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CHAPTER-IV

0-1 DISTRIBUTIVE LATTICES

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O-1 DISTRIBUTIVE LATTICES

4.1 INTRODUCTION

A O-1 distributive lattice is a lattice which is both O-distributive and 1-distributive. A suitable example of a O-1 distributive lattice is a lattice N_5 sketched in the following diagram.

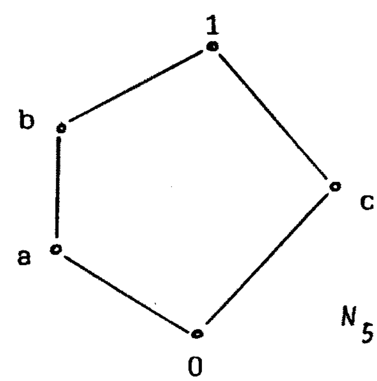

Fig.8

This lattice N_5 is pseudocomplemented as well as cuasicomplemented but not distributive.

In this chapter mainly we will characterize complemertedness in O-1 distributive lattices.

Ramana Murty [9] has proved the following

Result : Let L be a distributive lattice with O and 1. Then the following statements are equivalent.

1) L is a Boolean algebra.

2) Complement of every ultrafilter in L is a maximal ideal.

3) Complement of every maximal ideal is an ultrafilter.

4) Every prime filter is an ultrafilter.

5) Every prime ideal is a maximal ideal.

Adams [1] has proved

Result : Let L be a lattice with greatest element 1 and least element O in which an ideal, or dual ideal, is maximal if and only if it is prime. Then L is complemented.

These results are generalized in Article 2 to characterize the complementedness in a 0-1 distributive lattice.

A very special result that "A 0-1 distributive lattice is distributive if it is weakly complemented" has been proved.

The lattice N_5 (see Fig.8) is a 0-1 distributive lattice but not weakly complemented. Hence it is remarked that it is not distributive.

Article 3 deals with a diagram which exhibits the total outline of our study.

4.2 PROPERTIES

It is well-known that in a Boolean algebra the set-theoretic complement of a maximal ideal is a maximal filter and vice versa. Ramana Murty [9] has proved that this is a characteristic property of the Boolean algebra by proving that a distributive lattice with 0 and 1 is a Boolean algebra if and only if the complement of every maximal ideal is a maximal filter. But it can be observed that the conditions given in [9, Theorem 1] characterizes complementedness in a bounded distributive lattice. The following result generalizes the result of Ramana Murty for a 0-1 distributive lattice.

4.2.1 Result : Let L be a 0-1 distributive lattice. Then the following conditions are equivalent.

1) L is complemented.

2) Every prime ideal is maximal.

3) Every prime filter is maximal.

4) Complement of a maximal filter is a maximal ideal.

5) Complement of a maximal ideal is a maximal filter.

Proof : Claim : (1) ⟹ (2)

Let P be a prime ideal of L.

Let $x \in L$ such that $x \notin P$.

As L is complemented, there exists an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$.

But $0 \in P$ (since P is an ideal) implies that $x \wedge y \in P$.

As $x \notin P$ we get $y \in P$ by primeness of P.

Thus for any $x \notin P$, there exists an element $y \in P$ such that $x \vee y = 1$.

Hence by Result 1.2.7, P is a maximal ideal proving (2).

Claim : (2) $\Longrightarrow$ (4)

Let F be a maximal filter.

As L is O-distributive, F is a prime filter (see Result 2.3.10).

Then L-F is a prime ideal (see Result 1.2.11).

By assumption L-F is a maximal ideal proving (4).

Claim : (4) $\Longrightarrow$ (1)

Assume that complement of a maximal filter is a maximal ideal.

We show that L is complemented.

Let $x \in L$ and let $\{x\}^* = \{y \in L / x \wedge y = 0\}$ and $\{x\}^\perp = \{y \in L / x \vee y = 1\}$.

As L is O-distributive and 1-distributive, $\{x\}^*$ is an ideal in L and $\{x\}^\perp$ is a filter in L (see Result 2.3.4 and Result 3.3.2).

If $\{x\}^* \cap \{x\}^\perp \neq \emptyset$, then we are through.

Suppose $\{x\}^* \cap \{x\}^\perp = \emptyset$.

Let $\mathcal{F} = \{D / D$ is a filter in L such that $D \supseteq \{x\}^\perp$ and $D \cap \{x\}^* = \emptyset\}$.

Clearly as $\{x\}^\perp \in \mathcal{F}, \mathcal{F} \neq \phi$.

Applying Zorn's lemma we get a maximal element, say F, in **F**.

Claim 1 : $x \in F$.

If $x \notin F$, then $[F \vee \{x\}] \cap \{x\}^* \neq \emptyset$.

Hence there exists an element $y \in \{x\}^*$ such that $y \geq f \wedge x$ for some $f \in F$.

Now $f \wedge x \leq y \wedge x = 0$ implies that $f \in \{x\}^*$, which is impossible since $F \cap \{x\}^* = \emptyset$.

Hence we get $x \in F$ proving the claim.

Claim 2 : F is a maximal filter .

Let $z \in L$ such that $z \notin F$.

Then $[F \vee \{z\}] \cap \{x\}^* \neq \emptyset$.

Hence there exists an element $y \in \{x\}^*$ such that

$y \geq f \wedge z$ for some $f \in F$.

Now $f \wedge z \wedge x \leq y \wedge x = 0$ implies that $f \wedge z \wedge x = 0$.

As $x \in F$ and $f \in F$ we have $x \wedge f \in F$ (since F is a filter).

Thus for $z \notin F$ there exists $x \wedge f \in F$ such that $z \wedge x \wedge f = 0$.

Hence F is a maximal filter (see Result 1.2.7) which proves the claim 2.

By assumption L-F is a maximal ideal and $x \notin L-F$.

Hence there exists an element $y \in L-F$ such that $x \vee y = 1$ (see Result 1.2.7), which is a contradiction since $\{x\}^{\perp} \subseteq F$.

Hence $\{x\}^* \cap \{x\}^{\perp} \neq \emptyset$ proving that L is complemented.

Similarly we can show that $(1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (1)$. ■

Adams [1] has proved the following result "Let L be a lattice with greatest element 1 and least element O in which an ideal, or dual ideal, is maximal if and only if it is prime. Then L is complemented." But when every maximal ideal, or filter, is prime then L will become a 0-1 distributive lattice. We generalize the result of Adams as

4.2.2 Result : Let L be a 0-1 distributive lattice. If every prime filter in L is maximal, then L is complemented.

Proof : Let $a \in L$ be any element such that it has no complement in L.

Consider $\{a\}^* = \{x \in L / a \wedge x = 0\}$.

As L is 0-distributive, $\{a\}^*$ is an ideal in L (see Result 2.3.4).

As a has no complement in L we have $a \vee x \neq 1$ for any $x \in \{a\}^*$
Define $T = \{t \in L \,/\, t \leq a \vee x,\ x \in \{a\}^*\}$.

1. $T \neq \emptyset$ (since $a \in T$)
2. $1 \notin T$ (since $a \vee x \neq 1$ for any $x \in \{a\}^*$)
3. $t_1 \leq t_2 \in T \Rightarrow t_1 \in T$ (since $t_2 \leq a \vee x$ and $t_1 \leq t_2$ implies that $t_1 \leq a \vee x$)
4. Let $t_1, t_2 \in T$.

Then $t_1 \leq a \vee x_1$ and $t_2 \leq a \vee x_2$ for some $x_1, x_2 \in \{a\}^*$.

As $\{a\}^*$ is an ideal, $x_1 \vee x_2 \in \{a\}^*$.

Then $t_1 \leq a \vee x_1 \vee x_2$ and $t_2 \leq a \vee x_1 \vee x_2$ gives that $t_1 \vee t_2 \leq a \vee x_1 \vee x_2$.

Therefore $t_1 \vee t_2 \in T$.

1-4 $\Rightarrow$ T is a proper ideal in L.

As $1 \in L$, T is contained in some maximal ideal, say M, in L (see Result 1.2.6).

As L is 1-distributive, M is a prime ideal (see Result 3.3.2).

Hence L-M is a prime filter in L (see Result 1.2.11).

By data L-M is a maximal filter and as $a \in T \subseteq M, a \notin L-M$.

Then there exists an element $z \in L-M$ such that $a \wedge z=0$(see Result 1.2.7).

This implies that $z \in \{a\}^*$.

As $\{a\}^* \subset M$, $z \in M$.

Thus $z \in (L-M) \cap M=\emptyset$, a contradiction.

Hence a must have complement in L proving that L is complemented. ■

It is observed that every 0-distributive/1-distributive lattice need not be distributive (see Remark 2.2.6 and Remark 3.2.6). 0-distributivity and 1-distributivity put together will not sufficient to prove that the lattice is distributive. A sufficient condition for a 0-1 distributive lattice to be distributive is established in the following result for which we need the

4.2.3 Lemma : Let L be a 0-distributive weakly complemented lattice. Then if $x,y \in L$ $(x,y \neq 1)$ and if

(*) $x \in P \Leftrightarrow y \in P$ for all prime ideals P then x=y.

Proof : Suppose that x and y satisfy the conditions of the lemma.

If $y < x \vee y$, then there exists an element c such that $c \wedge y = 0$ and $c \wedge (x \vee y) \neq 0$ since by hypothesis L is weakly complemented (see Def. 2.4.8).

Then there exists a maximal filter F in L such that $c \wedge (x \vee y) \in F$ (see Result 1.2.6).

As L is O-distributive, F is prime (see Result 2.3.10).

Therefore M=L-F is a prime ideal in L (see Result 1.2.11).

As $c \wedge (x \vee y) \in F$ we get $c \wedge (x \vee y) \notin M$.

As $c \wedge y = 0$ we have $c \wedge y \in M$.

This implies that $c \in M$ or $y \in M$ as M is a prime ideal.

But $c \in M$ implies that $c \wedge (x \vee y) \in M$, which is not possible.

Thus $c \notin M$ and hence $y \in M$.

If $x \in M$, then $x \vee y \in M$ and hence $c \wedge (x \vee y) \leqslant x \vee y$ implies that $c \wedge (x \vee y) \in M$, which is not possible.

Thus $x \notin M$.

Hence we get that M is a prime ideal which contains y but not x.

This contradicts the assumption (*) of the lemma.

Therefore $y = x \vee y$.

Similarly we can show that $x = x \vee y$, and then we have $x = y$. ■

4.2.4 Result : Every 0-1 distributive lattice is distributive if it is weakly complemented.

Proof : Let a,x,y be arbitrary elements in L.

Let $s = a \wedge (x \vee y)$ and $t = (a \wedge x) \vee (a \wedge y)$.

We shall show that $s = t$.

Obviously $s \geqslant t$.

If $t = 1$, $s \geqslant t$, then $s=t=1$ and we are through.

Suppose $t \neq 1$. Then there exists a maximal ideal P such that $t \in P$ (see Result 1.2.6).

As L is 1-distributive, P is a prime ideal (see Result 3.3.2).

Now $(a \wedge x) \vee (a \wedge y) \in P$ implies that $a \wedge x \in P$ and $a \wedge y \in P$ (see Def.1.1.12).

Then either $a \in P$ or if it doesn't, x and y both belong to P, by primeness of P.

Thus either $a \in P$ or $x \vee y \in P$.

Hence in either the case $a \wedge (x \vee y) \in P$ i.e.; $s \in P$.

Thus we have shown that s belongs to a prime ideal that t does.

As $s \geqslant t$, t belongs to a prime ideal containing s.

Thus we get $s \in P$ if and only if $t \in P$ for any prime ideal P in L.

Then by Lemma 4.2.3 we get $s=t$.

i.e; $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$ for all $a, x, y \in L$.

Hence L is distributive. ■

As a conclusion we get the following

4.2.5 Remark : A lattice L is a Boolean algebra if and only if

1) L is O-1distributive;

2) Every prime filter in L is maximal filter and

3) L is weakly complemented.

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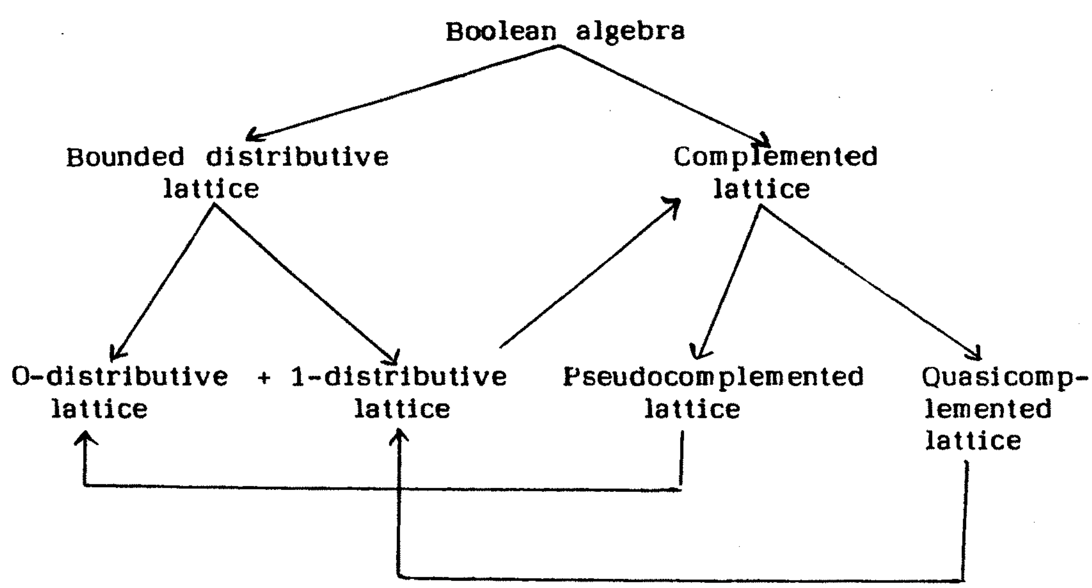
4.3 A DIAGRAM

The following diagram, indicating various possible generalizations of Boolean algebra,



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