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CHAPTER-1

P R E L I M I N A R I E S

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CHAPTER-I

PRELIMINARIES

In this chapter we give some basic definitions and results which will be used in subsequent chapters.

1.1 DEFINITIONS

Def.1.1.1 Partially ordered set or Poset [5]: Let P be a nonvoid set. Define a relation $\leqslant$ on P satisfying the following for all $a,b,c \in P$.

i) $a \leqslant a$ (reflexivity)

ii) $a \leqslant b$ and $b \leqslant a \implies a=b$ (antisymmetry)

iii) $a \leqslant b$ and $b \leqslant c \implies a \leqslant c$ (transitivity)

The ordered pair $(P, \leqslant)$ is called a partially ordered set or a poset.

A poset $(P, \leqslant)$ is called as chain (or totally ordered set or linearly ordered set) if it satisfies following condition :

iv) $a \leqslant b$ or $b \leqslant a$ for all $a,b, \in P$ (linearity).

Def.1.1.2 Zero element and unit element of poset [5] : A zero of a poset P is an element $0 \in P$ with $0 \leqslant x$ for $x \in P$. A unit element of a poset P is an element $1 \in P$ with $x \leqslant 1$ for all $x \in P$.

Def.1.1.3 Lattice as poset [5] : A poset $(L, \leqslant)$ is a lattice if sup { a,b }(or $a \vee b$) and inf{ a,b} (or $a \wedge b$) exists in L for all $a,b \in L$.

Def.1.1.4 Lattice as an algebra [5] : An algebra $< L; \wedge \vee >$ is called a lattice if L is a nonvoid set with two binary operations $\wedge$ and $\vee$ satisfying following properties for all $a,b,c \in L$:

i) $a \wedge a = a,\ a \vee a = a$ (idempotency)

ii) $a \wedge b = b \wedge a,\ a \vee b = b \vee a$ (commutativity)

iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, (associativity)
$(a \vee b) \vee c = a \vee (b \vee c)$

iv) $a \wedge (a \vee b) = a$, (absorption identities)
$a \vee (a \wedge b) = a$

Def.1.1.5 Distributive lattice [5] : A lattice $< L; \wedge , \vee >$ is said to be distributive if, for all $a,b,c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Def.1.1.6 Modular lattice [5] : A lattice $< L; \wedge , \vee >$ is said to be modular if $a,b \in L$ and $a \leq b \Longrightarrow a \vee (b \wedge c) = b \wedge (a \vee c)$ for all $c \in L$.

Def.1.1.7 Bounded lattice [5] : A lattice $< L; \wedge , \vee >$ is said to be bounded if both the least and the greatest elements, denoted by **0** and 1 respectively, are in L.

Def. 1.1.8 Complemented lattice [5] : A bounded lattice $< L; \wedge , \vee >$ is called complemented if, for every a in L, there exists b in L such that $a \wedge b = 0$ and $a \vee b = 1$.

Def.1.1.9 Boolean algebra [5] : A Boolean algebra is an algebra : $< L; \wedge, \vee, ', 0, 1 >$, where $< L; \wedge , \vee >$ is a distributive lattice, the complementation ',' is a unary operation and 0,1 are nullary operations.

Def.1.1.10 Pseudocomplemented lattice [5] : A lattice L with O is called Pseudocomplemented lattice if a^* exists for every $a \in L$, where a^* is the pseudocomplement of a in L, which is the largest element satisfying the condition $a \wedge a^* = 0$.

Def.1.1.11 Quasicomplemented lattice [5] : A lattice L with 1 is called quasicomplemented lattice if $a^\perp$ exists for every $a \in L$, where $a^\perp$ is the quasicomplement of a in L, which is the smallest element satisfying the condition $a \vee a^\perp = 1$.

Def.1.1.12 Ideal in a lattice [5] : A nonvoid subset I of a lattice L is said to be an ideal in L if

i) $a, b \in I \Longrightarrow a \vee b \in I$ and

ii) $a \in I$, $b \in L$, $b \leq a \Longrightarrow b \in I$.

Def.1.1.13 Filter in a lattice [5] : A nonvoid subset F of a lattice L is said to be filter in L if

i) $a, b \in F \Longrightarrow a \wedge b \in F$ and

ii) $a \in F$, $b \in L$, $a \leq b \Longrightarrow b \in F$.

Def.1.1.14 Prime ideal in a lattice [5] : A proper ideal I of L is said to be prime if $a, b \in L$ and $a \wedge b \in I$ imply that $a \in I$ or $b \in I$.

Def.1.1.15 Prime filter in a lattice [5] : A proper filter F of L is said to be prime if $a,b \in L$ and $a \vee b \in F$ imply that $a \in F$ or $b \in F$.

Def.1.1.16 Ideal generated by H [5] : The ideal generated by a nonempty subset H of L is the smallest ideal in L containing H and is denoted by (H].

Def.1.1.17 Filter generated by H [5] : The filter generated by a nonempty subset H of L is the smallest filter in L containing H and is denoted by [H).

Def.1.1.18 Principal ideal in a lattice [5] : Given an element a in L the ideal generated by $\{a\}$, denoted by $(a](=\{x \in L / x \leq a\})$, is called a principal ideal of L.

Def.1.1.19 Principal filter in a lattice [5] : Given an element a in L the filter generated by $\{a\}$, denoted by $[a)(=\{x \in L / x \geq a\})$, is called a principal filter of L.

Def.1.1.20 Maximal ideal in a lattice [5] : A proper ideal of a lattice L is called maximal if it is not contained in any other proper ideal of L.

Def.1.1.21 Maximal filter in a lattice [5] : A proper filter of a lattice L is called maximal if it is not contained in any other proper filter of L.

1.2 RESULTS :

Result 1.2.1 [4] : In a distributive lattice L with 0, $\{a\}^*$ $(=\{x \in L / x \wedge a = 0\})$ is an ideal for every a in L.

Result 1.2.2 [5] : The intersection of any number of ideals is an ideal.

Result 1.2.3 [5] : The set of all ideals in L, denoted by I(L), forms a lattice under the operations $\bar{\wedge}$ and $\underline{\vee}$ where i) $I_1 \bar{\wedge} I_2 = I_1 \cap I_2$ and (ii) $I_1 \underline{\vee} I_2 = (I_1 \cup I_2]$ for $I_1, I_2 \in I(L)$.

Result 1.2.4 [5] : For a,b $\in$ L, $(a] \cap (b] = (a \wedge b]$ and $(a] \underline{\vee} (b] = (a \vee b]$.

Result 1.2.5 [5] : For a,b $\in$ L, $(a] = (b] \Leftrightarrow a=b$.

Result 1.2.6 [12] : Any proper ideal (filter) of a lattice L with 1(0) is contained in a maximal ideal (filter).

Result 1.2.7 [1] : Let L be a lattice with 1(0). A proper ideal (filter) M in L is maximal if and only if for any element $a \notin M$, $(a \in L)$, there exists an element $b \in M$ such $a \vee b=1$ $(a \wedge b = 0)$.

Result 1.2.8 [5] :Stone's Theorem : Let L be a distributive lattice, let I be an ideal and D be a filter of L such that $I \cap D = \emptyset$. Then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \emptyset$.

Result 1.2.9 [5] : Every ideal I of a distributive lattice is the interesection of all prime ideals containing it.

Result 1.2.10 [5] : Let L be a distributive lattice, a,b $\in$ L, $a \neq b$. Then there is a prime ideal containing exactly one of a and b.

Result 1.2.11 [5] : An ideal P is a prime ideal of L if and only if L-P is a prime filter.

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