CHAPTRE II

A Conformally Flat Electromagnatic Mass Models In Einstein - Cartan Theory

A CONFORMALLY FLAT ELECTROMAGNATIC MASS MODELS IN EINSTEIN CARTAN THEORY.

1. INTRODUCTION

Recently Trautman by considering a Friedmann type of universe in Einstein Cartan theory has shown that the gravitational singularities may be averted . Hehl et al have discussed the generalization of Einstein's general theory of relativity in which the spin of matter couples to Cartan's torsion tensor. The predictions of Einstein-Cartan theory differ from those of general relativity only for matter filled regions. An important application for Einstein-Cartan theory is relativistic Astrophysics which deals with the interiors of stellar objects like neutron stars. With this view Prasanna, Kuchowicz, Hehl, Kerlick, Singh et. al. have considered the problems of static fluid spheres in Einstein Cartan theory. Non-singular cosmological models have been constructed by Kopezynski, Kuchowiez and Tsoubelis. Static charged fluid spheres in Einstein-Cartan theory have been considered by Krori et. al and Naduka.

Here we consider a static, charged conformally flat perfect fluid distribution in Einstein-Cartan theory and obtain the new solutions which are free from singularities.

2. CARTAN EQUATIONS OF STRUCTURE AND THE METRIC WITH CURVATURE

The cartan equations of structure are

$$\Theta^i = D\theta^i$$

$$= d\theta^i + w^i_j \wedge \theta^j$$

$$= \frac{1}{2} Q^i_{jk}\, \theta^j \wedge \theta^k \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

$$\Omega^i_j = dw^i_j + w^i_k \wedge w^k_j$$

$$= \frac{1}{2} R^i_{jkl}\, \theta^k \wedge \theta^l \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

$$Q^i_{jk} - \delta^i_j\, Q^l_{lk} - \delta^i_k\, Q^l_{jl} = -K\, S^i_{jk} \quad \ldots \quad (2.3)$$

Here D denotes the exterior covarient derivative and Q^i_{jk} is the torsion tensor and K = 8Π.

The classical description of spin, we define by the relation

$$S^i_{jk} = u^i S_{jk} \quad \text{with} \quad S_{ij} u^j = 0 \qquad (2.4)$$

Where u^i is the velocity four-vector.

We consider a static conformally flat spherically symmetric metric in the form

$$ds^2 = -e^{2\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 - dt^2) \quad \ldots \quad (2.5)$$

Where μ is a function of r only.

We have then the orthonormal tetrad

$$\left.\begin{array}{ll} \theta^1 = e^{\mu}\, dr, & \theta^2 = re^{\mu} d\theta, \\ \theta^3 = r\sin\theta\, e^{\mu} d\phi, & \theta^4 = e^{\mu} dt \end{array}\right\} \quad \ldots \quad (2.6)$$

The metric (2.5) now takes the form

$$ds^2 = -\left[(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 - (\theta^4)^2 \right] \quad \ldots \quad (2.7)$$

So that

$$g_{ij} = \text{diag}\ \{ -1, -1, -1, 1 \}$$

We assume that the spins of the particles composing the fluid are all aligned in the radial direction only and then we get the only indepedent non-zero component of the spin tensor S_{ij} to be $S_{23} = K$ (say).

The velocity vector $u^i = \delta^i_4$, since from the static condition. Thus the components of S^i_{jk} which are non-zero,

$$S^4_{23} = - S^4_{32} = K \quad \ldots \quad \ldots \quad (2.8)$$

Hence from Cartan's equations (2.3), we get the non-zero components of S^i_{jk} as

$$Q^4_{23} = - Q^4_{32} = - kK \quad \ldots \quad \ldots \quad (2.9)$$

From (2.1) on using (2.8) we have

$$\Theta^1 = 0, \qquad \Theta^2 = 0, \qquad \Theta^3 = 0 \qquad \text{and}$$

$$\Theta^4 = - kK\, \theta^2 \wedge \theta^3 \quad \ldots. \quad \ldots \quad \ldots \quad (2.10)$$

Following Kalyanshetti and Waghmode the Ricci tensor R_{ij} and the curvature scalar R are given by

$$R_{11} = e^{-2\mu} (3\mu'' + 2\frac{\mu'}{r}) \quad \ldots \quad \ldots \qquad (2.11)$$

$$R_{22} = R_{33} = e^{-2\mu} (\mu'' + 2\mu'^2 + 4\frac{\mu'}{r}) \quad \ldots \qquad (2.12)$$

$$R_{44} = - e^{-2\mu} (\mu'' + 2\mu'^2 + 2\frac{\mu'}{r}) - \frac{1}{2}k^2K^2 \qquad (2.13)$$

and

$$R = -[e^{-2\mu} (6\mu'' + 6\mu'^2 + 12\frac{\mu'}{r}) + \frac{1}{2}k^2K^2] \qquad (2.14)$$

3. ENERGY MOMENTUM-TENSOR AND THE EQUATIONS :

The Einstein - Maxwell equations for the perfect fluid are

$$R_{ij} - \frac{1}{2} Rg_{ij} = - 8\pi T_{ij} \quad \ldots \quad \ldots \qquad (3.1)$$

$$[(-g)^{1/2} F^{ij}]_{,j} = 4\pi J^i (-g)^{1/2} \quad \ldots \qquad (3.2)$$

$$F_{[ij;k]} = 0 \quad \ldots \quad \ldots \qquad (3.3)$$

where R_{ij} is the Ricci tensor, T_{ij} is the energy momentum tensor and F_{ij} is the electromagnatic field tensor and J^i is the current four vector.

For the system under the study energy momentum tensor T^i_j splits into two parts t^i_j and E^i_j for the matter part and for the charge part respectively.

$$T^i_j = t^i_j + E^i_j \quad \ldots \quad \ldots \qquad (3.4)$$

The nonvanishing components of t^i_j are

$$t^1_1 = t^2_2 = t^3_3 = -P \quad \text{and} \quad t^4_4 = \rho \qquad \dots \qquad \dots \qquad (3.5)$$

The non vanishing component of F^{ij} is

$$F^{14} = - F^{41}$$

Therefore the non-zero components of E^i_j are

$$E^4_4 = E^1_1 = - E^2_2 = - E^3_3 = - \frac{1}{8\pi} \; g_{44}g_{11}(F^{41})^2$$

Equation (3.3) is obviously satisfied by this choice of F^{ij} whereas (3.2) reduces to

$$F^{41} = \frac{Q(r)}{r^2} \, e^{-2\mu} \qquad \dots \qquad \dots \qquad (3.6)$$

where Q(r) is the charge upto redius r,

$$Q(r) = 4\pi \int_0^r J^4 \, r^2 \, e^{-2\mu} \, dr \qquad \dots \qquad \dots \qquad (3.7)$$

From the equation (3.7) we see that Q(r) is a constant Qo (say) out side the fluid sphere. Then from (3.6) we find the asymptotic form of the electric field as Qo/r^2.

The Einstein-Maxwell equation for the metric (2.5) using (2.11), (2.12), (2.13), (2.14) along with (3.4) and (3.5) give us

$$e^{-2\mu} \left(3\mu'^2 + 4\frac{\mu'}{r}\right) + \frac{1}{4} k^2K^2 = 8\pi P - 8\pi E^1_1 \qquad (3.8)$$

$$e^{-2\mu} \left(2\mu'' + \mu'^2 + 2\frac{\mu'}{r}\right) + \frac{1}{4} k^2K^2$$
$$= 8\pi P - 8\pi E^2_2 \qquad (3.9)$$

$$- e^{-2\mu} \left(2\mu'' + 2\mu'^2 + 4\frac{\mu'}{r}\right) + \frac{1}{4} k^2K^2$$
$$= 8\pi\rho + 8\pi E^4_4 \qquad (3.10)$$

where dashes denote differentiation with respect to r.

These field equations take the form

$$e^{-2\mu}\left[3\mu'^{2}+\frac{4\mu'}{r}\right]=8\pi\bar{P}-8\pi\,E_{1}^{1} \qquad \ldots \qquad (3.11)$$

$$e^{-2\mu}\left[2\mu''+\mu'^{2}+2\frac{\mu'}{r}\right]=8\pi\bar{P}-8\pi\,E_{2}^{2} \qquad \ldots \qquad (3.12)$$

and

$$-e^{-2\mu}\left[2\mu''+2\mu'^{2}+4\frac{\mu'}{r}\right]=8\pi\bar{\rho}+8\pi\,E_{4}^{4} \qquad (3.13)$$

After redefining pressure and density as $\bar{\bar{P}} = (P - 2\pi K^2)$ and $\bar{\rho} = (\rho - 2\pi K^2)$ by following Hehl.

4. SOLUTIONS OF THE FIELD EQUATIONS.

Eliminating $\bar{P}$ between the equations (3.11) and (3.12) we get

$$\mu''-\mu'^{2}-\frac{\mu'}{r}+8\pi\,E_{2}^{2}=0 \qquad \ldots \qquad (4.1)$$

But

$$E_{2}^{2}=\frac{1}{8\pi}\,g_{44}g_{11}(F^{41})^{2} \qquad \ldots \qquad (4.2)$$

Hence in view of this relation the equation (4.1) reduces to

$$\mu''-\mu'^{2}-\frac{\mu'}{r}-\frac{Q^{2}}{r^{4}}=0 \qquad (4.3)$$

To solve the equation (4.3) we assume that

$$Q = Ar^{3} \qquad (4.4)$$

where A is a constant of proportionality, we have then

$$\mu''-\mu'^{2}-\frac{\mu'}{r}-A^{2}r^{2}=0 \qquad \ldots \qquad (4.5)$$

Equation (4.5) is non-linear differential equation. To solve this non-linear differential equation we make the substitution

$$y = e^{-\mu} \qquad (4.6)$$

Then the equation (4.5) takes the form

$$y'' - \frac{y'}{r} + A^2r^2y = 0$$

i.e. $$\frac{d^2y}{dr^2} + R_1 \frac{dy}{dr} + R_2 y = 0 \qquad (4.7)$$

where $R_1 = -1/r$ and $R_2 = A^2r^2$,

Which is a second order linear differential equation.

We solve it by changing the independent variable from r to z, the relation between which is to be obtained. The equation (4.7) may be written in the form

$$\frac{d^2y}{dz^2} + T_1 \frac{dy}{dz} + T_2 y = 0 \qquad (4.8)$$

where

$$T_1 = \frac{R_1 \frac{dz}{dr} + \frac{d^2z}{dr^2}}{\left(\frac{dz}{dr}\right)^2}$$

and $$T_2 = \frac{R_2}{\left(\frac{dz}{dr}\right)^2} \qquad \ldots \qquad (4.9)$$

We assume the relation between r and z such that T_2 = constant = A^2.

Therefore,

$$\frac{A^2r^2}{\left(\frac{dz}{dr}\right)^2} = A^2 \quad \text{or} \quad \frac{dz}{dr} = r ,$$

which on integration gives

$$Z = \frac{r^2}{2} \qquad \ldots \qquad \ldots \qquad (4.10)$$

Then T_1 turns out to be zero and the equation (4.8) will become

$$\frac{d^2y}{dz^2} + A^2y = 0 \quad \ldots \quad (4.11)$$

Equation (4.11) is a linear differential equation with constant coefficients whose solution is

$$y = e^{-\mu} = (C'_1\cos AZ + C'_2 \sin AZ) \quad \ldots \quad (1.12)$$

where C'_1 and C'_2 are arbitrary constants. The metric (2.5), now can be written in the form

$$ds^2 = - \frac{1}{(C'_1 \cos AZ + C'_2 \sin AZ)^2} \times$$

$$\times\left\{ dr^2 + r^2d\theta^2 + r^2\sin^2\theta \, d\phi^2 - dt^2\right\} \quad \ldots . \quad (4.13)$$

5. <u>THE MODEL SOLUTION (When r << 1).</u>

When $r \ll 1$, $\cos AZ = 1$ and $\sin AZ = AZ$. The metric (4.13) then will take the form

$$ds^2 = - \frac{1}{(A_1 + B_1r^2)^2} \times$$

$$\times\left\{ dr^2 + r^2d\theta^2 + r^2\sin^2\theta \, d\phi^2 - dt^2\right\} \quad \ldots . \quad (5.1)$$

where we have set

$$A_1 = C'_1 \quad , \quad B_1 = \frac{1}{2} A \, C'_2 \quad \ldots \quad (5.2)$$

The pressure in the model is giben by the equation (3.11)

Thus we have

$$8\pi P = 4 B_1^2 r^2 + A^2 r^2 - 8B_1 A_1 + 16\pi^2 K^2 \quad \ldots \quad (5.3)$$

The density in the model is

$$8\pi\rho = 12 B_1 A_1 - A^2 r^2 + 16\pi^2 K^2 \quad \ldots \quad (5.4)$$

The spin density K is given by

$$K = A_2 e^{-\mu}$$

$$= A_2 (A_1 + B_1 r^2) \quad \ldots \quad \ldots \quad (5.5)$$

and $8\pi E_4^4 = 8\pi E_1^1 = -8\pi E_2^2 = -8\pi E_3^3 = A^2 r^2$ (5.6)

The costants A_1, B_1 appearing in the solutions can be evaluated by using the central and the boundry conditions.

The constant A_2 appearing in the solution can be evaluated in terms of density ρ_o at the centre from the equation (5.4).

At r = 0, we have

$$(e^{-2\mu})_{r=0} = 1$$

Hence $A_1 = 0$ ... . (5.7)

Pressure $\bar{P}$ at the boundary $r = r_o$, must vanish.

Therefore from (5.3) we get

$$4 B_1^2 r_o^2 - 8 B_1 + A^2 r_o^2 = 0 \quad \ldots \quad (5.8)$$

where $A = Qo / r_o^3$ ... (5.9)

and Q_o is the total charge on the sphere of redius r_o. On solving the equation (5.8) for B_1 ,we have

$$B_1 = \left\{ 2 \pm (4 - r_o^4 A^2)^{1/2} \right\} \Big/ 2 r_o^2 \quad \ldots \quad (5.10)$$

The constant A_2 can be evaluated from the equation (5.4)

$$8\pi\rho_o = 12 B_1 + 16\pi^2 A_2^2$$

Hence $$A = \frac{1}{2\pi} (2\pi\rho_o - 3B_1)^{1/2} \quad \ldots \quad (5.11)$$

6. PROPERTIES OF THE SOLUTIONS .

We have obtained the non-singular solutiions for a static charged conformally flat fluid sphere in Einstein -Cartan theory. The pressure and the density are both finite for the model even at $r = 0$, leading to a satisfactory model for point charge.

In obtaining the solutions we have assumed that the spins of the particles composing the fluid are all aligned in the radial direction only and the charge $Q = Ar^3$, where A is arbitrary constant. Tiwari et al have obtained the result. $q(r) = \frac{4}{3} \pi \rho_o r^3$ in their electromagnatic mass models in GR where pressure is negative. Wang recently has given a class conformally flat solutions for a charged sphere by assuming the mass density. If we get a set A = 0 and B = 1 in his solutions for the case n = 2, we get

$$e^{\nu} = (1 - cr^2)^2 \qquad \text{where } X = cr^2$$

The pressure and density at r = 0 in his case are

$$8\pi P = -8C \quad \text{and} \quad 8\pi\rho = 12C$$

Here pressure is negative and density is positive .

In our case the pressure and density at r = 0 are given by

$$8\pi P_o = -8B_1A_1 + 16\pi^2A_2^2A_1^2$$

$$8\pi\rho_o = 12\,B_1A_1 + 16\pi^2A_2^2A_1^2$$

The second term is due to the spin of the charge particle. The pressure is positive if $16\pi^2A_2^2A_1^2 - 8B_1A_1 \geq 0$. Thus the pressure and density are both positive satisfying the reality conditions $P \geq 0$ and $\rho \geq 0$. In absence of spin our solutions are particular solutions of Wang.

In the absence of charge we have A = 0, $A_2 = 0$ and on taking

$$B_1 = \left\{ 2 - (4 - r_o^4\, A^2)^{1/2} \right\} \Big/ 2\, r_o^2$$

We get $B_1 = 0$. The vanishing of the charge implies the vanishing of spin, the gravitational mass, the pressure and the density. Thus there exists no sphere and every thing would be electromagnatic in origin and we get electromagnatic mass model in Einstein-Cartan theory.

It is interesting to note that in the absence of spin i.e. $A_2 = 0$, we get $B_1 = 0$ and $\rho_0 = 0$. This leads to the important conclusion that the charged particle at the centre of the sphere will be equilibrium if it is spinning.