# CHAPTER 0

## DEFINITIONS AND TERMINOLOGY

In this intorudctory chapter we present some definitions concerning Boolean algebra and Fuzzy set theory, and also some terminology which we are going to use in this context. It also contains statements of some results which we need in the course of investigation.

**Definition 0.1 :** Let U be a set. A fuzzy set in U ( or fuzzy subset of U ) is a function A from U into the unit interval [0,1] .

Many times the membership function is denoted by $\mu$ and different fuzzy subsets are specified as $\mu_A$ , $\mu_B$.

**Definition 0.2 :** If A and B are two fuzzy subsets of U , their union and intersection are defined by the following fuzzy subsets of U :

$$(A \cup B)x = \max(A(x), B(x)) \text{ for all } x \in U$$
$$(A \cap B)x = \min(A(x), B(x)) \text{ for all } x \in U.$$

**Remarks :** The unit interval [0,1] here, is called valuation set. Note that an ordinary subset A of a set U can be expressed as a characteristic function.

$$\chi_A : U \longrightarrow \{0, 1\}$$

where $\chi_A(x) = 1$ if $x \in A$

$= 0$ if $x \notin A$

Thus a fuzzy subset of U is generalised subset where it is allowed to assume intermediate values.

Definition 0.3 : If A is fuzzy subset of U , the complement $\bar{A}$ of A is a fuzzy subset of U given by

$$\bar{A}(x) = 1 - A(x) \qquad \forall x \in U$$

Definition 0.4 : A set E closed under two binary operations $\cup$ and $\cap$ is called a lattice if the two operations are commutative, associative idempotent and, satisfy

$$a \cup (a \cap b) = a$$

$$a \cap (a \cup b) = a \quad \text{for } a, b \in E$$

Definition 0.5 : A set E closed under a binary operation * is called Semilattice if * is associative, commutative and idempotent.

Definition 0.6 : An element 0 of a lattice E is called zero element of the lattice if

$$a \cup 0 = a \qquad \text{for all } a \in E$$

An element 1 of a lattice E is called unit element of the lattice if

$$a \cap 1 = a \quad \text{for all } a \in E$$

A lattice may or may not have zero or unit elements.

If it has, it is called ' Lattice with 0 and 1 '.

**Definition 0.7 :** A lattice E with 0 and 1 is said to be complemented if for every element $a \in E$ there exists an element a' in E, called complemenent of a, such that,

$$a \cup a' = 1 \quad \text{and } a \cap a' = 0$$

**Definition 0.8 :** Boolean algebra :

A set E closed under two binary operations $\cup$ and $\cap$, is called a Boolean algebra if the following axioms are satisfied :

For all $a, b, c \in E$,

I(a) : $a \cup b = b \cup a$. I (b) : $a \cap b = b \cap a$.

That is, the two operations are commitative.

II (a) : $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$

II (b) : $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$

That is, either of the two operations is distributive over the other.

III (a) : There is an element 0 ( called a zero element) having the property that

$$a \cup 0 = 0 \cup a = a$$

III (b) : There is an ~~unit~~ element1 (called unit element) in E having the property that

$$a \cap 1 = 1 \cap a = a.$$

Thus 0 is identity for the $\cup$ operation and 1 is identity for $\cap$ operation.

IV : Corresponding to every element a there is an element a' in E such that $a \cup a' = 1$ and $a \cap a' = 0$

We call a' complement of a.

V. : $0 \neq 1$. That is 0 and 1 are distrinct.

Now let us revise some known results.

5/ **Theorem 0.1** : The two operations $\cup$ and $\cap$ in the Boolean algebra are idempotent. That is

$$a \cup a = a \text{ and } a \cap a = a \quad \text{for all } a \in E$$

**Theorem 0.2** : Laws of absorption.

For all $a, b \in E$

$$a \cup (a \cap b) = a \text{ and } a \cap (a \cup b) = a.$$

Remark : From above two results it is clear that $(E, \cup)$ & $(E, \cap)$ are semi-lattices, Moreover, E is a distributive complemented lattice with 0 and 1.

**Theorem 0.3** : In a Boolean algebra the identity elements 0 and 1 are unique.

**Theorem 0.4** : For all elements 'a' of a Boolean algebra,

$a \cup 1 = 1$ and $a \cap 0 = 0$

In other words, 1 and 0 are the 'absorbing element' of $\cup$ and $\cap$ respectively.

<u>Theorem 0.5</u> : The two operations $\cup$ and $\cap$ of a Boolean algebra E are associative. That is,

$$a \cup (b \cup c) = (a \cup b) \cup c$$
$$a \cap (b \cap c) = (a \cap b) \cap c \qquad a, b, c \in E.$$

<u>Theorem 0.6</u> : For each element a of a Boolean algebra, the complement a' is uniquely defined.

<u>Theorem 0.7</u> : Involution.

For all a in a Boolean algebra,

$$(a')' = a.$$

<u>Theorem 0.8</u> : De Morgan's Laws.

For all elements a,b in a Boolean algebra,

$$(a \cup b)' = a' \cap b' \text{ and } (a \cap b)' = a' \cup b'$$

<u>Theorem 0.9</u> : In a Boolean algebra

$$0' = 1 \text{ and } 1' = 0$$

<u>Example 0.1</u> : The smallest Boolean algebra consists of two elements 0 and 1 where the two operations $\cup$ and $\cap$ are given by

$$a \cup b = \max(a, b)$$
$$a \cap b = \min(a, b) \quad \text{where } a, b \in \{0, 1\}$$

This smallest Boolean algebra is of paramount importance in Boolean algebra theory. It is denoted usually by $B_2$.

<u>Theorem 0.11</u> : For any set X, the power set of X, $\wp(x)$ is a Boolean algebra under usual operations of union and intersection, in which zero element is empty set $\emptyset$ and unit is whole set X.

<u>Note</u> : Theorem 0.10 is a consequence of Theorem 0.11. Indeed, if X is a finite set of cardinality n then its power set $\wp(x)$ has $2^n$ elements in it. In fact $\wp(x)$ is the set of all characteristic functions on X.

<u>Theorem 0.12</u> : The cross product of two ( or finitely many ) Boolean algebras is also a Boolean algebra. ( The two operations in the product algebra are induced by the corresponding operations in the factor algebras).

<u>Note</u> : $(B_2)^n = B_2 \times B_2 \times \dots \times B_2$ n times is a Boolean algebra of cardinality $2^n$.

<u>Definition 0.9</u> : Two Boolean algebras $\langle E_1, \cup_1, \cap_1, 0_1, 1_1, / \rangle$ and $\langle E_2, \cup_2, \cap_2, 0_2, 1_2, / \rangle$ are said to be isomorphic, if there is a bijection

$$\emptyset : E_1 \longrightarrow E_2 \quad \text{satisfying}$$

I. $\emptyset(a \cup_1 b) = \emptyset(a) \cup_2 \emptyset(b)$

$\emptyset(a \cap_1 b) = \emptyset(a) \cap_2 \emptyset(b) \qquad a, b \in E_1$

II. $\emptyset ( a' ) = \emptyset ( a )'$  $a \in E_1$

<u>Note</u> : Identities 0 and 1 are preserved by any isomorphism between $E_1$ and $E_2$ . For

$$\emptyset ( 0_1 ) = \emptyset ( 0_1 \cdot ( 0_1 )' )$$

$$= \emptyset ( 0_1 ) \cdot \emptyset ( ( 0_1 )' ) \quad \text{by I}$$

$$= \emptyset ( 0_1 ) \cdot (\emptyset ( 0_1 ) )' \quad \text{by II}$$

$$= 0_2 \quad \text{by def. of complement}$$

Similarly $\emptyset ( 1_1 ) = 1_2$.

<u>Theorem 0.13</u> : Every finite Boolean algebra is isomorphic to the Boolean algebra $( B_2 )^n$ for some integer $n \geqslant 1$.

<u>Remark</u> : The above theorem shows that any finite Boolean algebra has cardinality $2^n$ for some n . Hence two finite Boolean algebras of equal cardinality are isomorphic. ( Since both are isomorphic to $B_2^n$ ).

<u>Notations</u> : If $f : A \rightarrow B$ is a map, the image $f(x)$ of an element x in A is denoted by $fx$ at some places.

After a proof ends, the symbol # is put.

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