# C H A P T E R - IV

## FREE GRAVITATIONAL FIELD OF INFINITELY CONDUCTING FERROFLUID

## 1. INTRODUCTION :

The physical interpretation of the Riemann Curvature tensor is given by Pirani (1956). The Riemann Curvature tensor emboids the effects of the gravitational field due to matter and free gravitational field (Jeorden et al.,1960). The interpretation of the Weyl conformal tensor as free gravitational field is due to Pirani and Shield (1961). The electric type and magnetic type components of the Weyl tensor are introduced and used to form Maxwell like equations by Hawking (1966). Kund and Trumper (1962) has investigated a property of radiative gravitational field through relativistic perfect fluid distribution. The interaction of the free gravitational field with the source is examined by Szekeres (1964). The formulation of Maxwell like equations by employing electric type and magnetic type components of the gravitationalfield is utilized by Glass (1975) and Date (1976). Asgeker (1979) to study relativistic magnetohydrodynamical aspects. Analagous to Weyl tensor the concept of Weyl projective tensor is introduced by R. R. shaha (1974) during the investigations of definite material scheme. The properties of Weyl conformal tensor are reexamined by Carmelli (1982) by using Newman Pensore formalism. It is proved by Barnes (1984) that if Weyl tensor is purely electric type or purely magnetic type then the flow vector is ~~irrational~~ irrotational unless the space-time has constant curvature.

The space-time in which the divergence of Weyl tensor vanishes is designated here as C-space. The kinematical and dynamical properties of infinitely conducting ferrofluid are examined in this section.

## 2. THE CONFORMAL CURVATURE TENSOR AND ITS PROPERTIES :

The conformal curvature tensor is defined through Riemann curvature tensor, Ricci tensor and Ricci scalar as given by Carmelli (1982).

$$C_{abcd} = R_{abcd} - \tfrac{1}{2}[g_{ac}R_{bd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac}] + \tfrac{1}{6}[g_{ad}g_{bc} - g_{ac}g_{bd}]R \ . \tag{2.1}$$

By employing Einstein Field Equations (I 4.1) we can also rewrite the conformal tensor in the form

$$C_{abcd} = R_{abcd} + T_{b[c}g_{d]a} + g_{b[c}T_{d]a} + \tfrac{2}{3}\, g_{b[d}g_{c]a} \ . \tag{2.2}$$

The Weyl conformal tensor has the same properties as the Riemann Curvature tensor, given by

$$C_{abcd} = -C_{bacd} = -C_{abdc} \ , \tag{2.3}$$

$$C_{abcd} = C_{cdab} \ , \tag{2.4}$$

$$C_{abcd} + C_{acdb} + C_{adbc} = 0 \ . \tag{2.5}$$

In addition, the Weyl tensor satisfies the important property that it is traceless,

i.e., $$C^{l}{}_{alb} = g^{lm}C_{malb} = 0 \ . \tag{2.6}$$

Thus it is irreducible.

The Weyl tensor can also be written in terms of the tracefree Ricci tensor (Carmelli, 1982)

$$S_{ab} = R_{ab} - \tfrac{1}{4}g_{ab}R, \qquad (2.7)$$

as in the form

$$R_{abcd} = C_{abcd} + \tfrac{1}{2}(g_{ac}S_{bd} - g_{ad}S_{bc} - g_{bc}S_{ad} + g_{ad}S_{bc}) + \tfrac{1}{2}(g_{ad}g_{bc} - g_{ac}g_{bd}). \qquad (2.8)$$

This is actually a statement of the fact that the Riemann Curvature tensor decomposes into its irreducible components like Weyl Conformal tensor Cabcd , the trace-free Ricci tensor $S_{ab}$ and Ricci Scalar curvature R. This decomposition can symbolically be written as

$$R_{abcd} = C_{abcd} \oplus S_{ab} \oplus R. \qquad (2.9)$$

No new quantities can be obtained from any of the above three irreducible components by contraction of their indices.

From the properties of Riemann curvature tensor and Weyl conformal tensor it follows that Riemann Curvature tensor has 20 independent components whereas the Weyl Conformal tensor has only ten independent components given by the nine independent components of the tracefree Ricci tensor $S_{ab}$ and the single components of the Ricci Scalar Curvature R.

The significance of the Weyl Conformal tensor can be seen from the following claim.

Claim : The Weyl Conformal Curvature tensor is preserved under conformal transformation.

Proof : Two Reimannian spaces V and $\bar{V}$ are called conformal spaces if their metric tensors $g_{ab}$ and $\underline{g}_{ab}$ are related by

$$\underline{g}_{ab}(x) = e^{2\sigma} g_{ab}(x), \tag{2.10}$$

where $\sigma$ is a real function of the co-ordinates. The correspondence between the spaces V and $\bar{V}$ is then called a conformal mapping.

The line elements of the two spaces V and $\bar{V}$ are related by

$$\underline{ds}^2 = e^{2\sigma} ds^2 ,$$

If one uses the same co-ordinates system in both spaces since the angle between two vector A and B is given by

$$\mathrm{Cos}(A,B) = \frac{A_a B^a}{(A_a A^a)(B_a B^a)} . \tag{2.11}$$

We see that the angles are preserved under a conformal mapping.

In the following we find the corresponding relations between the Riemann Curvature tensor and the Weyl Conformal tensor in the two spaces V and $\bar{V}$ .

From equation (2.10) we see that the contravariant metric forms in the two spaces V and $\overline{V}$ are related by

$$\underline{g}^{ab}(x) = e^{-2\sigma} g^{ab}(x) . \qquad (2.12)$$

Christoffel symbol of the first kind in the two spaces are related by

$$\underline{\Gamma}_{lmn} = e^{2\sigma} ( \Gamma_{lmn} + \Delta_{lmn} ) , \qquad (2.13)$$

where

$$\Delta_{lmn} = ( g_{lm} \frac{\partial \sigma}{\partial x^n} + g_{ln} \frac{\partial \sigma}{\partial x^m} - g_{mn} \frac{\partial \sigma}{\partial x^l} ). \qquad (2.14)$$

The Christoffel symbols of the second kind are related by

$$\underline{\Gamma}^a_{bc} = \underline{g}^{a\lambda} \underline{\Gamma}_{\lambda bc} = \Gamma^a_{bc} + \Delta^a_{bc} , \qquad (2.15)$$

where

$$\Delta^a_{bc} = g^{a\lambda} \Delta_{\lambda bc} ,$$

i.e.,

$$\Delta^a_{bc} = \delta^a_b \frac{\partial \sigma}{\partial x^c} + \delta^a_c \frac{\partial \sigma}{\partial x^b} - g_{bc} g^{a\lambda} \frac{\partial \sigma}{\partial x^\lambda} . \qquad (2.16)$$

We may now calculate the Riemann Curvature tensor $\underline{R}_{abcd}$ of the space $\overline{V}$.

It has the same expression as that of $R_{abcd}$ of the space V, except the metric tensor $\underline{g}_{ab}$ and the Christoffel symbols $\underline{\Gamma}^a_{bc}$ replacing $g_{ab}$ and $\overline{\Gamma}^a_{bc}$ in the equation

$$R_{abcd} = \frac{1}{2} ( \frac{\partial^2 g_{ad}}{\partial x^b \partial x^c} + \frac{\partial^2 g_{bc}}{\partial x^a \partial x^d} - \frac{\partial^2 g_{bd}}{\partial x^a \partial x^c} - \frac{\partial^2 g_{ac}}{\partial x^b \partial x^d} ) +$$

$$+ g_{lm}( \Gamma^{l}{}_{ad} \Gamma^{m}{}_{bc} - \Gamma^{l}{}_{ac} \Gamma^{m}{}_{bd} ) . \tag{2.17}$$

We find then

$$\underline{R}_{abcd} = e^{2\sigma} \Big[ R_{abcd} + (g_{ad}\sigma_{bc}+g_{bc}\sigma_{ad}-g_{ac}\sigma_{bd}-g_{bd}\sigma_{ac})$$

$$+ (g_{ad}g_{bc}-g_{ac}g_{bd}) ( \nabla_a\sigma \; \nabla^a\sigma ) \Big] . \tag{2.18}$$

In the above equation we have been using the notation according to which

$$\sigma_{ab} = \sigma_{ba} = \nabla_a \nabla_b\sigma - ( \nabla_a \sigma )( \nabla_b \sigma ), \tag{2.19}$$

$$\nabla_a \sigma \; \nabla^a \sigma = g^{ab} \nabla_a \sigma \; \nabla_b \sigma ,$$

i.e.,

$$\nabla_a \sigma \; \nabla^a \sigma = g^{ab} \frac{\partial \sigma}{\partial x^a} \frac{\partial \sigma}{\partial x^b} . \tag{2.20}$$

The Ricci tensor is consequently given by

$$\underline{R}_{ab} = \underline{g}^{lm} \underline{R}_{lamb} .$$

i.e.,

$$\underline{R}_{ab} = R_{ab}-2\sigma_{ab} - \Big[ \Box\sigma+2 \nabla_c \sigma \; \nabla^c \sigma \Big] g_{ab} , \tag{2.21}$$

where $$\Box\sigma = \nabla_c \nabla^c\sigma = g^{ab} \nabla_a \nabla_b \sigma . \tag{2.22}$$

The Ricci scalar curvature can now be calculated from the Ricci tensor as

$$\underline{R} = \underline{g}^{ab} \underline{R}_{ab} ,$$

i.e.,

$$\underline{R} = e^{2\sigma} \Big[ R - 6\Box\sigma - 6 \nabla_a \sigma \; \nabla^a \sigma \Big] . \tag{2.23}$$

From equations (2.21) to (2.23) we can eliminate the expression of $\sigma_{ab}$ and find

$$\sigma_{ab} = -\tfrac{1}{2}\,(\underline{R}_{ab} - R_{ab}) + \tfrac{1}{12}(\underline{R}\,\underline{g}_{ab} - R\,g_{ab})$$
$$+ \tfrac{1}{2}(\nabla_c\,\sigma\nabla^c\,\sigma)\,g_{ab}\,. \qquad (2.24)$$

Raising now the first index of the Riemannian Curvature tensor, given by (2.18) gives the following relation between the mixed components of curvature tensor.

$$\underline{R}^a{}_{bcd} = R^a{}_{bcd} + \delta^a_d\,\sigma_{bc} + \delta^a_c\,\sigma_{bd} +$$
$$+ g^{al}(g_{bc}\,\sigma_{ld} - g_{bd}\,\sigma_{lc}) +$$
$$+ (\delta^a_d\,g_{bc} - \delta^a_c\,g_{bd})\,(\nabla_a\sigma\nabla^a\sigma)\,. \qquad (2.25)$$

On substituting the expression for $\sigma_{ab}$ derived in (2.24) we finally obtain

$$\underline{R}^a{}_{bcd} - \tfrac{1}{2}(\delta^a_c\,\underline{R}_{bd} - \delta^a_d\,\underline{R}_{bc} - g_{bc}\underline{R}^a{}_d + g_{bd}\underline{R}^a{}_c)$$
$$- \tfrac{1}{6}(\delta^a_d\,\underline{g}_{bc} - \delta^a_c\,\underline{g}_{bd})\underline{R}$$
$$= R^a{}_{bcd} - \tfrac{1}{2}(\delta^a_c R_{bd} - \delta^a_d\,R_{bc} - g_{bc}R^a{}_d + g_{bd}R^a{}_c)$$
$$- \tfrac{1}{6}(\delta^a_d\,g_{bc} - \delta^a_c\,g_{bd})R. \qquad (2.26)$$

Equation (2.26) expresses the relation between the Riemann tensor, the Ricci tensor and the Ricci Scalar in the two spaces V and $\overline{V}$ .

Comparing now the equation (2.26) with the definition of Weyl conformal tensor $C^{a}{}_{bcd}$ obtained (2.2) by raising the first index l , we see that the left hand and right hand sides of the above equation are equal to

$\underline{C}^{a}{}_{bcd}$ and $C^{a}{}_{bcd}$ respectively.

In other words, we obtain

$$\underline{C}^{a}{}_{bcd} = C^{a}{}_{bcd} \quad . \tag{2.27}$$

Thus under the conformal mapping the Weyl conformal tensor is preserved.

## 3. <u>SOME ASPECTS OF FREE GRAVITATIONAL FIELD SIMILAR TO ELECTROMAGNETIC FIELD</u> :

(A) <u>Matter Current</u> : This concept of Matter Current is introduced by Szakers (1964) as similar to the electric current in electromagnetic field. He defined it as the divergence of the conformal curvature tensor,

i.e.,
$$J^{*}{}_{bcd} = 2C^{a}{}_{bcd\,;\,a} \; . \tag{3.1}$$

If we use the expression of $C^{a}{}_{bcd}$ given by (2.2) then we derive the expression for the matter current as follows. We start with the expression

$$C^{a}{}_{bcd} = R^{a}{}_{bcd} - g^{a}{}_{[d}R_{c]b} - g_{b[c}R^{a}{}_{d]} + \frac{R}{3}\, g^{a}{}_{[d}R_{c]b} \; , \tag{3.2}$$

i.e.,

$$C^a{}_{bcd} = R^a{}_{bcd} - \tfrac{1}{2}\left[g^a{}_d\, R_{cb} - g^a{}_c R_{db}\right] - \tfrac{1}{2}\left[g_{bc} R^a{}_d - g_{bd} R^a{}_c\right]$$
$$+ \frac{R}{6}\left[g^a{}_d g_{cb} - g^a{}_c g_{db}\right] \quad ,$$

This with Einstein Field Equations (I 4.1) produces

$$2C^a{}_{bcd} = 2\, R^a{}_{bcd} - \Big[g^a{}_d\,(\,T_{bc} - \tfrac{1}{2}T\, g_{bc}) -$$
$$-g^a{}_c\,(\,T_{bd} - \tfrac{1}{2}\,T\, g_{bd})\Big] + \tfrac{T}{3}(\,g^a{}_d\, g_{cb} - g^a{}_c\, g_{db})$$
$$- \Big[g_{bc}(\,T^a{}_d - \tfrac{1}{2}\,Tg^a{}_d) - g_{bd}(\,T^a{}_c - \tfrac{1}{2}T\, g^a{}_c)\Big] \ .$$

i.e.,

$$2C^a{}_{bcd} = 2\, R^a{}_{bcd} + g^a{}_d(\,T_{bc} - \tfrac{1}{2}T\, g_{bc}) - g^a{}_c(\,T_{bd} - \tfrac{1}{2}T\, g_{bd})$$
$$+ g_{bc}(\,T^a{}_d - \tfrac{1}{2}T\, g^a{}_d) - g_{bd}(T^a{}_c - \tfrac{1}{2}T\, g^a{}_c)$$
$$+ \tfrac{T}{3}(g^a{}_d g_{cb} - g^a{}_c\, g_{db}) \ .$$

From this we write

$$2C^a{}_{bcd;a} = 2\, R^a{}_{bcd;a} + g^a{}_d(T_{bc;a} - \tfrac{1}{2}T_{;a}\, g_{bc})$$
$$- g^a{}_c(T_{bd;a} - \tfrac{1}{2}T_{;a}\, g_{bd}) + g_{bc}(T^a{}_{d;a} - \tfrac{1}{2}T_{;a} g^a{}_d)$$
$$- g_{bd}(T^a{}_{c;a} - \tfrac{1}{2}T_{;a} g^a{}_c) + \tfrac{T}{3}{}_{;a}\,(g^a{}_d g_{cb} - g^a{}_c g_{db}) \ ,$$

i.e.,

$$2C^a{}_{bcd;a} = 2\, R^a{}_{bcd;a} + T_{bc;d} - \tfrac{1}{2}T_{;d}\, g_{bc} - T_{bd;c} + \tfrac{1}{2}T_{;c}\, g_{bd}$$
$$+ T^a{}_{d;a}\, g_{bc} - \tfrac{1}{2}T_{;d}\, g_{bc} - g_{bd}T^a{}_{c;a} + \tfrac{1}{2}T_{;c}\, g_{bd}$$
$$+ \frac{1}{3}\, g_{cb}\, T_{;d} - \frac{1}{3}\, T_{;c}\, g_{db} \ . \tag{3.3}$$

We simplify this by using the relation

$$R^{a}{}_{bcd;a} + R^{a}{}_{bda;c} + R^{a}{}_{bac;d} = 0 ,$$

i.e.,
$$R^{a}{}_{bcd;a} = -(R^{a}{}_{bda;c} + R^{a}{}_{bac;d}) ,$$

i.e.,
$$R^{a}{}_{bcd;a} = R_{bc;d} - R_{bd;c} . \qquad (3.4)$$

in the form

$$2C^{a}{}_{bcd;a} = 2\left[R_{bc;d} - R_{bd;c}\right] + T_{bc;d} - T_{bd;c} - \tfrac{1}{2}T_{;d}\, g_{bc} + \tfrac{1}{2}T_{;c}\, g_{bd} + g_{bc}\, T^{a}{}_{d;a} - g_{bd}\, T^{a}{}_{c;a} - \tfrac{1}{2}T_{;d}\, g_{bc} + \tfrac{1}{2}g_{bd}\, T_{;c} + \tfrac{1}{3}T_{;d}\, g_{cb} - \tfrac{1}{3}T_{;c}\, g_{db} . \qquad (3.5)$$

Again by using Field equations in I$^{st}$ term we get

$$2C^{a}{}_{bcd;a} = 2\left[(T_{bc} - \tfrac{1}{2}T\, g_{bc})_{;d} + (T_{bd} - \tfrac{1}{2}T\, g_{bd})_{;c}\right] + T_{bc;d} - T_{bd;c} - \tfrac{1}{2}T_{;d}\, g_{bc} + \tfrac{1}{2}T_{;c}\, g_{bd} + \tfrac{1}{2}g_{bd}T_{;c} + \tfrac{1}{3}T_{;d}\, g_{cb} - \tfrac{1}{3}T_{;c}\, g_{db} ,$$

i.e.,
$$2C^{a}{}_{bcd;a} = -2\, T_{bc;d} + T_{;d}\, g_{bc} + 2T_{bd;c} - T_{;c}\, g_{bd} + T_{bc;d} - T_{bd;c} - \tfrac{1}{2}T_{;d}\, g_{bc} + \tfrac{1}{2}T_{;c}\, g_{bd} + g_{bc}T^{a}{}_{d;a} - g_{bd}\, T^{a}{}_{c;a} - \tfrac{1}{2}T_{;d}\, g_{bc} + \tfrac{1}{2}\, g_{bd}\, T_{;c} + \tfrac{1}{3}\, g_{cb}\, T_{;d} - \tfrac{1}{3}T_{;c}\, g_{db} ,$$

This after simplification can be put as

$$2C^{a}{}_{bcd;a} = T_{bc;d} - T_{bd;c} - \tfrac{1}{3}T_{;d}\, g_{cb} + \tfrac{1}{3}T_{;c}\, g_{bd} .$$

Hence we write the expression for the matter current through stress energy tensor.

$$J^*_{bcd} = T_{bc;d} - T_{bd;c} - \frac{1}{3}T_{;d}\, g_{cb} + \frac{1}{3}T_{;c}\, g_{bd} \,. \tag{3.6}$$

This for infinitely conducting ferrofluid yields

$$\begin{aligned} J^*_{bcd} = {} & \left[A U_b U_c - B\, g_{bc} - \mu h_b h_c\right]_{;d} \\ & - \left[A U_b U_d - B\, g_{bd} - \mu h_b h_d\right]_{;c} \\ & - \frac{1}{3}\, g_{bc} \left[A-4B+\mu h^2\right]_{;d} + \frac{1}{3}\, g_{bd} \left[A-4B+\mu h^2\right]_{;c} \end{aligned}$$

i.e.,

$$\begin{aligned} J^*_{bcd} = {} & A_{;d}\, U_b U_c - A_{;c} U_b U_d + A U_{b;d} U_c \\ & - A U_{b;c} U_d + A U_b U_{c;d} - A U_b U_{d;c} \\ & + \mu_{;c} h_b h_d - \mu_{;d} h_b h_c + \mu h_{b;c} h_d - \mu h_{b;d} h_c + \\ & + \mu h_b h_{d;c} - \mu h_b h_{c;d} - \frac{1}{3}T_{;d}\, g_{bc} + \frac{1}{3}T_{;c}\, g_{bd} + \\ & + B_{;c}\, g_{bd} - B_{;d}\, g_{bc} \end{aligned} \tag{3.7}$$

where the value of the rest mass is

$$T = (r-3p-2\mu h^2+2\mu^2 h^2) \,. \tag{3.8}$$

<u>Claim</u> : The Matter Current is always conservative .

<u>Proof</u> : For the fluid with particles at zero rest mass (Radhakrishna 1973), we have T=0 .

This constraint with equation (3.6) yields

$$J^*_{bcd} = T_{bc;d} - T_{bd;c} \,. \tag{3.9}$$

This gives

$$J^{*b}{}_{cd;b} = T^{b}{}_{c;db} - T^{b}{}_{d;cb} \quad . \tag{3.10}$$

It follows from contracted Bianchi identities and Ricci identities that

$$T^{b}{}_{c;mn} - T^{b}{}_{c;nm} = T^{b}{}_{d}R^{d}{}_{cmn} - T^{d}{}_{c}R^{b}{}_{dmn} \quad . \tag{3.11}$$

Hence from equations (3.9)and (3.11) we get

$$J^{*b}{}_{cd;b} = T^{b}{}_{m}\,(R^{m}{}_{cdb} - R^{m}{}_{dcb}) + T^{m}{}_{d}R_{m}{}^{a}{}_{ca}$$
$$- T^{m}{}_{e}\,R_{m}{}^{a}{}_{da} \quad . \tag{3.12}$$

As the stress energy tensor is symmetric we have

$$J^{*b}{}_{cd;b} = T_{mc}R^{m}{}_{d} - T^{m}{}_{d}R_{mc} \quad ,$$

i.e., $$J^{*b}{}_{cd;b} = R_{mc}R^{m}{}_{d} - R_{md}R^{m}{}_{c} \quad ,$$

$$J^{*b}{}_{cd;b} = R_{md}R^{m}{}_{c} - R_{mc}R^{m}{}_{d} \quad .$$

Consequently,

$$J^{*b}{}_{cd;b} = 0 \quad .$$

Hence the proof is complete.

(B) <u>C - Space</u> : The concept of C-Space has been introduced by Szekers (1964) as the space in which the matter current vanishes.

Mathematically C-Space is characterized by

$$J^{*}{}_{bcd} = 0 \quad . \tag{3.13}$$

The well known Bianchi identities

$$R_{ab[cd;e]} = 0. \tag{3.14}$$

imply

$$J^*_{abc} = R_{c[a;b]} - \frac{1}{6} g_{c[a} R_{;b]}. \tag{3.15}$$

On using the Einstein Field Equations (I. 4.1) the above expression becomes

$$J^*_{abc} = - T_{c[a;b]} + \frac{1}{3} g_{c[a} T_{;b]} . \tag{3.16}$$

For the Infinitely Conducting Ferrofluid described by the stress energy tensor (I. 2.9), the value of the matter current is given by

$$\begin{aligned} J^*_{bcd} = {} & A_{;d} U_b U_c - A_{;c} U_b U_d + AU_{b;d} U_c - AU_{b;c} U_d \\ & + AU_b U_{c;d} - AU_b U_{d;c} + \mathcal{M}_{;c} h_b h_d - \mathcal{M}_{,d} h_b h_c \\ & + \mu h_{b;c} h_d - \mu h_{b;d} h_c + \mu h_b h_{d;c} - \mu h_b h_{c;d} \\ & - C_{;d} g_{bc} + C_{,c} g_{bd} + B_{;c} g_{bd} - B_{;d} g_{bc}, \end{aligned} \tag{3.17}$$

where

$$C = \frac{1}{3}(r - 3p - 2\mu h^2 + 2\mu^2 h^2) .$$

Hence the condition of C-Space for Infinitely Conducting Ferrofluid by using equation (3.13)and (3.17) gives,

$$\begin{aligned} & A_{;d} U_b U_c - A_{;c} U_b U_d + AU_{b;d} U_c - AU_{b;c} U_d + AU_b U_{c;d} \\ & - AU_b U_{d;c} + \mathcal{M}_{;c} h_b h_a - \mathcal{M}_{;d} h_b h_c + \mu h_{b;c} h_d - \mu h_{b;d} h_c \\ & + \mu h_b h_{d;c} - \mu h_b h_{c;d} - C_{;d} g_{bc} + C_{,c} g_{bd} + B_{;c} g_{bd} \\ & - B_{;d} g_{bc} = 0 . \end{aligned} \tag{3.18}$$

If we contract this equation with $U^b U^c h^d$ , $h^d g^{bc}$ we get

$$(A-3B-3C)_{;d}h^d - A\dot{U}_d h^d - \dot{\mu} h^2 h^c{}_{;c} + \frac{\mu h^2{}_{;d}h^d}{2} = 0,$$

i.e.,

$$(p+ \frac{\mu^2 h^2}{2})_{;d}h^d - A\dot{U}_d h^d - \mu h^2 h^c{}_{;c} + \frac{\mu h^2{}_{;d}h^d}{2} = 0 . \qquad (3.19)$$

and $$\frac{1}{3}(2r+3p+2\mu h^2+\frac{\mu^2 h^2}{2})_{;d}h^d - (r+p)\dot{U}_d h^d = 0 . \qquad (3.20)$$

Also we recall the equation

$$T^{ab}{}_{;b}\, h_a = 0 \text{ (vide, II 29)},$$

i.e., $$(r+p)\dot{U}^d h_d - \left[p+\mu(1-\frac{\mu}{2})h^2\right]_{;d}h^d + \frac{\mu h^2{}_{;d}h^d}{2} = 0. \qquad (3.21)$$

And the consequence of Maxwell equations (vide, II 7),

$$\mu_{;d}h^d+\mu(h^d{}_{;d}+\dot{U}^d h_d) = 0. \qquad (3.22)$$

<u>Theorem</u> : For the C-Space of Infinitely Conducting Ferrofluid the Magnitude of the magnetic field remains invariant along magnetic lines if and only if the magnetic permeability is invariant along these lines .

<u>Proof</u> : The substraction of equation (3.19) from (3.21) gives ,

$$(\mu h^2-\mu^2 h^2)_{;d}h^d +\mu h^2\dot{U}_d h^d -\mu h^2 h^c{}_{;c} -\mu h^2{}_{;d}h^d = 0.$$

This after simplification yields,

$$(\mu^2 h^2)_{;d}h^d = 0 . \qquad (3.23)$$

This then provides

$$\mu_{;d}h^{d}h^{2} + \frac{\mu}{2}h^{2}{}_{;d}h^{d} = 0 . \qquad (3.24)$$

This proves the required result.

Theorem 2 : For the C-Space time of Infinitely Conducting Ferrofluid the Isotropic pressure is preserved along the magnetic lines if and only if the ~~flow~~ four acceleration is normal to these lines.

Proof : By substracting the value given by (3.23) in (3.19) and then using the Maxwell equation (3.22) we obtain

$$p_{;d}h^{d} = (r+p)\,\dot{U}_{d}h^{d} . \qquad (3.25)$$

This result proves the theorem.

Theorem 3 : For the C-Space of Infinitely Conducting Ferrofluid, the matter energy density is conserved along magnetic lines if and only if the magnetic permeability is consreved along these lines.

Proof : By utilising the value given by (3.23), the equations (3.20) and (3.21) provides the equation

$$\frac{1}{3}(2r+3p+2\mu h^{2})_{;d}h^{d} - (p+\mu h^{2})_{;d}h^{d} + \frac{\mu h^{2}{}_{;d}h^{d}}{2} = 0,$$

i.e.,

$$(2r-\mu h^{2})_{;d}h^{d} + 3\frac{\mu}{2}h^{2}{}_{;d}h^{d} = 0 ,$$

i.e., $2r_{;d}h^d - \mu_{;d}h^d h^2 + \frac{\mu}{2}h^2{}_{;d}h^d = 0$ , (vide 3.23)

i.e., $$r_{;d}h^d = \mu_{;d}h^d h^2 \;. \tag{3.26}$$

From this equation the theorem follows.

NOTE : We observe from the above three theorems that the dynamical quantities like matter density r, isotropic pressure p and magnitude of the magnetic field $h^2$ all become conserved quantities along magnetic lines if and only if the magnetic permeability is conserved along these lines. This shows the direct effect of the conservation of the magnetic permaebility on the conservation of the above stated dynamical quantities.

(C) Electric type and Magnetic type components : ~~the~~

The electric type component $\underline{E}_{ab}$ and magnetic type component $H_{ab}$ are defined by Glass (1975) in the form

$$\underline{E}_{ac} = C_{abcd}\, U^b U^d \;, \tag{3.27}$$

$$H_{ab} = \tfrac{1}{2}\,(C_a{}^{pqr})(\eta_{qrbs}\, U_p U^s) \;. \tag{3.28}$$

We note from these definitions, the properties

$\underline{E}_{ab} = \underline{E}_{ba}$ , $H_{ab} = H_{ba}$ [Symmetry Property]

$\underline{E}^a{}_a = H^a{}_a = 0$ , [Trace free]

$H_{ab}U^a = \underline{E}_{ab}U^a = 0$ . [U-orthogonal property]

Hence we can express the Weyl tensor through these components as

$$C_{abcd} = (g_{abef}\, g_{cdgh} - \eta_{abef}\,\eta_{cdgh})U^e U^g E^{fh} -$$
$$- (g_{abef}\,\eta_{cdgh} + \eta_{abef}\, g_{cdgh})U^e U^g H^{fh} , \qquad (3.29)$$

where

$$g_{abcd} = g_{ac}\, g_{bd} - g_{ad}\, g_{bc} . \qquad (3.30)$$

Theorem : For the essentialy expanding flow, ~~of~~ the electric type component is given by E.N. Glass (1975),

$$E_{ab} = \tfrac{1}{2}\,(\perp R_{ab} - \tfrac{1}{3}\, P_{ab}\, P^{cd} R_{cd}) .$$

i.e.,

$$E_{cd} = \tfrac{1}{2}\,(P^a{}_c P^b{}_a R_{ab} - \tfrac{1}{3} P_{cd} P^{ab} R_{ab}) ,$$

i.e.,

$$E_{cd} = \tfrac{1}{2} R_{ab}\,[P^a{}_c P^b{}_d - \tfrac{1}{3} P_{cd} P^{ab}] . \qquad (3.31)$$

But we have

$$P^a{}_c P^b{}_d = (g^a{}_c - U^a U_c)(g^b{}_d - U^b U_d) ,$$

i.e.,

$$P^a{}_c P^b{}_d = g^a{}_c\, g^b{}_d + U^a U^b U_c U_d - g^b{}_d U^a U_c - g^a{}_c U^b U_d .$$

So that (3.31) reduces to

$$E_{cd} = \tfrac{1}{2} R_{ab}\,[g^a{}_c\, g^b{}_d + U^a U^b U_c U_d - g^a{}_c U^b U_d - g^b{}_d U^a U_c$$
$$- \tfrac{1}{3}\,(g_{cd} - U_c U_d)\,(g^{ab} - U^a U^b)],$$

i.e.,

$$E_{cd} = \tfrac{1}{2} R_{ab}\,[g^a{}_c\, g^b{}_d - \tfrac{2}{3}\, U^a U^b U_c U_d - g^a{}_c U^b U_d - \tfrac{1}{3}\, g_{cd} g^{ab}$$
$$+ \tfrac{1}{3}\, g_{cd} U^a U^b + \tfrac{1}{3}\, U_c U_d\, g^{ab}] , \qquad (3.32)$$

For Infinitely Conducting Ferrofluid we have

$$R_{ab} = A\, U_a U_b - \tfrac{1}{2}B\, g_{ab} - \mu h_a h_b \quad . \tag{3.33}$$

So that (3.32) produces

$$\begin{aligned} \underline{F}_{cd} = A\Big[ & U_c U_d + \tfrac{2}{3} U_c U_d - U_c U_d - U_c U_d - \tfrac{1}{3} g_{cd} \\ & + \tfrac{1}{3} g_{cd} + \tfrac{1}{3} U_c U_d\Big] - \tfrac{1}{2}B\Big[ g_{cd} + \tfrac{2}{3} U_c U_d - U_c U_d \\ & - U_c U_d - \tfrac{4}{3} g_{cd} + \tfrac{1}{3} g_{cd} + \tfrac{4}{3} U_c U_d\Big] . \end{aligned}$$

This after simplification implies that

$$\underline{F}_{cd} = -\frac{\mu}{2} h_c h_d \quad . \tag{3.34}$$

So that

$$\underline{F}_{cd} = 0 \Leftrightarrow h_a = 0 \quad .$$

Thus the claim is proved.

<u>REMARK</u> : It is proved by Asgekar and Date (1979) for Relativistic Magnetofluid that the gravitational tidal force is due to magnetic field only. Hence we note that the same result remains true even for relativistic Infinitely Conducting Ferrofluid.