CHAPTER - III

LIE TRANSPORT

1. INTRODUCTION :

The fact that the Lie transport is of paramount importance to continuum mechanics can be gauged from the following remark of Schutz (1980) P.182 "From the geometric point of view the existence of a flow suggests immediately the use of Lie derivative, ... the local conservation laws become much more transparent when framed with Lie derivatives."

Among all transports the Lie transport has the distinguishing property that it is independent of the Christoffel symbols of the Riemannian space. For instance, we have

$$\pounds_v A_{ab} = A_{ab;k}v^k + A_{kb}v^k_{;a} + A_{ak}v^k_{;b} = A_{ab,k}v^k + A_{kb}v^k_{,a} + A_{ak}v^k_{,b}$$

where comma denotes partial derivative and semicolon denotes covariant derivative.

The most popular spherically symmetric space-times studied in general theory of relativity are expressed through (Eiesland, 1925) the three parameter group of Killing vectors $k^a_{(1)}$, $k^a_{(2)}$, $k^a_{(3)}$, that is,

$$\pounds_{\underset{(i)}{k}} g_{ab} = 0, \quad i = 1, 2, 3 \quad a,b = 1, 2, 3, 4$$

where $$k^a_{(i)} = \begin{bmatrix} 0 & x^3 & -x^2 & 0 \\ -x^3 & 0 & x^1 & 0 \\ x^2 & -x^1 & 0 & 0 \end{bmatrix}.$$

Pirani, (1964), has given the physical significance of the Lie transport of the three dimensional projection operator $(g_{ab} - u_a u_b)$ in the general theory of relativity as characterizing RIGID time-like congruence u^a, where $u^a u_a = 1$. His theorem reads

$$£_u (g_{ab} - u_a u_b) = 0 \quad \text{iff} \quad \theta = 0, \ \sigma_{ab} = 0$$

where θ is the expansion and σ_{ab} is shear.

Lie derivative (coined by Van-Dantzing) provides an intrinsic method of comparing the values of geometrical objects at different points of a manifold. Many research workers, Taub (1951), Takeno (1961), Davis and Katzin (1962), Rosen (1962), Stachel (1962), Pirani (1964), Katzin, Levine and Davis (1969), Yano (1970), Collinson (1970a,b), Aminova (1971), Audretsch (1971), worked on applications of Lie derivative to the general theory of relativity. The role of Lie derivative in the classification of spaces has been comprehensively described by Petrov (1969) in his treatise on Einstein spaces.

Lie derivative of a tensor field along an aribtrary vector field is presented in books on differential geometry (Yano 1955, 1970, Schouten 1954). In this dissertation we specialize the arbitrary vector field to coincide with the flow vector field of a continuum in relativistic continuum mechanics.

<u>Definition</u> : The tensor field x^a_b is defined to be <u>Lie-transported</u> if

$$£_u x^a_b = 0 ,$$

where $£_u x^a_b = x^a_{b;k} u^k - x^k_b u^a_{;k} + x^a_k u^k_{;b}$.

Recently Radhakrishna (1988) has considered Lie transport along the common propagation vector, (a special null vector n^a) of a null electromagnetic field interacting with a null gravitational field and obtained the gravitational potentials satisfying the following conditions separately

i) $\pounds_n R_{ab} = 0$ but $\pounds_n R^h_{abk} \neq 0$.

ii) $\pounds_n R^h_{abk} = 0$ but $\pounds_n \Gamma^a_{bc;d} \neq 0$. (Γ^a_{bc} Christoffel symbol)

iii) $\pounds_n (\Gamma^a_{bc})_{;d} = 0$ but $\pounds_n \Gamma^a_{bc} \neq 0$.

iv) $\pounds_n \Gamma^a_{bc} = 0$ but $\pounds_n g_{ab} \neq 0$.

v) $\pounds_n g_{ab} = 0$.

<u>Higher order Lie-Transports</u> :

(i) An interesting identity

$$(\pounds_u \pounds_v - \pounds_v \pounds_u) \omega^k = \pounds_{\pounds_v u} \omega^k$$

for any 3 vector fields $\bar{u}$, $\bar{v}$, $\bar{\omega}$ exists (Schouten, 19[illegible]) in Ricci Calculus.

(ii) An interesting restriction on the constitutive equation of matter in relativistic continuum mechanics has been reported by Kute (198[illegible]) in the form

$$\pounds_u \pounds_u (g_{ab} - u_a u_b) = \gamma^c_a \gamma^d_b \dot{u}_{(c;d)} + u^{\overset{\perp}{c}}_{;a} u_{\overset{\perp}{c};\overset{\perp}{b}} - \dot{u}_a \dot{u}_b - u^c u^d R_{acbd}$$

$$\overset{\perp}{u}{}^k = \gamma^k_m u^m , \quad \gamma_{ab} = g_{ab} - u_a u_b .$$

(iii) In 1989 Katkar has shown that

$$\pounds_n \pounds_n \pounds_n g_{ab} = 0$$

for Petrov type N fields, where $\bar{n}$ is the common propagation vector of the null electromagnetic fields and the null gravitational field.

2. <u>LIE TRANSPORT OF COVARIANT VECTOR FIELDS</u> (1-forms)

<u>Theorem 1</u> : If v_a is a material tensor, then, $£_u v_a$ is also a material tensor.

<u>Proof</u> : We know that, the Lie derivative of a covariant vector v_a is given by

$$£_u v_a = v_{a;k} u^k + v_k u^k_{;a}$$

consider inner product of $£_u v_a$ with u^a.

$$u^a £_u v_a = u^a (v_{a;k} u^k + v_k u^k_{;a})$$

$$u^a £_u v_a = u^a v_{a;k} u^k + u^a v_k u^k_{;a} \quad \text{... (3.1)}$$

Since v_a is a material vector we have $u^a v_a = 0$ and so

$$(v_a u^a)_{;k} = 0$$

$$v_{a;k} u^a + v_a u^a_{;k} = 0$$

$$v_a u^a_{;k} = - v_{a;k} u^a \quad \text{... (3.2)}$$

Substitute (3.2) in (3.1).

$$u^a £_u v_a = - v_a (u^a_{;k}) u^k + u^a v_k u^k_{;a}$$

$$= - v_a u^k u^a_{;k} + u^k v_a u^a_{;k}$$

$$u^a £_u v_a = 0 .$$

Thus $£_u v_a$ is also a material tensor.

<u>Note</u> : If v^a is a material tensor then $£_u v^a$ is not ingeneral a material tensor. This can be established in the following way.

We know that, the Lie derivative of a contravariant vector v^a is given by

$$\pounds_u v^a = v^a_{;k} u^k - v^k u^a_{;k}$$

Consider inner product of $\pounds_u v^a$ with u_a

$$(\pounds_u v^a) u_a = (v^a_{;k} u^k - v^k u^a_{;k}) u_a .$$

$$= \dot{v}^a u_a , \text{ since } u^a_{;k} u_a = 0, \; v^a_{;k} u^k = \dot{v}^a$$

$$u_a \pounds_u v^a = - v^a \dot{u}_a , \text{ since } v^a \text{ is material tensor, } v^a \dot{u}_a = - \dot{v}^a u_a . \quad ..(3.3)$$

This shows that $\pounds_u v^a$ is <u>not</u> ingeneral material tensor.

<u>Remark</u> : If $v^a \dot{u}_a = 0$ or $\dot{v}^a u_a = 0$

then $\pounds_u v^a$ is a material tensor. In RSF $\pounds_u Q^a$ is a material tensor.

3. <u>COMMUTATIVITY OF TRANSPORTS</u> :

The aim is to get the necessary and sufficient conditions for the Fermi and Lie transports to commute. The result is derived in the following. From the definition of $\pounds_u$ and F_u

$$\pounds_u x^a = \dot{x}^a - x^k u^a_{;k} \quad \ldots (3.4)$$

$$F_u x^a = \dot{x}^a + x^k (\dot{u}^a u_k - \dot{u}_k u^a) \quad \ldots (3.5)$$

$$F_u \pounds_u x^a = (\dot{x}^a - x^k u^a_{;k})^{\cdot} + (\dot{x}^i - x^k u^i_{;k})^k [\dot{u}^a u_\ell - \dot{u}_\ell u^a]$$

$$F_u \pounds_u x^a = \ddot{x}^a - \dot{x}^k u^a_{;k} - x^k \dot{u}^a_{;k} + (\dot{x}^i - x^k u^i_{;k})[\dot{u}^a u_\ell - \dot{u}_\ell u^a] \quad \ldots (3.6)$$

$$\pounds_u F_u x^a = \pounds_u [\dot{x}^a + x^k (\dot{u}^a u_k - \dot{u}_k u^a)]$$

$$= [\dot{x}^a + x^k(\dot{u}^a u_k - \dot{u}_k u^a)]^{\cdot} - [\dot{x}^p + x^k(\dot{u}^p u_k - \dot{u}_k u^p)] u^a_{;p}$$

$$\pounds_u F_u x^a = \ddot{x}^a + \dot{x}^k(\dot{u}^a u_k - \dot{u}_k u^a) + x^k(\dot{u}^a u_k - \dot{u}_k u^a)^{\cdot}$$

$$- [\dot{x}^p u^a_{;p} + x^k(\dot{u}^p u_k - \dot{u}_k u^p) u^a_{;p}] \quad \ldots (3.7)$$

Now, consider

$$(\pounds_u F_u - F_u \pounds_u)x^a = -\ddot{x}^a + \dot{x}^k u^a_{;k} + x^k \dot{u}^a_{;k} - (\dot{x}^i - x^k u^i_{;k})[\dot{u}^a u_\ell - \dot{u}_\ell u^a]$$

$$+\{\ddot{x}^a + \dot{x}^k(\dot{u}^a u_k - \dot{u}_k u^a) + x^k(\dot{u}^a u_k - \dot{u}_k u^a)^{\cdot} - [\dot{x}^p u^a_{;p} + x^k(\dot{u}^p u_k - \dot{u}_k u^p)u^a_{;p}]\}$$

since from (3.6) and (3.7)

$$= x^k[\dot{u}^a u_k - \dot{u}_k u^a - (\dot{u}^p u_k - \dot{u}_k u^p)u^a_{;p} + \dot{u}^a_{;k} + u^l_{;k}(\dot{u}^a u_l - \dot{u}_l u^a)]$$

where $\dot{u}^a_{;k} = (u^a_{;k})_{;p}u^p \neq (\dot{u}^a)_{;k}$

or $(u^a_{;k})^{\cdot} \neq (\dot{u}^a)_{;k}$

We note the absence of terms in $\dot{x}^k$ and $\ddot{x}^k$ in the expression for $(\pounds F - F\pounds)_u x^a$

If $(\pounds_u F_u - F_u \pounds_u)x^a = 0$, for arbitrary x^a, then

$$\dot{u}^a u_k - \dot{u}_k u^a - u^a_{;p}(\dot{u}^p u_k - \dot{u}_k u^p) + \dot{u}^a_{;k} + u^p_{;k}(\dot{u}^a u_p - \dot{u}_p u^a) = 0 .$$

$$k_1 p^a u_k - k_1 p_k u^a - u^a_{;p}(k_1 p^p u_k - k_1 p_k u^p) + \dot{u}^a_{;k} + u^p_{;k}(k_1 p^a u_p - k_1 p_p u^a) = 0,$$

since $\dot{u}^a = k_1 p^a$.

$$k_1[p^a u_k - p_k u^a - u^a_{;\ell}(p^\ell u_k - p_k u^\ell) + u^\ell_{;k} u_\ell p^a - u^\ell_{;k} p_\ell u^a + \dot{u}^a_{;k}] = 0.$$

$$k_1[p^a u_k - p_k u^a + u^a_{;\ell} u^\ell p_k - u^a_{;\ell} p^\ell u_k - u^\ell_{;k} p_\ell u^a] + \dot{u}^a_{;k} = 0$$

For convenience we write this as $B^a_m = 0$ where

$$B^a_m = k_1[p^a u_{\cdot m} - p_m u^a + \dot{u}^a p_{\cdot m} - u^a_{;\ell} p^\ell u_{\cdot m} - u^\ell_{;m} p_\ell u^a] + \dot{u}^a_{;m}$$

$$= k_1[p^a u_{\cdot m} - p_m u^a + k_1 p^a p_m + (\gamma_{122} p^a + \gamma_{132} Q^a + {}_{142} R^a)u_{\cdot m}$$

$$+ (k_1 u_{\cdot m} + \gamma_{122} p_{\cdot m} + \gamma_{123} Q_{\cdot m} + \gamma_{124} R_{\cdot m})u^a] + (k_1 p^a)_{;m}$$

$$= k_1 (p^a + \gamma_{122} p^a + \gamma_{132} Q^a + \gamma_{142} R^a + k_1 u^a) u_m$$

$$+ (-u^a + k_1 p^a + \gamma_{122} u^a) p_m + \gamma_{123} Q_m u^a + \gamma_{124} u^a R_m + k_{1;m} p^a + k_1 p^a{}_{;m}$$

$$B^a{}_m = k_1\{[k_1u^a + (1+\gamma_{122})P^a + \gamma_{132}Q^a + \gamma_{142}R^a]u_m + [(\gamma_{122}-1)u^a + k_1P^a]P_m + [(\gamma_{123}+\gamma_{124})u^a]Q_m + \gamma_{124}u^aR_m\} + [\dot{k}_1u_m - (k_{1;\ell}P^\ell)P_m - (k_{1;\ell}Q^\ell)Q_m - (k_{1;\ell}R^\ell)R_m]P^a + k_1[\gamma_{211}u^au_m - \gamma_{231}Q^au_m - \gamma_{212}u^aP_m + \gamma_{232}Q^aP_m - \gamma_{213}u^aQ_m + \gamma_{233}Q^aQ_m - \gamma_{214}u^aR_m + \gamma_{234}Q^aR_m - \gamma_{241}R^au_m + \gamma_{242}R^aP_m + \gamma_{243}R^aQ_m + \gamma_{244}R^aR_m],$$

on expressing $p^a_{;k}$ as a linear combination of 12 outer products and

$$(k_1)_{;m}P^a = [\dot{k}_1u_m - k_{1;\ell}p^\ell)P_m - (k_{1;\ell}Q^\ell)Q_m - (k_{1;\ell}R^\ell)R_m]P^a .$$

$$B^a_m = k_1\{[(k_1+\gamma_{211})u^a + (\frac{\dot{k}_1}{k_1} + 1 + \gamma_{122})P^a + (\gamma_{132}-\gamma_{231})Q^a + (\gamma_{142}-\gamma_{241})R^a]u_m + [(2\gamma_{122}-1)u^a + (k_1 - \frac{k_{1;\ell}P^\ell}{k_1})P^a + \gamma_{232}Q^a + \gamma_{242}R^a]P_m + [(\gamma_{123}+\gamma_{124}-\gamma_{213})u^a - \frac{k_{1;\ell}Q^\ell P^a}{k_1} + \gamma_{233}Q^a + \gamma_{243}R^a]Q_m + [(\gamma_{124}-\gamma_{214})u^a - \frac{k_{1;\ell}R^\ell P^a}{k_1} + \gamma_{234}Q^a + \gamma_{244}R^a]R_m .$$

Now $B^a_m = 0$ implies that the co-efficient of each outer product like P^au_m, Q^au_m, u^aP_m, . . . must vanish, i.e.,

$$\dot{k}_1/k_1 + 1 + \gamma_{122} = 0,\ \gamma_{132} - \gamma_{231} = 0,\ 2\gamma_{122} - 1 = 0, \text{ etc.}$$

<u>Theorem 2</u> : TFAE

1) $F_u\pounds_u = \pounds_uF_u$

2) (a) $1 + \frac{\dot{k}_1}{k_1} + \gamma_{122} = \frac{k_{1;\ell}Q^\ell P^a}{k_1} = \frac{k_{1;\ell}R^\ell P^a}{k_1}$

$$= k_1 - \frac{k_{1;\ell}P^\ell}{k_1} = k_1 + \gamma_{211} = 0 .$$

(b) $\gamma_{132} - \gamma_{231} = \gamma_{142} - \gamma_{141} = 2\gamma_{122} = \gamma_{232} = \gamma_{242}$

$= \gamma_{123} + \gamma_{124} - \gamma_{213} = \gamma_{233} = \gamma_{243} = \gamma_{124}$

$= \gamma_{234} = \gamma_{244} = 0 .$

4. <u>THE LIE TRANSPORT OF THE RELATIVISTIC SERRET-FRENET TETRAD</u> :

i) $\pounds_u u^a = \dot{u}^a - u^k u^a_{;k}$

$= \dot{u}^a - \dot{u}^a$

$\pounds_u u^a = 0$

ii) $\pounds_u p^a = \dot{p}^a - p^k u^a_{;k}$

$= (k_1 u^a + k_2 Q^a) - [-\gamma_{122} p^a + \gamma_{132} Q^a + \gamma_{142} R^a] .$

by (RSF-2) and computational aids (VI).

$= k_1 u^a + (k_2 + \gamma_{132}) Q^a + \gamma_{122} p^a + \gamma_{142} R^a .$

$\pounds_u p^a = k_1 u^a + \gamma_{122} p^a + (k_2 + \gamma_{132}) Q^a + \gamma_{142} R^a .$

It follows that $\pounds_u p^a = 0$ iff

$k_1 = \gamma_{122} = \gamma_{142} = 0 .$

$k_2 = \gamma_{312} .$

iii) $\pounds_u Q^a = \dot{Q}^a - Q^k u^a_{;k}$

$= (-k_2 p^a + k_3 R^a) - [-(\gamma_{133} Q^a + \gamma_{123} p^a + \gamma_{143} R^a)$

, by (RSF-3) and computational aids (VII).

$$\pounds_u Q^a = (-k_2 + \gamma_{123})P^a + \gamma_{133}Q^a + (k_3 + \gamma_{143})R^a$$

Obviously, $\pounds_u Q^a = 0$ implies and implied by

$$k_2 = \gamma_{123}, \quad \gamma_{133} = 0, \quad k_3 = -\gamma_{143}.$$

iv) $\pounds_u R^a = \dot{R}^a - R^k u^a_{;k}$

$$= -k_3 Q^a - [-(\gamma_{143}Q^a + \gamma_{124}P^a + \gamma_{144}R^a)]$$

, by (RSF-4) and computational aids (VIII).

$$\pounds_u R^a = \gamma_{124}P^a + (-k_3 + \gamma_{143})Q^a + \gamma_{144}R^a.$$

Consequently, we have,

$\pounds_u R^a = 0$ When and only when $\gamma_{124} = \gamma_{144} = 0$, $k_3 = \gamma_{143}$.

5. LIE TRANSPORT IN CAUSAL THERMODYNAMICS :

Classical thermodynamics suffers from the defect of the prediction of infinite speed of heat propagation. But according to relativity no interaction can propogate faster than light. Eckart (1940) Landau & Lifshitz (1958) developed relativistic thermodynamics but unfortunately the infinite speed of heat propagation defect remained. The credit of introducing a flawless theory of relativistic thermodynamics goes to Carter (1988).

In this 'regular' theory of thermodynamics he introduced four vectors constructed from the flow vector u^a, namely

particle current	: $n^a = nu^a$
entropy current	: $s^a = su^a$
chemical momentum	: $\chi^a = \chi u^a$
thermal momentum	: $\theta^a = \theta u^a$.

where n is particle density, s is entropy, χ is chemical potential and θ is temperature.

In an attempt to extend the study of Lie Transports to 'regular' thermodynamics, we investigate

$$\pounds_{\lambda u} g_{ab} = 0, \qquad \pounds_{\lambda u} \zeta^{a}_{(k)} = 0$$

where λ is any nonzero scalar and the Serret-Frenet tetrad

$$\zeta^{a}_{(k)} = \{ u^a, P^a, Q^a, R^a \}$$

<u>Theorem 3</u> : TFAE

1) $\pounds_{\lambda u} g_{ab} = 0$.

2) $\dot{\lambda} = 0$.

$\lambda_{,a} Q^a = 0$.

$\lambda_{,a} R^a = 0, \quad \lambda_{,a} P^a = -k_1$.

$\gamma_{144} = \gamma_{133} = \gamma_{122} = 0$.

$\gamma_{123} + \gamma_{132} = 0$.

$\gamma_{134} + \gamma_{143} = 0$.

$\gamma_{124} + \gamma_{142} = 0$.

where λ is a scalar function.

<u>Proof</u> :

$$\pounds_{\lambda u} g_{ab} = (\lambda u_a)_{;b} + (\lambda u_b)_{;a} = 0$$

$$\lambda_{,a} u_b + \lambda_{,b} u_a + \lambda (u_{a;b} + u_{b;a}) = 0 \qquad \text{... (3.8)}$$

Suppose this equation is written as $x_{ab} = 0$

where $x_{ab} = \lambda_{,a} u_b + \lambda_{,b} u_a + \lambda (u_{a;b} + u_{b;a})$

These are 10 equations because a, b take the values 1, 2, 3, 4 and $x_{ab} = x_{ba}$.

To get all the ten conditions clearly, we consider the ten contractions separately.

1) $x_{ab}u^a u^b = 0$ implies $\lambda_{,a}u^a + \lambda_{,b}u^b + \lambda(u_{a;b} + u_{b;a}) = 0$
implies $2\dot{\lambda} = 0$ implies $\dot{\lambda} = 0$.

2) $x_{ab}u^a P^b = 0$ implies $\lambda_{,b}P^b + \lambda \dot{u}_b P^b = 0$
implies $\lambda_{,b}P^b = k_{1\lambda}$, by (RSF-1).
implies $(\frac{\lambda_{,b}}{\lambda})P^b = k_1$.

3) $x_{ab}u^a Q^b = 0$ implies $\lambda_{,b}Q^b + \lambda(u_{a;b}u^a Q^b + \dot{u}_b Q^b) = 0$
implies $\lambda_{,b}Q^b = 0$

4) $x_{ab}u^a R^b = 0$ implies $\lambda_{,b}R^b = 0$

5) $x_{ab}P^a Q^b = 0$ implies $\lambda(u_{a;b}P^a Q^b + u_{a;b}P^b Q^a) = 0$
since $\lambda \neq 0$ implies $u_{a;b}(P^a Q^b + P^b Q^a) = 0$
implies $\gamma_{123} + \gamma_{132} = 0$.

6) $x_{ab}P^a R^b = 0$ implies $\lambda(u_{a;b}P^a R^b + u_{b;a}P^b R^a) = 0$
since $\lambda \neq 0$ implies $u_{a;b}(P^a R^b + R^a P^b) = 0$
implies $\gamma_{124} + \gamma_{142} = 0$.

7) $x_{ab}Q^a Q^b = 0$ implies $\lambda(u_{a;b}Q^a Q^b + u_{b;a}Q^a Q^b) = 0$.
since $\lambda \neq 0$ implies $u_{a;b}Q^a Q^b = 0$.
implies $\gamma_{133} = 0$.

8) $x_{ab}Q^a R^b = 0$ implies $\lambda(u_{a;b}Q^a R^b + u_{b;a}Q^a R^b) = 0$
since $\lambda \neq 0$ implies $u_{a;b}(Q^a R^b + R^a Q^b) = 0$
implies $\gamma_{134} + \gamma_{143} = 0$

9) $x_{ab}R^a R^b = 0$ implies $\gamma_{144} = 0$.

10) $x_{ab}P^a P^b = 0$ implies $u_{a;b}P^a P^b = 0$
implies $\gamma_{122} = 0$.

Theorem 4 : The tetrad $\{u^a, P^a, Q^a, R^a\}$ will be Lie transported along λu^a

iff (1) $\dot{\lambda} = 0, \quad (\lambda \neq 0)$

(2) $k_1 = P^k \frac{\lambda_{;k}}{\lambda}$

(3) $\lambda_{;k} R^k = \lambda_{;k} Q^k = 0$

(4) $k_2 = \gamma_{312} = \gamma_{123}$

(5) $k_3 = 0$.

(6) $\gamma_{124} = \gamma_{133} = \gamma_{122} = \gamma_{142} = 0$

$\gamma_{144} = \gamma_{413}$

Proof (I) : $\pounds_{\lambda u} u^a = u^a_{;k} (\lambda u^k) - u^k(\lambda u^a)_{;k}$

$= \lambda u^k u^a_{;k} - \lambda_{;k} u^k u^a - u^a_{;k} u^k \lambda$

$\pounds_{\lambda u} u^a = - \dot{\lambda} u^a$

Therefore, the necessary and sufficient condition for

$\pounds_{\lambda u} u^a = 0$ is $\dot{\lambda} = 0$.

(II) $\pounds_{\lambda u} P^a = P^a_{;k}(\lambda u^k) - P^k (\lambda u^a)_{;k}$, by definition of Lie derivative.

$= \lambda \dot{P}^a - (P^k \lambda_{,k}) u^a - P^k (u^a_{;k}) \lambda$

$= (k_1 \lambda - P^k \lambda_{;k}) u^a + \lambda k_2 Q^a - P^k u^a_{;k} \lambda$, by (RSF-2).

$= (k_1 \lambda - P^k \lambda_{;k}) u^a + \lambda k_2 Q^a - [-(\gamma_{122} P^a + \gamma_{132} Q^a + \gamma_{142} R^a)] \lambda$

, by computational aids (VI) .

$\pounds_{\lambda u} P^a = (k_1 \lambda - P^k \lambda_{;k}) u^a + \gamma_{122} P^a + \lambda (k_2 + \gamma_{132}) Q^a + \lambda \gamma_{142} R^a$.

Therefore, the necessary and sufficeint conditions for

$\pounds_{\lambda u} P^a = 0$ are $k_1 \lambda = P^k \lambda_{;k}$ implies $k_1 = P^k \frac{\lambda_{;k}}{\lambda}$

$\gamma_{122} = \gamma_{142} = 0$

$k_2 = - \gamma_{132}$

(III) $\pounds_{\lambda u}Q^a = Q^a_{;k}(\lambda u^k) - Q^k(\lambda u^a)_{;k}$, by definition of Lie derivative

$= \lambda\dot{Q}^a - (Q^k{}_{\lambda,k})u^a - \lambda Q^k(u^a_{;k})$

$= \lambda(-k_2P^a + k_3R^a) - Q^k{}_{\lambda,k}u^a - \lambda Q^k u^a_{;k}$ by (RSF - 3).

$= -Q^k{}_{\lambda,k}u^a - \lambda(k_2P^a + k_3R^a + u^a_{;k}Q^k)$

$= -(Q^k{}_{\lambda,k})u^a + \lambda(-k_2P^a + K_3R^a)$,

$-\lambda[-(\gamma_{133}Q^a + \gamma_{123}P^a + \gamma_{143}R^a)]$

by computational aids (VII)

$= (-Q^k{}_{\lambda,k})u^a + \lambda(-k_2 + \gamma_{123})P^a + \lambda\gamma_{133}Q^a + \lambda(k_3 + \gamma_{143})R^a.$

Therefore, the necessary and sufficient conditions for

$\pounds_{\lambda u}Q^a = 0$ are $Q^k{}_{\lambda,k} = \gamma_{133} = 0$

$k_2 = \gamma_{123}$.

$k_3 = \gamma_{413}$.

(IV) $\pounds_{\lambda u}R^a = R^a_{;k}(\lambda u^k) - R^k(\lambda u^a)_{;k}$, by definition of Lie derivative.

$= \lambda\dot{R}^a - (R^k{}_{\lambda,k})u^a - R^k(u^a_{;k})\lambda$.

$= -\lambda k_3Q^a - R^k_{\lambda,k}u^a - \lambda[-(\gamma_{143}Q^a + \gamma_{124}P^a + \gamma_{144}R^a)]$

by computational aids (VIII) and (RSF-4).

$= -R^k{}_{\lambda,k}u^a + \gamma_{124}P^a + \lambda(-k_3 + \gamma_{143})Q^a + \lambda\gamma_{144}R^a.$

Consequently the four cinditions for $\pounds_{\lambda u}R^a = 0$ are equivalent to

$R^k{}_{\lambda;k} = \gamma_{124} = \gamma_{144} = 0.\quad k_3 = \gamma_{143}$.

Hence from (I) to (IV) the tetrad $\{u^a, P^a, Q^a, R^a\}$ will be Lie transported iff

1) $\dot{\lambda} = 0, \quad (\lambda \neq 0)$.

2) $k_1 = P^k \dfrac{\lambda_{;k}}{\lambda}$.

3) $\lambda_{;k} R^k = \lambda_{;k} Q^k = 0$.

4) $k_2 = \gamma_{312} = \gamma_{123}$.

5) $k_3 = 0$.

6) $\gamma_{124} = \gamma_{133} = \gamma_{122} = \gamma_{142} = 0$.

$\gamma_{144} = \gamma_{413}$.

REFERENCES

AMINOVA, A.V. (1971). Gravitational fields that admit groups of projective motions, Soviet Physics, Doklady, 16, 4, P. 294.

AUDRETSCH, J. Inertial Reference Frames in Einstein's Theory of Gravitation. Int. J. of Theor. Phy. 21 No.1, 1-9.

BARRABES, C. (1984). Causal Relativistic Thermodynamics of Transports Processes in Electromagnetic Continuous Media, Lecture notes in Maths. 212, pp. 54-56, Springer Verlag.

CARTER, B. (1988), Conductivity with causality in Relativistic hydrodynamics, The regular solution to Eckarts Problem. "Highlights in Gravitation and Cosmology" Cambridge University Press (Ed. Iyer B.R. et al.) p. 58-66.

COLLINSON, C.D. (1970a). Conservation Laws in General Relativity upon the Existence of Preferred Collineations, Gen. Rel. Grav.1, 137-142.

COLLINSON, C.D. (1970). Curvature Collineations in Empty Space times, J. Math. Phys. 11, 818-819.

DANTZIG, O. Van (1932). Cur allgelinen differential Geometric I, Proc. Kon. Akad. Amsterdam, pp. 35, 524.

DAVIS, W.R. and KATZIN,G.H. (1962). Mechanical Conservation Laws and the Physical Properties of Groups of Motions in Flat and Curved Space-times, Amer. J. Phys., 30, 750-764.

ECKART, C. (1940). Phys. Rev. 58, p. 919.

EIESLAND, J. (1925). The Groups of Motions of an Einstein Space, Trans. Amer. Math. Soc., 27, 213.

KATZIN, G.H. et al. (1969). Curvature Collineations : A Fundamental Symmetry Property of the space-time of General Relativity Defined by the Non Vanishing Lie Derivatives of the Riemann Curvature Tensor, J. Math. Phys., 10, 617-629.

LANDAU, L. and LIFSHITZ, E.M. (1958). Fluid Mechanics, Ch. XV : Relativistic Fluid Dynamics, pp. 499-506.

PETROV, A.N. (1969). Einstein Spaces. (Oxford : Pergman Press) Ed. J. Woodrow.

PIRANI, F.A.E. (1964). Lectures on General Relativity, Ed. S. Deser and K.W. Ford, Prentice Hall, Inc, New Jersey.

RADHAKRISHNA, L. et al. (1985). Rheometro dynamics, Proc. Workshop in Solid Mechanics, Rurkee University, pp. 197-203.

RADHAKRISHNA, L. (1988). The Five Models of Pure Radiation Fields with certain Invariant Tensor Fields, in Proc. International Conference on Mathematical Modelling in Science and Technology, pp. 305-311. (Indian Institute of Technology, Madras).

ROSEN, G. (1962). Symmetries of the Einstein-Maxwell Equations. J. Math. Phys. 3, 313-318.

SCHOUTEN, J.A. (1954). Ricci Calculus (Berlin : Springer Verlag).

SCHUTZ, B.F. (1980). Geometrical Methods of Mathematical Physics. (Cambridge University Press) P. 182.

STACHEL, J.J. (1962). Lie derivative and the cauchy problem in the general theory of relativity, thesis submitted to Stevens Institute of Technology, Castle Point, Hoboken, New Jerrsey.

TAKENO, H. (1966). The theory of Spherically symmetric space-times : Revised edition, Res. Inst. for Theor. Phy. Hiroshima, Japan.

TAUB, A.H. (1951). Empty Space-Times Admitting a Three Parameter Group of Motions,. Ann. Phys. 53, 472-490.

YANO, K. (1955). The Theory of Lie Derivatives and Its Application (Amsterdam : North-Holland Publ. Co.).

YANO, K. (1970). Integral formulae in Riemannian geometry, P. 44, Marcel Dekker, Inc. New York.