CHAPTER - IV

Approximate Solution of Pohlhausens Problem of Free Convection on a Heat Vertical Plate

Introduction

Approximate solution of Pohlhausen's problem of free convection on a heat vertical plate was obtained by Pohlhausen [4]. Further the equation are solved by Squire [2] and Eckert [3].

In this problem the equation of motion is solved and calculated the Nusselt numbers for different Prandtl numbers.

Also for different Prandtl numbers the boundary Layer thickness is obtained.

It is also shown by graphically that as Prandtl number increases the boundary layer thickness decreases and Nusselt numbers increases.

<u>Nomenclature</u>

$a = K/\rho C_p$: thermal diffusivity

K : coefficient of thermal conductivity

C_p : Specific heat at constant pressure

Pr : Prandtl number $\mu C_p/K$

$$Nu(x) : \frac{-\left(\frac{\partial T}{\partial y}\right)_{y=0} x}{T_w - T_\infty} \quad : \text{Nusselt number}$$

ν : Kinematic viscosity

η : y/δ

u and v : Velocity components in x and y directions.

U_∞ : Free-stream velocity in x-direction

$$\theta : \frac{T - T_\infty}{T_w - T_\infty}, \text{ dimensionless temperature.}$$

δ : Boundary layer thickness

$U_1(x)$: Any arbitrary function

C_1, C_2 : Any two constants.

$$Gr = \frac{g(T_w - T_\infty)}{\nu^2 T_\infty} \cdot x^3, \text{ Grassof's number}$$

II) Solution of Problem

The equatioh governing the motion of the fluid in the neighbourhood heated vertical plate are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\alpha\theta \qquad \ldots (2)$$

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = a \frac{\partial^2 \theta}{\partial y^2} \qquad \ldots (3)$$

where $\theta = \frac{T-T_\infty}{T_w-T_\infty}$ and $\alpha = \frac{T_w-T_\infty}{T_\infty}$

with the boundary conditions

$$\left.\begin{array}{llll} y = 0, & u = 0, & v = 0 & \theta = 1 \\ y = \delta, & u = 0, & \theta = 0 & \end{array}\right\} \qquad \ldots (4)$$

We have assumed that the thickness of the thermal boundary Layer is same as that of the velocity boundary Layer

Integrating equation (2) and (3) with respect to y between the limits $y = 0$ to $y = \delta$ we get from equation (2)

$$\int_0^\delta u \frac{\partial u}{\partial x} dy + \int_0^\delta v \frac{\partial u}{\partial y} dy = \nu \int_0^\delta \frac{\partial^2 u}{\partial y^2} dy +$$

$$+ g\alpha \int_0^\delta \theta \, dy \qquad \ldots (5)$$

Consider

$$\int_0^\delta v \frac{\partial u}{\partial y} dy = vu \Big]_{y=0}^{y=\delta} - \int_0^\delta u \frac{\partial v}{\partial y} dy$$

$$= v.u_{y=\delta} - v\, u_{y=0} - \int_0^\delta u \frac{\partial v}{\partial y} dy$$

by equation (1)

$$\int_0^\delta v \frac{\partial u}{\partial y} dy = \int_0^\delta u \frac{\partial u}{\partial x} dy$$

Consider the first part of R.H.S. of equation (5)

$$\nu \int_0^\delta \frac{\partial^2 u}{\partial y^2} dy = \nu \left(\frac{\partial u}{\partial y} \right)_{y=0}^{y=\delta}$$

$$= -\nu \left(\frac{\partial u}{\partial y} \right)_{y=0} + \nu \left(\frac{\partial u}{\partial y} \right)_{y=\delta}$$

$$= -\nu \left(\frac{\partial u}{\partial y} \right)_{y=0}$$

Then equation (5) will take the following form

$$\int_0^\delta u \frac{\partial u}{\partial x} dy + \int_0^\delta u \frac{\partial u}{\partial x} dy = -\nu \left(\frac{\partial u}{\partial y} \right)_{y=0} + g\alpha \int_0^\delta \theta \, dy$$

$$\text{i.e.} \quad \frac{d}{dx} \int_0^{\delta} u^2 dy = -\nu \left(\frac{\partial u}{\partial y} \right)_{y=0} + g\alpha \int_0^{\delta} \theta \, dy \quad \dots (6)$$

Integrating the equation (3) between the limits $y = 0$ to $y = \delta$ we obtain

$$\int_0^{\delta} u \frac{\partial \theta}{\partial x} dy + \int_0^{\delta} v \frac{\partial \theta}{\partial y} dy = a \int_0^{\delta} \frac{\partial^2 \theta}{\partial y^2} dy \quad \dots (7)$$

Consider $\int_0^{\delta} v \frac{\partial \theta}{\partial y} dy = v\theta \Big]_{y=0}^{y=\delta} - \int_0^{\delta} \theta . \frac{\partial v}{\partial y} dy$

$$= v\theta_{y=\delta} - v\theta_{y=0} - \int_0^{\delta} \theta \frac{\partial v}{\partial y} dy$$

$$= \int_0^{\delta} \theta \frac{\partial u}{\partial x} dy \quad \text{by (1)}$$

and

$$a \int_0^{\delta} \frac{\partial^2 \theta}{\partial y^2} dy = \left[a \frac{\partial \theta}{\partial y} \right]_{y=0}^{y=\delta}$$

$$= \left[a \frac{\partial \theta}{\partial y} \right]_{y=\delta} - \left[a \frac{\partial \theta}{\partial y} \right]_{y=0}$$

$$= -a \left(\frac{\partial \theta}{\partial y} \right)_{y=0}$$

Substituting this in equation we get

$$\int_0^{\delta} u \frac{\partial u}{\partial x} dy + \int_0^{\delta} \theta \frac{\partial u}{\partial x} dy = -a \left(\frac{\partial \theta}{\partial y} \right)_{y=0}$$

$$\text{i.e.} \quad \frac{d}{dx} \int_0^{\delta} (u\theta)\,dy = -a \left(\frac{\partial\theta}{\partial y} \right)_{y=0} \qquad \ldots (8)$$

solving the above integral equations by taking the following polynomals in $\eta = y/\delta$ for the distributions of u and θ, satisfying the respective boundary conditions.

$$\left.\begin{aligned} u &= u_1(x)\ \eta(1-\eta)^2 \\ \text{and} \quad \theta &= (1-\eta)^2 \end{aligned}\right\} \qquad \ldots (9)$$

where $u_1(x)$ is an arbitrary function which has the dimension of velocity to be determined.

Now putting the values of u and θ in equation (6) we get

$$\frac{1}{105} \frac{d}{dx} (u_1^2(x)\delta) = \frac{1}{3} g\alpha\delta - \nu \frac{u_1}{\delta} \qquad \ldots (10)$$

Now putting the values of u and θ in equation (8) we get

$$\frac{d}{dx} \left[u_1(x) \frac{12}{360} \right] \delta = \frac{2a}{\delta}$$

$$\text{i.e.} \quad \frac{1}{30} \frac{d}{dx} (u_1\delta) = \frac{2a}{\delta} \qquad \ldots (11)$$

Now we will find the solution of above equation in terms of

$$u_1 = c_1 x^m \text{ and } \delta = c_2 x^n \qquad \ldots (12)$$

Putting the values of u_1 and δ in equation (10) and (11) we get

$$\frac{1}{105} \frac{d}{dx} [u_1^2 \delta] = \frac{1}{3} g\alpha\delta - \nu \frac{u_1}{\delta}$$

$$\text{i.e.} \quad \frac{1}{105} \frac{d}{dx} [c_1^2 x^{2m} c_2 x^n] = \frac{1}{3} g\alpha c_1 x^n - \nu \frac{c_1 x^m}{c_2 x^n}$$

$$\text{i.e.} \quad \frac{1}{105} [(2m + n) x^{2m+n-1} c_1^2 c_2] = \frac{1}{3} g\alpha c_1 x^n - \nu \frac{c_1}{c_2} x^{m-n}$$

$$\text{i.e.} \quad \frac{2m+n}{105} c_1^2 c_2 x^{2m+n-1} = \frac{1}{3} g\alpha c_2 x^n - \frac{c_1}{c_2}) x^{m-n} \qquad \ldots (13)$$

equation (11) becomes

$$\text{i.e.} \quad \frac{1}{30} \frac{d}{dx} (u_1 \delta) = \frac{2a}{\delta}$$

$$\frac{1}{30} \frac{d}{dx} [x^{m+n} c_1 c_2] = \frac{2a}{\delta}$$

$$\frac{m+n}{30} c_1 c_2 x^{m+n-1} = \frac{2a}{c_2} x^{-n} \qquad \ldots (14)$$

must be identically satisfied.

This gives

$$2m + n - 1 = n = m - n$$

$$2m + n - 1 = n$$

$$\therefore m = 1/2, \quad n = 1/4$$

We find the value of C_1 and C_2 by putting m and n.

i.e. $C_1 = 5.164\ \nu \left(\frac{g\alpha}{\nu^2} \right)^{1/2} \left(\frac{20}{21} + Pr \right)^{-1/2}$ ... (15)

and

$C_2 = 3.94\ Pr^{-1/2} \left(Pr + \frac{20}{21} \right)^{1/4} \left(\frac{g\alpha}{\nu^2} \right)^{-1/4}$ ... (16)

from $\delta = C_2 x^n$

$$\Longrightarrow \frac{\delta}{x^n} = C_2 \quad \text{put } n = 1/4 \text{ in this we get}$$

$$\frac{\delta}{x^{1/4}} = 3.94\ Pr^{-1/2} (Pr + 0.95)^{1/4} \cdot \left(\frac{g\alpha}{\nu^2} \right)^{-1/4}$$

Dividing by $x^{3/4}$ we get

$$\frac{\delta}{x} = 3.94\ Pr^{-1/2} (Pr + 0.95)^{1/4} \cdot \left(\frac{g\alpha\, x^3}{\nu^2} \right)^{-1/4}$$

$$\text{But } Gr = \frac{g(T_w - T_\infty)\, x^3}{\nu^2 T_\infty} = \frac{g\alpha\, x^3}{\nu^2}$$

i.e. $\frac{\delta}{x} = 3.94\ Pr^{-1/2} (Pr + 0.95)^{1/4} (Gr)^{-1/4}$

The temperature gradient of the wall is given by

$$\theta = (1 - \eta)^2$$

$$\therefore \frac{\partial \theta}{\partial \eta} = 2(1-\eta)(-1)$$

$$\left(\frac{\partial \theta}{\partial \eta} \right)_{\eta=0} = -2.$$

The local Nusselt number for the heat transfer, in the present case is given by

$$Nu(x) = \frac{-\left(\frac{\partial T}{\partial y} \right)_{y=0} . x}{T_w - T_\infty} = -\left(\frac{\partial \theta}{\partial \eta} \right)_0 . \frac{x}{\delta}$$

$$Nu(x) = 2. \frac{x}{\delta}$$

[illegible] and Conclusions

From the table 1 as Prandtl number increases the Nusselt number increases (Gr is constant). Hence the temperature of the plate decreases as Prandtl number increases and Rate of heat transfer increases as Prandtl number increases.

Therefore, rate of heat transfer in water is greater than that in air.Result is also shown by graphically [Fig.1 and Fig.2].

Table 1

Sr.No.	Pr	$\delta^* = \frac{\delta}{x} (Gr)^{1/4}$	$Nu(x) = 2 \times x/\delta$
1	0.6	5.66108	0.3534
2	0.7	5.32371	0.3757
3	0.8	5.05367	0.3957
4	0.9	4.83130	0.4140
5	1.0	4.64409	0.4306
6	1.1	4.59179	0.4355
7	1.2	4.54681	0.4398
8	2	3.64194	0.5491
9	3	3.19875	0.6252
10	4	2.93098	0.6823
11	5	2.74496	0.7286
12	6	2.60503	0.7677
13	7	2.49422	0.8018
14	8	2.40327	0.8321
15	9	2.32662	0.85[illegible]
16	10	2.26071	0.9078

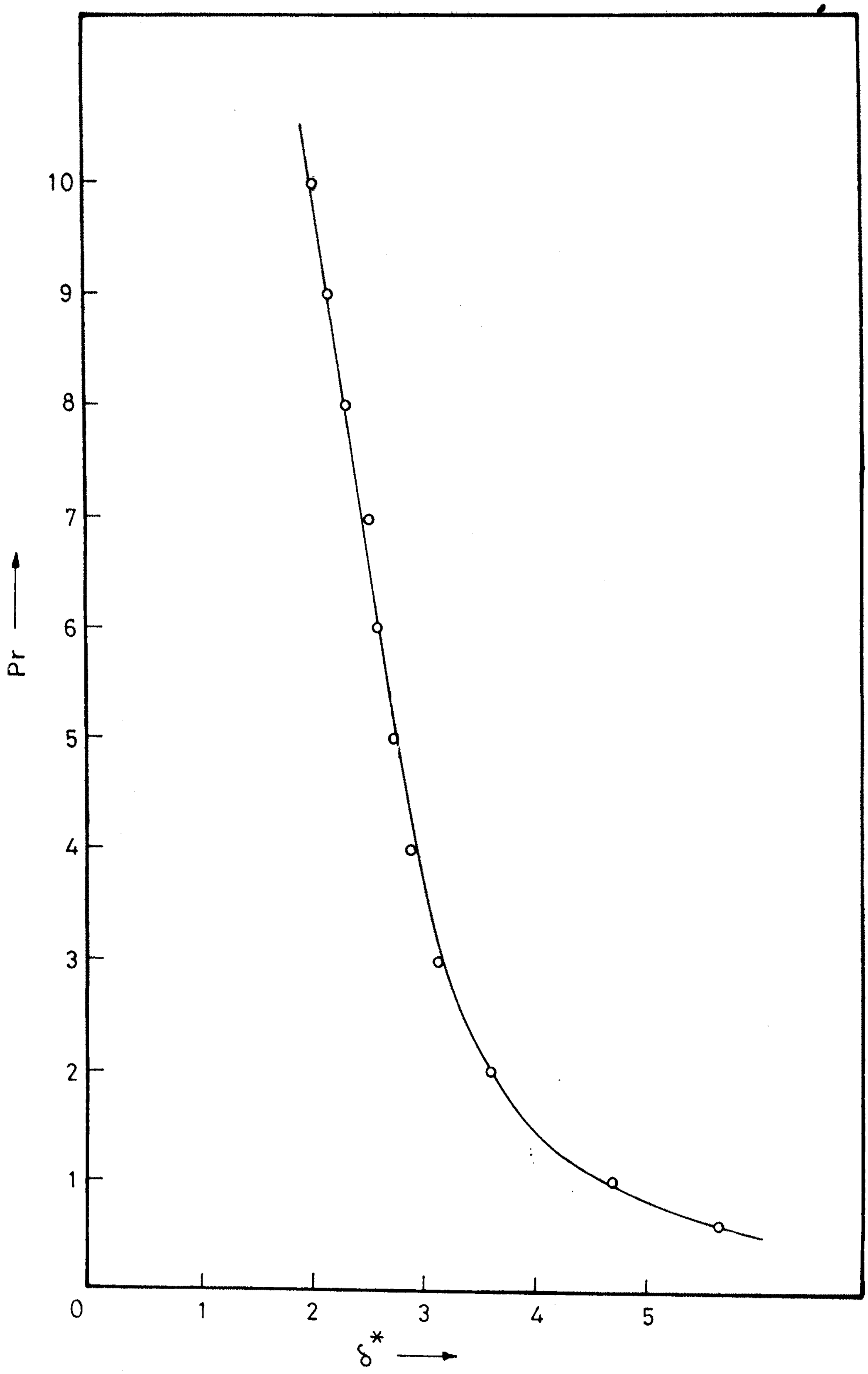

Fig. 1

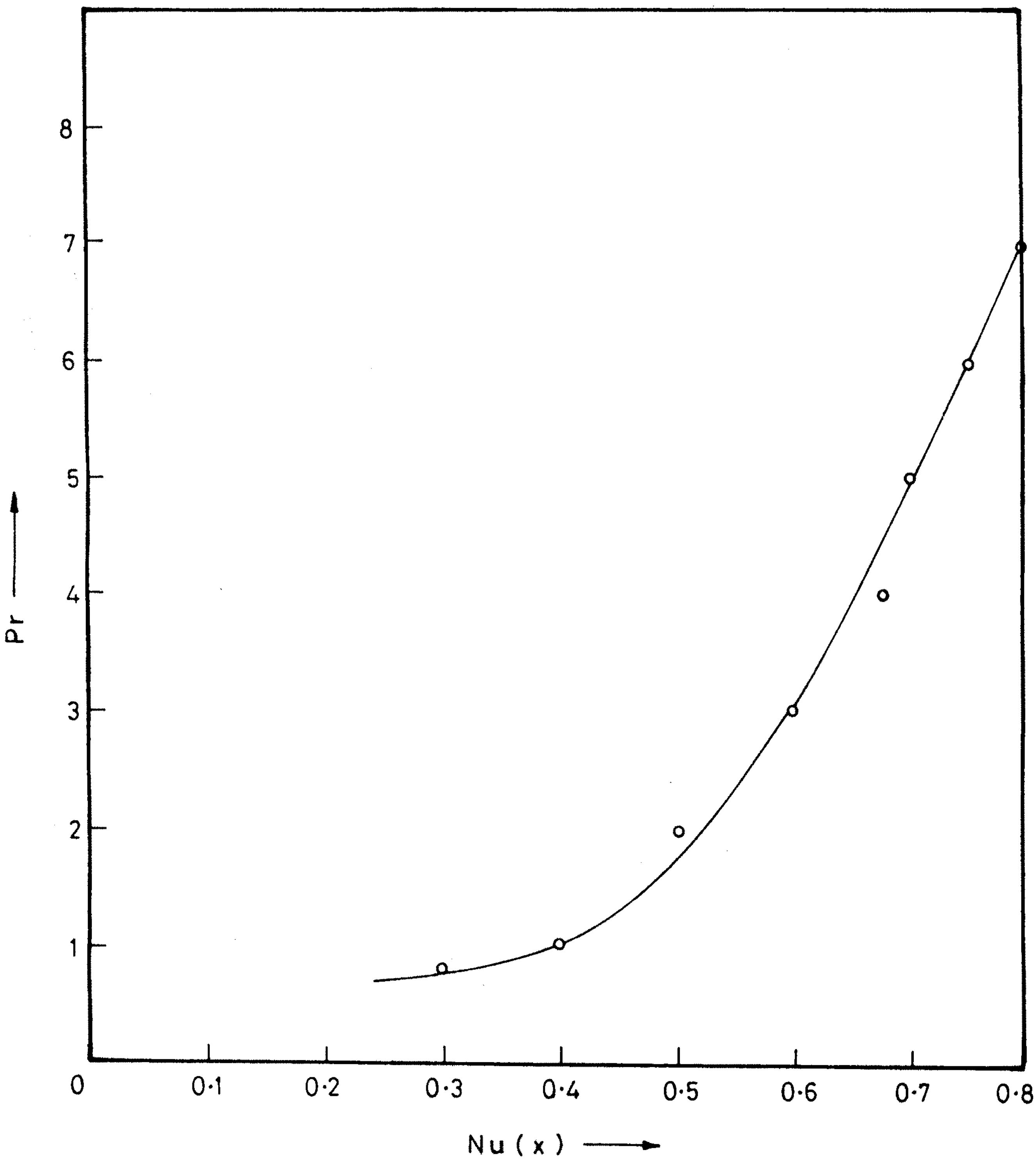

Fig. 2

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