# CHAPTER - II

## DIFFERENTIAL IDENTITIES PERTINENT TO INFINITELY CONDUCTING FERROFLUID

The well known Bianchi identities produce after contraction the local conservation laws

$$G^{ab}{}_{;b} = T^{ab}{}_{;b} = 0 \, . \qquad (1)$$

The solution of equation (1) do not form the general solutions of Einstein Field Equations . So that equations (1) are known as Weak Conservation Laws. From (1) we write by using the definition of covariant derivative,

$$T^{b}{}_{a,b} + T^{k}{}_{a} \Gamma^{b}_{kb} - T^{b}_{k} \Gamma^{k}_{ab} = 0 \, . \qquad (2)$$

This may be written as

$$t^{b}{}_{k,b} + \frac{1}{2} t_{ab} \, g^{ab}{}_{,k} = 0 \, , \qquad (3)$$

where we have

$$t_{ab} = (-g)^{\frac{1}{2}} \, T_{ab} \, . \qquad (4)$$

we know that

$$T^{b}{}_{a,b} = 0 \, , \qquad (5)$$

is the energy momentum conservation law in Special Relativity.

Here equation (3) can be interpreted by Einstein as the energy momentum law of matter in the presence of gravitational field. We observe that the equation (3) cannot be intergrated due to presence of the term $\frac{1}{2} T_{ab} \, g^{ab}{}_{,k}$ . This non-integrability is the main hurdle in understanding the conservation laws in General Relativity.

These conservation laws (1) are used to derive

(i) Equation of Continuity

(ii) Equations of Stream Lines

(iii) Geodesic Deviation Equations

(iv) Heat Generating Equations

pertaining to given infinitely conducting Ferrofluid described by the stress energy tensor (I.2.9).

We know that Bianchi identities yield via field equations ,

$$T^{ab}{}_{;b} = 0 \,.$$

This for infinitely conducting Ferrofluid produces

$$(\rho f + \mu h^2)_{;b} U^a U^b + (\rho f + \mu h^2)(U^a{}_{;b} U^b + U^a U^b{}_{;b})$$
$$- \left[p + \mu\left(1 - \frac{\mu}{2}\right)h^2\right]_{;b} g^{ab} - \mu_{;b} h^a h^b - \mu h^a{}_{;b} h^b$$
$$- \mu h^a h^b{}_{;b} = 0,$$

i.e.,

$$(\rho f + \mu h^2)_{;b} U^a U^b + (\rho f + \mu h^2)(\dot{U}^a + \theta U^a)$$
$$- \left[p + \mu\left(1 - \frac{\mu}{2}\right)h^2\right]_{;b} g^{ab} - \mu_{;b} h^a h^b - \mu h^a{}_{;b} h^b$$
$$- \mu h^a h^b{}_{;b} = 0 \,. \qquad (6)$$

ELECTROMAGNETIC FIELD EQUATIONS :

(i) The Maxwell equations (I.4.7) generate the result when contracted by $U_a$

$$\mu_{,b}h^b - \mu_{,b}U^b h^a U_a + \mu[U^a{}_{;b}h^b U_a + h^b{}_{;b} - U^b{}_{;b}U_a h^a - U^b h^a{}_{;b}U_a] = 0 .$$

By using (I.2.3) and (I.2.4) in this equation we get

$$\mu_{,b}h^b + \mu(h^b{}_{;b} - h^a{}_{;b}U_a U^b) = 0 . \qquad (7)$$

(ii) If we transvect the Maxwell equations (I.4.7) with $h_a$, then we get an interesting result

$$h^2\mu_{,b}U^b + \mu[U^a{}_{;b}h_a h^b + h^2 U^b{}_{;b} + \tfrac{1}{2}h^2{}_{;b}U^b] = 0 . \qquad (8)$$

By transvecting the energy balance equation (6) with $U_a$ we obtain

$$(\rho f+\mu h^2)_{;b}U^a U^b U_a + (\rho f+\mu h^2)(\dot{U}^a U_a + \theta U^a U_a)$$
$$-[p+\mu(1-\tfrac{\mu}{2})h^2]_{;b}\, g^{ab}U_a - \mu_{;b}h^a h^b U_a - \mu h^a{}_{;b}h^b U_a$$
$$-\mu h^a h^b{}_{;b}U_a = 0 . \qquad (9)$$

The unitary character $U^a U_a = 1$ implies $\dot{U}^a U_a = 0$ so that the above equation reduces to

$$(\rho f+\mu h^2)_{;b}U^b + (\rho f+\mu h^2)\theta - [p+\mu(1-\tfrac{\mu}{2})h^2]_{;b}U^b$$
$$-\mu h^a{}_{;b}h^b U_a = 0 . \qquad (10)$$

By substituting the value $\varrho f = r+p$ , the equation (10) gives

$$[r+p+\mu h^2-p-\mu(1-\frac{\mu}{2})h^2]_{;b}U^b + (r+p)\theta + \mu h^2 h^b{}_{;b}$$

$$+\mu U_{a;b}h^a h^b = 0 . \qquad (11)$$

If we make use of Maxwell equation (8) in the above equation we get

$$[r+\frac{\mu^2 h^2}{2}]_{;b}U^b + (r+p)\theta - \frac{\mu}{2}h^2{}_{;b}U^b - \mu h^2 U^b{}_{;b} = 0,$$

i.e. ,

$$[r+\frac{\mu^2 h^2}{2} - \frac{\mu h^2}{2}]_{;b}U^b + (r+p)\theta - \frac{\mu}{2}h^2{}_{;b}U^b = 0 . \qquad (12)$$

This the equation of Continuity for Infinitely Conducting Ferrofluid.

REMARKS::

(a) If the magnetic permeability is constant then this result (12) reduces to

$$\dot{r} + (r+p)\theta + \frac{\mu}{2}h^2{}_{;b}U^b = 0 . \qquad (13)$$

This result is due to Maugin (1972).

(b) If we put $\mu$ ~~is~~ equal to Unity then the result matches with the continuity equation derived by Lichnerowicz (1967)

(c) In the absence of magnetic field the equation (12) gives the equation of continuity for perfect fluid obtained by Ellis (1972).

NOTE:

In case of Relativistic Magnetofluid considered by Lichnerowicz the continuity equation does not involve magnetic field in the explicit form. But for infinitely conducting ferrofluid we observe the clear effects of magnetic field in the equation of continuity.

When we substitute the equation of continuity in the equation (6) we get

$$(\rho f+\mu h^2)_{;b}U^aU^b + (\rho f+\mu h^2)\dot{U}^a + \theta\rho f U^a + \mu h^2\theta U^a$$
$$- \left[p+\mu(1-\tfrac{\mu}{2})h^2\right]_{;b}g^{ab} - \mu_{;b}h^ah^b - \mu_{;b}h^ah^b - \mu h^a{}_{;b}h^b$$
$$-\mu h^a h^b{}_{;b} = 0,$$

i.e. ,

$$(\rho f+\mu h^2)_{;b}U^aU^b + (\rho f+\mu h^2)\dot{U}^a - \left[r-\tfrac{\mu}{2}(1-\mu)h^2\right]_{;b}U^aU^b$$
$$+\tfrac{1}{2}\mu_{,b}U^bU^ah^2 - \mu h_c U^c{}_{;b}h^bU^a - \tfrac{\mu}{2}h^2{}_{;b}U^aU^b - h^2\mu_{,b}U^aU^b$$
$$-\left[p+\mu(1-\tfrac{\mu}{2})h^2\right]_{;b}g^{ab} - \mu_{,b}h^ah^b - \mu h^a{}_{;b}h^b - \mu h^ah^b{}_{;b} = 0 ,$$

i.e.,

$$\left[r+p+\mu h^2 - r + \tfrac{\mu}{2}(1-\mu)h^2 - \tfrac{\mu h^2}{2}\right]_{;b}U^aU^b + (\rho f+\mu h^2)\dot{U}^a$$
$$-\mu h_{b;c}h^bh^cU^a - \left[p+\mu(1-\tfrac{\mu}{2})h^2\right]_{;b}g^{ab} - \mu(h^ah^b)_{;b} = 0 ,$$

i.e.,

$$\left[p+\mu(1-\tfrac{\mu}{2})h^2\right]_{;b}P^{ab} + (\rho f+\mu h^2)\dot{U}^a - \mu U_{b;c}h^bh^ch^a$$
$$-(\mu h^ah^b)_{;b} = 0. \qquad (14)$$

<u>These are equations of stream lines for infinitely conducting ferrofluid describing the deviation of</u>

fluid path from geodesic path.

We have the equation of continuity for infinitely conducting ferrofluid in the form,

$$\dot{r} + (r+p)\theta + (\mu-1)\left(\frac{\mu \dot{h}^2}{2} + \dot{\mu} h^2\right) = 0 . \tag{15}$$

We recall the relations interlinking the thermodynamical variables for polorised and magnetised fluid due to Maugin (1970) as

$$r = r_o\,(1+\epsilon) , \tag{16}$$

$$T_o dS = d\epsilon + p\,d\left(\frac{1}{r_o}\right) - \frac{1}{2r_o}\mu(1-\mu)\,dh^2 , \tag{17}$$

$$\chi = 1+\epsilon+\frac{p}{r_o} , \tag{18}$$

where $\chi$ is the generalised density.
So we write from equation (16)

$$\dot{r} = \left[r_o(1+\epsilon)\right]_{;a} U^a , \tag{19}$$

and also

$$(r+p)\theta - r_o\left(1+\epsilon+\frac{p}{r_o}\right)U^a{}_{;a} = 0 . \tag{20}$$

When we add equations (19) and (20) then we get

$$\dot{r}+(r+p)\theta - (r_o U^a)_{;a} + r_o \epsilon_{;a} U^a + r_{o;a}\left(\frac{p}{r_o}\right)U^a = 0 . \tag{21}$$

Further from relation (17) it follows that

$$r_o T_o dS = r_o d\epsilon - \frac{p}{r_o}dr_o - \frac{\mu}{2}(1-\mu)\,dh^2 . \tag{22}$$

Consequently we get

$$U^a r_o T_o dS = r_o U^a d\epsilon - \left(\frac{p}{r_o}\right) U^a dr_o - \tfrac{1}{2}\mu(1-\mu) U^a dh^2 \ . \qquad (23)$$

$$r_o T_o S_{;a} U^a = r_o \epsilon_{,a} U^a - \left(\frac{p}{r_o}\right) r_{o;a} U^a -$$

i.e. ,

$$-\tfrac{1}{2}\mu(1-\mu) h^2_{\ ;a} U^a \ . \qquad (24)$$

We combine this equation with equation (21) to derive

$$\dot{r} + (r+P)\theta - (r_o U^a)_{;a} + r_o T_o S_{;a} U^a + \tfrac{1}{2}\mu(1-\mu) h^2_{\ ;a} U^a = 0. \qquad (25)$$

On employing this equation (25) in the equation of Continuity (12) we obtain

$$\chi(r_o U^a)_{;a} + r_o T_o S_{;a} U^a + \tfrac{1}{2}\mu(1-\mu) h^2_{\ ;a} U^a - \left(\dot{\mu} h^2 + \frac{\mu \dot{h}^2}{2}\right) \times$$

$$\times (1-\mu) = 0. \qquad (26)$$

The law of conservation of baryon numbers which describes the equation of continuity for the rest mass ( Misner and Sharp , 1964 ) is described by ,

$$(r_o U^a)_{;a} = 0 \ . \qquad (27)$$

So that the equation (26) can finally be put in the form

$$r_o T_o S_{;a} U^a + \tfrac{1}{2}\mu(1-\mu) h^2 + (\mu - 1)\left(\frac{\mu \dot{h}^2}{2} + \dot{\mu} h^2\right) = 0,$$

i.e.,

$$r_o T_o S_{;a} U^a + (\mu - 1)\dot{\mu} h^2 = 0 \ . \qquad (28)$$

Here we conclude that in the space-time of infinitely conducting ferrofluid following the law of conservation of baryon numbers, the flow is adiabatic if and only if the magnetic permeability is constant.

Hence we conclude that the flow of infinitely conducting ferrofluid cannot be adiabatic.

NOTE :

Following the result (6) we calculate the transvected equation $T^{ab}{}_{;b}h_a = 0$ which implies

$$(\rho f+\mu h^2)_{;b}U^aU^bh_a + (\rho f+\mu h^2)(\dot{U}^ah_a+\Theta U^ah_a)-[p+\mu(1-\frac{\mu}{2})h^2]_{;b} \times g^{ab}h_a - \mu_{,b}h^ah^bh_a - \mu h^a{}_{;b}h^bh_a - \mu h^a{}_{;b}h^bh_a - \mu h^ah^b{}_{;b}h_a = 0,$$

i.e.,

$$(\rho f+\mu h^2)\dot{U}^ah_a - [p+\mu(1-\frac{\mu}{2})h^2]_{;b}h^b + \mu_{,b}h^2h^b - \mu h^a{}_{;b}h^bh_a + \mu h^2h^b{}_{;b} = 0 .$$

This with Maxwell equation (7) gets reduced to

$$(r+p)\dot{U}^ah_a + \mu h^2[-h^b{}_{;b} - \mu_{,b}\frac{h^b}{\mu}] - [p+\mu(1-\frac{\mu}{2})h^2]_{;b}h^b + \mu_{;b}h^bh^2 + \mu h^2{}_{;b}\frac{h^b}{2} + \mu h^2h^b{}_{;b} = 0 ,$$

i.e. ,

$$(r+p)\dot{U}^ah_a - [p+\mu(1-\frac{\mu}{2})h^2]_{;b}h^b + \frac{\mu}{2}h^2{}_{;b}h^b = 0. \qquad (29)$$

This result is equivalent to the result obtained by Asgekar (1978) which states that the acceleration is orthogonal to the magnetic field if and only if the pressure is invarient along the magnetic field. (when $\mu$ is constant)

We write the famous Ricci identities with respect to time-like vector as given in the form

$$U_{a;bc} - U_{a;cb} = R_{dabc}U^d . \qquad (30)$$

This with simplification yields

$$R_{ab}U^aU^b = \dot{\theta} + \frac{1}{3}\theta^2 + 2\sigma^2 - 2W^2 - \dot{U}^a{}_{;a} \ . \qquad (31)$$

But for infinitely conducting ferrofluid we have

$$R_{ab}U^aU^b = \tfrac{1}{2}(r+3p+2\mu h^2-\mu^2 h^2) \ . \qquad (32)$$

Hence the Ray Choudhri equation for infinitely conducting ferrofluid takes the form

$$\dot{\theta}+\frac{1}{3}\theta = \tfrac{1}{2}(r+3p+2\mu h^2-\mu^2 h^2) - 2\sigma^2 + 2W^2 + \dot{U}^a{}_{;a} \ . \qquad (33)$$

This equation is known as the RayChoudhri's equation for Infinitely Conducting Ferrofluid and clearly shows the effects of kinematical entities on dynamical quantities of infinitely conducting ferrofluid.