamics. The main aim of all investigations in gravitational physics is the construction of gravitational potentials satisfying Eienstein's field equations for a given distribution of matter. The solutions of these field equations are usually tried by imposing restrictions either on geometry (symmetries) or on dynamics. Apart from the symmetries of the space-time like spherical symmetry, plane symmetry, axial symmetry, cylindrical symmetry etc. Davis and his collaborators (1961 and 1974) found 16 types of symmetries for the gravitational field and obtained the corresponding weak conservation laws as the integral of geodesic equations. The first dynamical symmetry known as Maxwell colineation is introduced by Collinson (1970). In connection with his symmetry Khade and Radhakrishna (1974) proved that a non-trivial Maxwell collineation with respect to the propogation vector of a self gravitating null electromagnetic field implies the expansion free vector. It is shown by Woolley (1973) that the Lie derivative of F_{ab} can be expressed interms ofdual of F_{ab} in case of sourcefree, non-null electromagnetic field admitting killing vectors. This result is further generalised by Michaski and Weinwright (1975) and proved that

F_{ab} is necessarily invarient under a 2-parameter orthogonally transitive ~~abellion~~ abelian group of isometries. In 1983 N.I.Singh has investigated some exact solutions of Einstein-Maxwell Field equations obeying non-trivial Maxwell Collineations.for sourcefree non-null electromagnetic fields.

The second type of dynamical symmetry known by the term as the Einstein-Collineation is first coined by Khade and Radhakrisha (1974). They introduced the concept of Einstein-Collineations by the relations $L\, T_{ab} = 0$ and studied some preferred symmetries of null electromagnetic field with respect to the propogation vector and polorization vector. It is proved by Radhakrishna and Rao (1975) that the Einstein-Collineation of field collapsing with neutrino emission, ~~to~~ the material energy density together with neutrino flux are conserved . By assuming the existance of $L\, T_{ab} = L\, F_{ab} = 0$, Harrera and Casigi (1983) has studied the influence of geometrical symmetry $L\, g_{ab} = 0$ on the expression of $L\, F_{ab}$ through the Newman Penrose Formalism

Our aim in this section is to examine the properties of Einstein-Collineation compactiable with the space-time of infinitely conducting ferrofluid. The necessary conditions (kinematical and dynamical) for this space-time to admit Einstein-Collineation with respect to flow vector and magnetic field vector are found.

2. SPACE-TIME ADMITTING EINSTEIN COLLINEATION :

It follows from the defining expression of Einstein Collineation given by (I.5.13) with the Einstein Field Equation(I.4.1)

$$\underset{k}{L}\, T_{ab} = 0, \tag{2.1}$$

where k is any arbitrary vector field.

CLAIM 1 : The necessary condition for the space-time admitting Einstein Collineation to imply Ricci Collineation with respect to arbitrary vector field k is that the space-time should admit groups of motion with respect to the vector field k.

Proof : We have to prove

$$\underset{k}{L}\, T_{ab} = 0 \Rightarrow \underset{k}{L}\, R_{ab} = 0 \text{ if } \underset{k}{L}\, g_{ab} = 0.$$

We write from field equations (I.4.1)

$$\underset{k}{L}\, R_{ab} = \underset{k}{L}\, (T_{ab} - \tfrac{1}{2}\, T\, g_{ab}). \tag{2.2}$$

If $L\, g_{ab} = 0$, then $\underset{k}{L}\, T_{ab} = 0$ gives $\underset{k}{L}\, T_{ab}\, g^{ab} = 0$,

i.e.,

$$\underset{k}{L}\, T = 0. \tag{2.3}$$

Hence from (2.2) and (2.3) we get

$$\underset{k}{L}\, R_{ab} = 0. \tag{2.4}$$

This is the required result.

This result shows the equivalence between the Dynammical Symmetry ($\underset{k}{L} T_{ab}=0$) and Geometrical Symmetry ($\underset{k}{L} R_{ab}=0$) under the only restriction that the space-time should admit groups of motiom ($\underset{k}{L} g_{ab}=0$).

3. EINSTEIN COLLINEATION WITH RESPECT TO TIME LIKE VECTOR FIELD $\bar{U}$:

Theorem 1 : For the space-time of infinitely conducting Ferrofluid admitting Einstein Collineation with respect to the flow vector implies that the flow is expansion free and geodesic.

Proof : The Einstein Collineation with respeqct to U implies

$$\underset{u}{L} T_{ab} = 0 \, . \qquad (3.1)$$

The conservation law generator for this symmetry (R.R.Shaha,1976) is

$$(T^{ab}U_a)_{;b} = 0.$$

This for Infinitely Conducting Ferrofluid gives

$$[A_{;b}U^aU^b + A(\dot{U}^a+\theta U^a) - B_{;b}g^{ab} - \mu_{;b}h^ah^b - \mu h^a{}_{;b}h^b$$

$$-\mu h^ah^b{}_{;b}]\, U_a + (AU^aU^b-Bg^{ab}-\mu h^ah^b)\, U_{a;b} = 0,$$

i.e., $(A-B)_{;b}U^b + (A-B)\theta = 0$,

i.e., $$(r+\frac{\mu^2 h^2}{2})^{\cdot} + (r+\frac{\mu^2 h^2}{2})U^b{}_{;b} = 0 \,. \qquad (3.2)$$

The result (3.1) for Infinitely Conducting Ferrofluid gives

$$(AU_aU_b-B\,g_{ab}-\mu h_ah_b)_{;c}U^c + (AU_cU_b-B\,g_{cb}-\mu h_ch_b)U^c{}_{;a}$$

$$+(AU_aU_c-B\,g_{ac}-\mu h_ah_c)U^c{}_{;b} = 0.$$

Rearranging the terms we get

$$\dot{A}U_aU_b + AU_{a;c}U^cU_b - AU_aU_{b;c}U^c - B_{;c}U^c\,g_{ab} - \mu_{,c}U^ch_ah_b$$

$$-\mu h_{a;c}h_bU^c - \mu h_{a;c}h_bU^c - \mu h_ah_{b;c}U^c + AU_cU_bU^c{}_{;a} -$$

$$- B\,g_{cb}U^c{}_{;a} - \mu h_ch_bU^c{}_{;a} + AU_aU_cU^c{}_{;b} - B\,g_{ac}U^c{}_{;b} -$$

$$-\mu h_ah_cU^c{}_{;b} = 0 \,,$$

i.e., $$\dot{A}U_aU_b + A(\dot{U}_aU_b+\dot{U}_bU_a) - \dot{B}\,g_{ab} - B(U_{a;b}+U_{b;a}) - \dot{\mu}h_ah_b$$

$$-\mu h_bU^c(h_{a;c}-h_{c;a}) - \mu h_aU^c(h_{b;c}-h_{c;b}) = 0. \qquad (3.4)$$

Transvecting equation (3.4) with U^a we get

$$\dot{A}U_b + A\dot{U}_b - \dot{B}U_b - B\dot{U}_b = 0 \,,$$

i.e., $$[r+\frac{\mu^2 h^2}{2}]_{;c}U^cU_b + [r+\frac{\mu^2 h^2}{2}]\dot{U}_b = 0. \qquad (3.5)$$

This when transvected with U^b gives

$$[r+\frac{\mu^2 h^2}{2}]_{;a}U^a = 0. \qquad (3.6)$$

It follows from equation (3.6) and equation (3.2) that

$$U^a{}_{;a} = 0 \,. \qquad (3.7)$$

Also the equation (3.6) and (3.5) yields

$$U_{b;a}U^a = 0 \,. \qquad (3.8)$$

Thus the results (3.7) and (3.8) establish the theorem.

Theorem 2 : The necessary conditions for Infinitely Conducting Ferrofluid to admit Einstein Collineation with respect to the flow vector are

(i) $(\dot{h})^2=0$ (ii) $\dot{\mu}=0$ (iii) $\dot{p}=0$ (iv) $\dot{r}=0$.

Proof : The Einstein Collineation with respect to the flow vector for infinitely conducting Ferrofluid provides

$$\underset{u}{L}\, T_{ab} = 0 \,,$$

i.e.,

$$\dot{A}U_aU_b + A(\dot{U}_aU_b+\dot{U}_bU_a) - \dot{B}\, g_{ab} - B(U_{a;b}+U_{b;a}) - \dot{\mu}\, h_a h_b$$

$$-\mu h_b U^c(h_{a;c}-h_{c;a}) - \mu h_a U^c(h_{b;c}-h_{c;b}) = 0 \,. \qquad (3.9)$$

On contracting this with U^a we get

$$[r+\frac{\mu h^2}{2}]_{;c}U^cU_b + [r+\frac{\mu h^2}{2}]\dot{U}_b = 0 \,. \qquad (3.10)$$

Also by inner multiplication of (2.13) with h^b gives

$$[r+\frac{\mu h^2}{2}]\, \dot{U}_b h^b = 0 \,.$$

This implies that

$$\dot{U}_b h^b = 0 \quad \text{as} \quad r + \frac{\mu^2 h^2}{2} \neq 0 \ . \tag{3.11}$$

Further the operation $\underset{u}{L} T_{ab} h^a = 0$ gives

$$A\dot{U}_a h^a U_b - \dot{B}h_b - Bh^a(U_{a;b} + U_{b;a}) + \dot{\mu} h^2 h_b + \mu h^2 U^c (h_{b;c} - h_{c;b})$$

$$+ \frac{\mu}{2} h^2{}_{;c} U^c U_b + \mu h_{c;a} U^c h^a h_b = 0 \ . \tag{3.12}$$

This equation i.e. (3.12) contracted by h^b produces

$$\dot{B}h^2 - 2Bh^a h^b U_{a;b} - \dot{\mu} h^4 - \mu h^2 h^2{}_{;c} U^c - 2\mu h^2 h_{c;a} U^c h^a = 0 . \tag{3.13}$$

Now the equation $\underset{u}{L} T_{ab} g^{ab} = 0$, implies

$$-3\dot{B} - 2B\theta + \dot{\mu} h^2 + \mu h^2{}_{;c} U^c + 2\mu h_{c;a} h^a U^c = 0 \ . \tag{3.14}$$

By using equation (3.12) in (3.14) we obtain

$$-3\dot{B} + \dot{\mu} h^2 + \mu \dot{h}^2 + 2\mu h_{c;a} h^a U^c = 0 \ . \tag{3.15}$$

Also from equation (3.14) and equation (3.15) we get

$$-2\dot{B}h^2 - 2Bh^a h^b U_{a;b} = 0 \ ,$$

i.e.,

$$\dot{B}h^2 = Bh^a h^b U_{a;b} \ . \tag{3.16}$$

The equations (3.14),(3.15) and (3.16) produce

$$\dot{B} = 0 \ . \tag{3.17}$$

From (3.16) and (3.17) we write

$$h^a h^b U_{a;b} = 0 \ . \tag{3.18}$$

Hence it follows from equations (3.10),(3.12),(3.13) and Maxwell equations (II 2.2)

$$h^2{}_{;c}U^c = 0 \ , \tag{3.19}$$

$$\mu_{,c}U^c = 0 \ . \tag{3.20}$$

If we substitute all these results (3.19) and (3.20) in equation (3.17) we get

$$p_{;c}U^c = 0 \ . \tag{3.21}$$

Finally by making use of the results $\dot{\mu}=0$, $\theta=0$, $(\dot{h^2})=0$ in the continuity equation (II 3.2) we obtain

$$\dot{r} = 0 \ . \tag{3.22}$$

Thus we have succeeded in deriving the necessary conditions of the theorem given by

(i) $(\dot{h^2})=0$ (ii) $\dot{\mu}=0$ (iii) $\dot{p}=0$ (iv) $\dot{r}=0$.

REMARK : We note that theorem 1 provides the kinematical conditions and theorem 2 provides the dynamical conditions necessary for the space-time of Infinitely Conducting Ferrofluid to admit the Einstein Collineation with respect to the flow vector.

Theorem 3 : The flow of the Infinitely Conducting Ferrofluid obeying Einstein Collineation with respect to flow lines is adiabatic i.e., $\underset{u}{L} T_{ab} = 0 \quad \dot{s} = 0$.

Proof : Equation of Heat transfer in Infinitely Conducting Ferrofluid is given by

$$r_o T_o S_{;a} U^a = \dot{\mu}(1-\mu) h^2 . \qquad (3.23)$$

The necessary condition for Einstein Collineation i.e.,

$\underset{u}{L} T_{ab} = 0$ leads to

$$\dot{\mu} = 0 . \qquad (3.24)$$

Consequently

$$S_{;a} U^a = 0 . \qquad (3.25)$$

This shows that the flow is adiabatic.

Corollary 1 : If the stress energy tensor of the Infinitely Conducting Ferrofluid is left invarient under lie dragging with respect to flow lines then the isotropic pressure remains invarient along magnetic lines, if the magnitude of the magnetic field is conserved along the magnetic lines.

Proof : From the equation $(T^{ab}{}_{;b} h_a) = 0$, we find

$$(r+p)\dot{U}^a h_a - [p+\mu(1-\frac{\mu}{2})h^2]_{;a} h^a + \frac{\mu}{2} h^2{}_{;a} h^a = 0 . \qquad (3.26)$$

By employing the results of theorem 1 and theorem 2 in (3.26) we get

$$p_{;a} h^a = 0 \Leftrightarrow [\mu(1-\frac{\mu}{2})h^2]_{;a} h^a - \frac{\mu}{2} h^2{}_{;a} h^a = 0 . \qquad (3.27)$$

Corollary 2 : From Maxwell Equations we have

$$\mu_{;a}h^{a} = 0 \Longleftrightarrow h^{a}{}_{;a} = 0 \; . \qquad (3.28)$$

Corollary 3 :

$$\underset{U}{L}\, g_{ab} = 0 \Longleftrightarrow [h_{[a;c]}h_{b} + h_{[b;c]}h_{a}]U^{c} = 0 \; .$$

Proof : We have from Einstein Collineation with respect to U

$$\dot{A}U_{a}U_{b} + A(\dot{U}_{a}U_{b}+\dot{U}_{b}U_{a}) - \dot{B}\, g_{ab} - B(U_{a;b}+U_{b;a})$$

$$-\dot{\mu}h_{a}h_{b} - \mu h_{b}U^{c}(h_{a;c}-h_{c;a}) - \mu h_{a}U^{c}(h_{b;c}-h_{c;b}) = 0 \; ,$$

i.e.,

$$B(U_{a;b}+U_{b;a}) = U^{c}[(h_{a;c}-h_{c;a})h_{b} + (h_{b;c}-h_{c;b})h_{a}] = 0 \; . \qquad (3.29)$$

i.e.,

$$[h_{[a;c]}h_{b} + h_{[b;c]}h_{a}]\, U^{c} = 0.$$

From this we get

$$\underset{u}{L}\, g_{ab} = 0 \Longleftrightarrow [h_{[a;c]}h_{b} + h_{[b;c]}h_{a}]\, U^{c} = 0 \; ,$$

as $$U_{a;b}+U_{b;a} = \underset{u}{L}\, g_{ab} = 0 \; . \qquad (3.30)$$

REMARK 1 : We observe from theorem 2 that   is constant along the flow line . This shows that the Ferromagnetic nature of the flow is not maintained. Hence we conclude that the space-time of the Infinitely Conducting F Ferrofluid admitting Einstein Collineation with respect to the flow is incompatible.

REMARK 2 : From theorem 1 and theorem 2 we see that the conditions obtained are not sufficient for space-time to admit Einstein Collineation.

4. Now we consider the features of Einstein Collineation with respect to magnetic field vector $\bar{h}$.

The Einstein Collineation with respect to the magnetic field lines imply

$$\underset{h}{L}\, T_{ab} = 0 \;.$$

This condition for Infinitely Conducting Ferrofluid gives

$$\underset{h}{L}\, T_{ab} = T_{ab;c}h^{c} + T_{cb}h^{c}{}_{;a} + T_{ac}h^{c}{}_{;a} = 0, \qquad (4.1)$$

i.e.,
$$(AU_aU_b - B\, g_{ab} - \mu h_a h_b)_{;c}h^{c} + (AU_cU_b - B\, g_{cb} - \mu h_c h_b)h^{c}{}_{;a} + (AU_aU_c - B\, g_{ac} - \mu h_a h_c)h^{c}{}_{;b} = 0 \;,$$

i.e.,
$$A_{;c}h^{c}U_aU_b + Ah^{c}\left[U_b(U_{a;c} - U_{c;a}) + (U_{b;c} - U_{c;b})U_a\right] - B_{;c}h^{c}\, g_{ab} - B(h_{a;b} + h_{b;a}) - \mu_{;c}h^{c}h_ah_b - \mu h_{a;c}h_bh^{c} + \tfrac{1}{2}\mu h^{2}{}_{;a}h_b + \tfrac{\mu}{2}h^{2}{}_{;b}h_a - \mu h_ah_{b;c}h^{c} = 0. \qquad (4.2)$$

On contracting equation (4.2) with U^a we get

$$A_{;c}h^{c}U_b + Ah^{c}(-\dot{U}_cU_b + U_{b;c} - U_{c;b}) - B_{;c}h^{c}U_b + BU^{a}(h_{a;b} + h_{b;a}) - \mu h_{a;c}h_bh^{c}U^{a} + \tfrac{\mu}{2}h^{2}{}_{;a}U^{a}h_b = 0,$$

i.e.,
$$\left[r + \frac{\mu^{2}h^{2}}{2}\right]_{;c}h^{c}U_b + Ah^{c}(-\dot{U}_cU_b + U_{b;c} - U_{c;b}) -$$

$$-BU^a(h_{a;b}+h_{b;a}) - \mu h_{a;c}h_b h^c U^a + \frac{\mu}{2} h^2_{\ ;a}U^a h_b = 0. \quad (4.3)$$

By taking inner multiplication of equation (4.3) with U^b we obtain

$$\left[r+\frac{\mu^2 h^2}{2}\right]_{;c}h^c+2Ah^c(-U^aU_{c;a}-U_{c;b}U^b)-B(\dot{h}_aU^a+\dot{h}_bU^b) = 0,$$

i.e.,

$$\left[r+\frac{\mu^2 h^2}{2}\right]_{;c}h^c + 2\left(r+\frac{\mu^2 h^2}{2}\right)\dot{h}_c U^c = 0 . \quad (4.4)$$

Also by transvecting equation (4.3) with h^b we get,

$$Ah^c h^b(U_{b;c}-U_{c;b}) - B(U^a h^b h_{a;b}-\tfrac{1}{2}h^2_{\ ;a}U^a) + \mu h^2 h_{a;c}h^c U^a$$

$$-\frac{\mu}{2}h^2_{\ ;a}U^a h^2 = 0,$$

i.e.,

$$\left(p-\frac{\mu^2 h^2}{2}\right)h_{a;c}h^c U^a + \tfrac{1}{2}\left(p+2\mu h^2-\frac{\mu^2 h^2}{2}\right)h^2_{\ ;a}h^a = 0. \quad (4.5)$$

Further $\underset{h}{L}\, T_{ab}h^a = 0$, provides

$$Ah^c(U_{a;c}-U_{c;a})U_b h^a - B_{;c}h^c h_b - B(\tfrac{1}{2}h^2_{\ ;b}+h^a h_{b;a})$$

$$+\mu_{,c}h^c h_b h^2 + \frac{\mu}{2}h^2_{\ ;c}h_bU^c + \frac{\mu}{2}h^2_{\ ;a}h_b h^a - \frac{\mu}{2}h^2_{\ ;b}h^2$$

$$+\mu h^2 h_{b;c}h^c = 0,$$

i.e.,

$$\left(\frac{\mu^2 h^2}{2}-p\right)\left(h_{b;a}h^a-\frac{h^2_{\ ;b}}{2}\right) - B_{;c}h^c h_b + \mu_{;c}h^c h^2 h_b$$

$$+\frac{\mu}{2}h^2_{\ ;a}h^a h_b = 0 . \quad (4.6)$$

On contracting equation (4.6) with h^b we get

$$\left(p-\frac{\mu^2 h^2}{2}-\mu h^2\right)h^2_{\ ;b}h^b + B_{;c}h^c h^2 - \mu_{,c}h^c h^4 = 0 . \quad (4.7)$$

If we transvect equation (4.2) with g^{ab}, then we derive

$$(r-3p-3\mu h^2-\frac{\mu^2 h^2}{2})_{;c}h^c - 2A\dot{U}_c h^c - 2Bh^c{}_{;c}$$

$$+\mu_{,c}h^c h^2 + 2\mu h^2{}_{;c}h^c = 0 . \qquad (4.8)$$

<u>Theorem</u> : For Infinitely Conducting Ferrofluid if

$\underset{h}{L}\, T_{ab} = 0$ and $h^2{}_{;c}h^c = 0$, then

(i) $h^c{}_{;c}=0$ (ii) $\dot{U}_c h^c=0$ (iii) $\mu_{,c}h^c=0$ (iv) $\Upsilon_{;c}h^c=0$

(v) $p_{;c}h^c=0$.

<u>Proof</u> : We apply the condition that the magnitude of the magnetic field remains invariant under magnetic field, i.e.,

$$h^2{}_{;c}h^c = 0 . \qquad (4.9)$$

By using this condition in equation (4.7) we get

$$B_{;c}h^c = \mu_{,c}h^c h^2 , \qquad (4.10)$$

i.e.,

$$(p+\mu h^2-\frac{\mu^2 h^2}{2})_{;c}h^c = (\mu h^2)_{;c}h^c - \mu h^2{}_{;c}h^c ,$$

i.e.,

$$(p-\frac{\mu^2 h^2}{2})_{;c}h^c = 0 . \qquad (4.11)$$

We deduce from conservation of law generator for the associated collineation $(T^{ab}h_a)_{;b}=0$ that

$$(p-\frac{\mu^2 h^2}{2})_{;c}h^c + (p-\frac{\mu^2 h^2}{2})h^c{}_{;c} = 0 . \qquad (4.12)$$

Hence (4.10) demands

$$h^{c}{}_{;c} = 0 \,, \tag{4.13}$$

From the equation (3.8) and equation (4.13) we derive

$$\left(r - \frac{\mu^2 h^2}{2}\right)_{;c} h^c - 2B_{;c} h^c - 2A\dot{U}_c h^c = 0 \,. \tag{4.14}$$

From equation (II§3.2) and (3.9) under condition (4.14) we get

$$(r+p)\dot{U}_c h^c = B_{;c} h^c \tag{4.15}$$

So that equation (4.14) and equation (4.15) yields

$$\left(r + \frac{\mu^2 h^2}{2}\right)_{;c} h^c - 2(\mu^2 h^2)_{;c} h^c - 2(r+p)\dot{U}_c h^c - 2A\dot{U}_c h^c = 0 \,. \tag{4.16}$$

This equation with (4.4) produces

$$\dot{U}_c h^c = 0 \,. \tag{4.17}$$

Equation (4.10) and equation (4.15) gives

$$\mu_{,c} h^c = 0. \tag{4.18}$$

Finally by employing the results (4.9),(4.17) and(4.18) in equations (4.10) we deduce

$$p_{;c} h^c = 0 \,. \tag{4.19}$$

and $r_{;c} h^c = 0 \,.$ (4.20)

Thus the necessary condition for the theorem are derived

REMARKS :

(i) Magnetic lines are divergence free

vide (4.13)

(ii) Magnetic lines are orthogonal to acceleration.

vide (4.17)

(iii) Magnetic permeability is invariant along the magnetic field. vide (4.18)

(iv) Matter density is conserved along the magnetic lines. vide (4.20)

(v) Isotropic pressure is conserved along the magnetic lines. vide (4.19)

NOTE :

The conditions desired for Einstein Collineation with respect to the flow vector and the magnetic field vector are not sufficient.