CHAPTER - IV

CARTER-QUINTANA'S TRANSPORT

1. INTRODUCTION :

In classical Continuum mechanics Oldroyd (1950) has invented the convective derivative as generalization of material derivative through the formula

$$\overset{*}{A}{}^{..j.}_{.i..} = \overset{o}{A}{}^{..j.}_{.i..} + v^{k}_{;i} A^{..j.}_{.k..} - v^{i}_{;k} A^{..k.}_{.i..}$$

where $$\overset{o}{A}{}^{..j.}_{.i..} = \frac{\delta}{\delta t} A^{..j.}_{.i..}$$

$$= \frac{\partial}{\partial t} A^{..j.}_{.i..} + v^{k} A^{..j.}_{.i..;k}$$

with $v^{k} = \frac{dx^{k}}{dt}$ and $\frac{\delta}{\delta t}$ is the material derivative (also called as intrinsic derivative). For the query, what is the physical significance of the operation of convective differentiation with respect to time, applied to a tensor intrinsically associated with a material of a moving continuum, Oldroyd's answer is that it is a kind of total differentiation following a material element which introduces no dependence on a fixed frame of reference or on the way the material is moving in space (vide p.42, Fredrickson 1964).

In relativistic continuum mechanics, the time has no absolute significance and hence the convective differentiation with respect to time has to be suitably modified. This extension was accomplished by Oldroyd in 1970. However, this extension had a limitation in the sense that the relativistic convective derivative was confined to material tensor, i.e., tensors orthogonal to the flow vector.

Through the convective operator $\overset{*}{C}_u$ with respect to the timelike flow vector u^a of a mixed tensor field A^a_b :

$$\overset{*}{C}_u A^a_b = A^a_{b;k}u^k + A^a_k u^k_{;b} - A^k_b (u^a_{;k} - u^a \dot{u}_k) \qquad \text{... (4.1)}$$

where $A^a_b u_a = 0, \quad A^a_b u^b = 0,$

we refer A^a_b as <u>convective transported</u> iff $\overset{*}{C}_u A^a_b = 0.$

This concept has been generalized to arbitrary (non-material) tensors by Carter and Quintana in 1972 in the following pattern :

$$C_u X^a_b = X^a_{b;c}u^c + X^b_c (u^c_{;a} - u^c \dot{u}_a) - X^c_a (u^b_{;c} - u^b \dot{u}_c) \ . \qquad \text{... (4.2)}$$

We observe that (4.2) reduces to (4.1) when $X^b_c u^c = 0$. Also

$C_u X^a_b = \pounds_u X^a_b$ in the case of geodesic flow i.e. $\dot{u}_a = 0$.

In this chapter we propose the idea of <u>Carter-Quintana transport</u> of a tensor field by means of the following characterization :

<u>Definition</u> : <u>Carter-Quintana transport</u> :

A tensor field $X^{.b..}_{..a.}(x^i)$ is said to be transported in the sense of Carter and Quintana, if and only if

$$C_u X^{.b..}_{..a.} = X^{.b..}_{..a;c} u^c + X^{.b..}_{..c.} (u^c_{;a} - u^c \dot{u}_a)$$
$$- X^{.c..}_{..a.} (u^b_{;c} - u^b \dot{u}_c) = 0 \ .$$

<u>Some special cases</u> : For ready reference we record the Carter-Quintana's convective derivative of the vector field A_a as

$$C_u A_a = \dot{A}_a + A_c (u^c_{;a} - u^c \dot{u}_a) \qquad \ldots (4.3)$$

Similarly, for the contravariant vector field B^b, we have,

$$C_u B^b = \dot{B}^b - B^c (u^b_{;c} - u^b \dot{u}_c) \quad . \qquad \ldots (4.4)$$

i) Putting $A_a = u_a$ in (4.3), we get,

$$C_u u_a = 0$$

ii) When $B^b = u^b$, we have,

$$C_u u^b = 0, \quad \text{identically.}$$

iii) <u>Covariant Material Vector Fields</u> : When A_a is a material vector, we have, $A_a u^a = 0$ and formula (4.3) gives us

$$C_u A_a = £_u A_a \quad .$$

Note : If $A_a = \dot{u}_a$, the acceleration vector field, we get

$$C_u \dot{u}_a = £_u \dot{u}_a = \ddot{u}_a + \dot{u}_k u^k{}_{;a} \quad .$$

2. <u>CARTER-QUINTANA TRANSPORT OF THE GRAVITATIONAL POTENTIALS</u> :

For brevity we use CQ-transport in place of Carter-Quintana's transport in this chapter henceforth.

We examine the CQ-transport of the gravitational potentials g_{ab}, a significant tensor in general relativity.

Theorem 1 : $C_u g_{ab} = 0$ iff $\Theta = 0$, $\sigma_{ab} = 0$.

Proof : We have the notation

$$\gamma_{ab} = g_{ab} - u_a u_b \text{ or } \gamma^b_a = \delta^b_a - u_a u^b \qquad \dots (4.5)$$

$$A_{\underset{a}{\perp}\underset{b}{\perp}} = \gamma^c_a \gamma^d_b A_{c;d} \qquad \dots (4.6)$$

We consider,

$$u_{a;b} = \delta^c_a \delta^d_b u_{c;d}$$

$$= (\gamma^c_a + u^c u_a)(\gamma^d_b + u^d u_b) u_{c;d}, \quad \text{by (4.5)}$$

$$= \gamma^c_a \gamma^d_b u_{c;d} + \gamma^c_a u^d u_b u_{c;d} \qquad \text{since } u^c u_{c;d} = 0$$

$$= u_{\underset{a;}{\perp}\underset{b}{\perp}} + \gamma^c_a \dot{u}_c \dot{u}_b \qquad \text{by (4.6)}$$

$$= u_{\underset{a}{\perp};\underset{b}{\perp}} + (\delta^c_a - u^c u_a)\dot{u}_c u_b \qquad \text{by (4.1)}$$

$$= u_{\underset{a}{\perp};\underset{b}{\perp}} + \dot{u}_a u_b, \qquad \text{since } u^c \dot{u}_c = 0$$

Hence, we get,

$$u_{a;b} - \dot{u}_a u_b = u_{\underset{a}{\perp};\underset{b}{\perp}}$$

The decomposition of flow gradient is

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + \tfrac{1}{3}\Theta\gamma_{ab} + \dot{u}_a u_b \qquad \dots (4.7)$$

and accordingly

$$u_{a;b} - \dot{u}_a u_b = u_{\underset{a}{\perp};\underset{b}{\perp}} = \sigma_{ab} + \omega_{ab} + \tfrac{1}{3}\Theta\gamma_{ab} \qquad \dots (4.8)$$

Now, the formula for the CQ-derivative with respect to flow of a second rank covariant tensor gives

$$C_u g_{ab} = g_{ab;k}u^k + g_{kb}\,(u^k_{;a} - u^k\dot{u}_a) + g_{ak}\,(u^k_{;b} - u^k\dot{u}_b)$$

$$= u_{b;a} - \dot{u}_a u_b + u_{a;b} - \dot{u}_b u_a, \quad \text{since } g_{ab} \text{ are covariant constant.}$$

$$= u_{\perp a;\perp b} + u_{\perp b;\perp a}, \quad \text{by (4.6)}$$

$$= \sigma_{ab} + \omega_{ab} + \tfrac{1}{3}\Theta\gamma_{ab} + \sigma_{ba} + \omega_{ba} + \tfrac{1}{3}\Theta\gamma_{ba} \quad \text{by (4.8)}$$

$$C_u g_{ab} = 2\,(\sigma_{ab} + \tfrac{1}{3}\Theta\gamma_{ab}), \quad \text{since } \omega_{ab} = -\,\omega_{ba} \qquad \dots (4.9)$$

Now, we consider,

$$g^{ab}C_u g_{ab} = 2g^{ab}\,(\sigma_{ab} + \tfrac{1}{3}\Theta\gamma_{ab})$$

$$= 2g^{ab}\sigma_{ab} + 2\,(g^{ab}\gamma_{ab})\,\frac{\Theta}{3}$$

Thus, we have

$$g^{ab}C_u g_{ab} = 2\Theta, \quad \text{since } \sigma^a_a = 0, \quad \gamma^a_a = 3.$$

Consequently,

$C_u g_{ab} = 0$ implies $\Theta = 0$ which implies

$$\sigma_{ab} = 0 \qquad \text{by (4.5)}$$

Conversely, when $\Theta = 0$, $\sigma_{ab} = 0$, we get

$$C_u g_{ab} = 0 .$$

This completes the proof of the theorem.

Interpretation : The physical significance of the theorem is that the gravitational potentials are CQ-transported iff the flow of the continuum is expansion-free and shearfree (i.e. rigid).

3. CQ-TRANSPORT OF THE RELATIVISTIC SERRET-FRENET FRAME :

We observe that,

i) $C_u u^a = 0$, identically.

ii) $C_u P^a = \dot{P}^a - P^k(u^a_{;k} - u^a \dot{u}_k)$, by definition of CQ-transport.

$$= K_1 u^a + K_2 Q^a + \gamma_{122} P^a + \gamma_{132} Q^a + \gamma_{142} R^a - K_1 u^a,$$

by(RSF - 2)

$$C_u P^a = \gamma_{122} P^a + (K_2 + \gamma_{132}) Q^a + \gamma_{142} R^a .$$

$$C_u P^a = 0 \quad \text{iff} \quad \gamma_{122} = \gamma_{142} = 0, \quad K_2 = -\gamma_{132}$$

iii) Similarly by using (RSF- 3), we get the relation

$$C_u Q^a = -(K_2 + \gamma_{123}) P^a - \gamma_{133} Q^a + (K_3 - \gamma_{143}) R^a.$$

and so $C_u Q^a = 0$ are equivalent to $K_2 = \gamma_{123}, \gamma_{133} = 0, K_3 = \gamma_{143}$.

iv) Adopting (RSF- 4) in the expression for $C_u R^a$, we get,

$$C_u R^a = -\gamma_{124} P^a - (K_3 + \gamma_{134}) Q^a - \gamma_{144} R^a.$$

Accordingly, $C_u R^a = 0$ implies and is implied by

$$\gamma_{124} = 0, \quad K_3 = -\gamma_{134}; \quad \gamma_{144} = 0$$

We have now proved the

Theorem 2 : The necessary and sufficient conditions that the relativistic Serret-Frenet tetrad $\{ u^a, P^a, Q^a, R^a \}$ is CQ-transported are

$$\gamma_{122} = \gamma_{133} = \gamma_{144} = \gamma_{142} = \gamma_{124} = 0$$

$$K_2 = \gamma_{123} = \gamma_{312}$$

$$K_3 = \gamma_{143} = \gamma_{314} .$$

4. CQ-TRANSPORT OF THE - 2 DIMENSIONAL PROJECTION OPERATOR

There is no interest in the CQ-transport of the 3-dimensional projection operator,

$$\gamma_{ab} = g_{ab} - u_a u_b$$

since $C_u u_a = 0$ and $C_u g_{ab}$ is already discussed. Now we turn our attention to the 2-dimensional projection operator

$$P_{ab} = g_{ab} - u_a u_b + P_a P_b \ .$$

(Note : $P^a_a = 2$, $P_{ab} P^b_{\cdot c} = P_{ac}$); Since $\{ u^a, P^a, Q^a, R^a \}$ is an orthonormal tetrad, we get a more convenient expression for P_{ab} by adopting the completeness relation

$$g_{ab} = u_a u_b - P_a P_b - Q_a Q_b - R_a R_b \ .$$

Therefore, we have,

$$P_{ab} = g_{ab} - u_a u_b + P_a P_b = - (Q_a Q_b + R_a R_b) \ .$$

We now obtain a theorem on the CQ-transport of P_{ab} .

Theorem 3 : TFAE

1) $C_u P_{ab} = 0$

2) $K_2 = - \gamma_{132}$, $\gamma_{134} = - \gamma_{143}$, $\gamma_{133} = \gamma_{144} = 0.$

Proof : Consider the two dimensional projejction operator

$$P_{ab} = - (Q_a Q_b + R_a R_b)$$

$$C_u P_{ab} = - C_u (Q_a Q_b + R_a R_b)$$

$$= -\{ Q_b [\dot{Q}_a + Q_c (u^c_{;a} - u^c \dot{u}_a)] + Q_a [\dot{Q}_b + Q_c (u^c_{;b} - u^c \dot{u}_b)]$$

$$+ R_b [\dot{R}_a + R_c (u^c_{;a} - u^c \dot{u}_a)] + R_a [\dot{R}_b + R_c (u^c_{;b} - u^c \dot{u}_b)]\}$$

, by definition of C_u .

$$= - [Q_b (-K_2 P_a + K_3 R_a) + Q_a (-K_2 P_b + K_3 R_b) + R_b(-K_3 Q_a) + R_a(-K_3 Q_b)$$

$$+ Q_c U^c_{;a} Q_b + u^c_{;b} Q_c Q_a + u^c_{;a} R_c R_b + u^c_{;b} R_c R_a] ,$$

by (RSF - (1) and (2)), $Q_c u^c = R_c u^c = 0$.

$$= K_2 (Q_b P_a + Q_a P_b) - (U^c_{;a} Q_c Q_b + u^c_{;b} Q_c Q_a + u^c_{;a} R_c R_b + u^c_{;b} R_c R_a).$$

$$= K_2 (Q_a P_b + Q_b P_a) + (\gamma_{132} P_a + \gamma_{133} Q_a + \gamma_{134} R_a) Q_b$$

$$+ (\gamma_{132} P_b + \gamma_{133} Q_b + \gamma_{134} R_b)Q_a + (\gamma_{142} P_a + \gamma_{143} Q_a + \gamma_{144} R_a) R_b$$

$$+ (\gamma_{142} P_b + \gamma_{143} Q_b + \gamma_{144} R_b) R_a$$, by computational aids, chapter-I.

$$C_u P_{ab} = (K_2 + \gamma_{132}) (P_a Q_b + P_b Q_a) + 2 \gamma_{133} Q_a Q_b + 2 \gamma_{144} R_a R_b$$

$$+ (\gamma_{134} + \gamma_{143}) (R_a Q_b + R_b Q_a)$$

Hence, we have,

$C_u P_{ab} = 0$ if and only if $K_2 + \gamma_{132} = 0$,

$$\gamma_{133} = \gamma_{144} = \gamma_{134} + \gamma_{143} = 0 .$$

5. NON-COMMUTATIVITY OF C_u WITH RAISING/LOWERING INDEX :

We note that ∇_u commutes with raising and lowering of indices, if ∇_u represents covariant derivative with respect to u, since

$$\nabla_u (g_{ab}A^b) = (\nabla_u g_{ab})A^b + g_{ab} (\nabla_u A^b)$$

$$= g_{ab}\nabla_u A^b, \text{ because } g_{ab;c} = 0 \text{ and}$$

$$\nabla_u g_{ab} = g_{ab;c}u^c = 0 .$$

Thus,

$$(\nabla_u g_{ab})A^b = (g_{ab}\nabla_u) A^b ,$$

which shows that ∇_u commutes with raising and lowering of indices. Also, we note

$$\nabla_u A^a = 0 \quad \text{iff } \nabla_u A_a = 0.$$

We investigate whether such nice property is shared by CQ-transport.

We consider the following question :

<u>If A^a is CQ-transported, then under what circumstances is A_a CQ - transported</u> ?

We have,

$$A_a = g_{ab}A^b$$

$$C_u A_a = C_u (g_{ab}A^b)$$

$$= (C_u g_{ab})A^b + g_{ab} (C_u A^b) , \text{ by Leibnitz rule.}$$

$$= 2\Theta_{ab}A^b , \text{ since } A^b \text{ is CQ-transported.}$$

Thus, $C_u A_a \neq 0$ in general. Also $\Theta_{ab} A^b = 0$ need not imply that $\Theta_{ab} = 0$. Obviously $u^a C_u A_a = 0$, since $\Theta_{ab} u^a = 0$.

Special case : When the continuous medium is rigid, then, $\Theta_{ab} = 0$ and $C_u g_{ab} = 0$ and consequently $C_u A_a = 0$.

Thus, in general $(C_u g_{ab}) A^b \neq g_{ab} (C_u A^b)$ which implies that C_u does not commute with raising or lowering of indices.

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