PREFACE

In classical continuum mechanics several transports or time rates expressed in terms of Jaumann derivative, Convective derivative, Material derivative are used to study the kinematics and dynamics of elastic, plastic materials, viscous fluids, materials with memory (Fredrickson, 1964). If we want to extend the study of these transports to the relativistic domain there are some hurdles. For instance time has no absolute status in the theory of relativity. Hence the time rate of kinematical variables in classical continuum mechanics has to be suitably changed. This is achieved by replacing the time rate, by covariant differentiation along the TIME-LIKE flow vector of the continuum, since time-like vectors have absolute character in special as well as in general relativity (Oldroyd, 1970). In relativistic mechanics the motion of a continuous medium is described by a congruence of time-like world lines in space-time or equivalently by a unit time-like vector field u^a. We use an overhead dot to denote the covariant derivative along u^a.

Definition : The set of orthonormal vectors $\{u^a, P^a, Q^a, R^a\}$ satisfying the conditions (Davis, 1970)

$$\dot{u}^a = k_1 P^a$$

$$\dot{P}^a = k_1 u^a + k_2 Q^a$$

$$\dot{Q}^a = -k_2 P^a + k_3 R^a$$

$$\dot{R}^a = - k_3 Q^a$$

is called the relativistic Serret-Frenet tetrad, where k_1 , k_2 , k_3 are the curvature, torsion, bitorsion scalar fields of the world line of a particle in space-time. Here P^a is the unit acceleration vector field of the particle and (Magdum, 1988)

$$Q^a = \frac{1}{k_2} \left(\frac{\ddot{u}^a}{k_1} + \frac{\dot{k}_1}{k_1^2} \dot{u}^a - k_1 u^a \right)$$

$$R^a = \frac{-1}{k_1^2 k_2} \eta^{abcd} u_b \dot{u}_c \ddot{u}_d .$$

Strong and rapidly changing gravitational fields are prevalent in neutron stars, which deal with high pressure ($P \simeq \rho C^2$), high speeds ($V \simeq C$) of supermassive bodies ($M \simeq \frac{RC^2}{2G}$). Here ρ is the density, M is the mass, R is the radius of the star, G is the constant of gravitation and C is the velocity of light(viz., 1,86,000 miles per second). It is in such astrophysical circumstances the general relativistic effects are significant.

In this dissertation we confine ourselves to the kinematical aspects of the general relativistic continuum mechanics. Each chapter is devoted to one type of transport; that is a special kind of derivative along the time-like flow vector. A new transport is constructed in the last chapter. After historical remarks in Section 1, of each chapter, we present theorems (believed to be new) which contain both the necessary as well as the sufficient conditions for the existence of the transports of certain famous tensor fields and the results seem elegant when couched in terms of the Ricci rotation coefficients constructed from the relativistic Serret-Frenet tetrad.

Chapter I, Section 1 starts with some historical remarks on relativistic continuum mechanics, exposition of relativistic Serret-Frenet tetrad. The section ends with the decomposition of the tensor gradient of the flow field in terms of the Ricci coefficients of rotation constructed from the relativistic Serret-Frenet tetrad. The concomitant expressions for the symmetric shear field, skew-symmetric rotation field and the expansion scalar field associated with the time-like flow vector field are evaluated as computational aids for later utilization.

The Section 2 of the first chapter is devoted to Jaumann transport which is superior in many respects to other transports as per the observations of Prager (1961). The formal extension of this operator to relativistic continuum mechanics was made by Radhakrishna et al. (1981).

Definition : A tensor field $A^{\cdot\cdot a\cdot}_{\cdot b\cdot\cdot}$ is said to be Jaumann transported if and only if

$$J_u A^{\cdot\cdot a\cdot}_{\cdot b\cdot\cdot} = 0$$

where $J_u A^{\cdot\cdot a\cdot}_{\cdot b\cdot\cdot} = (A^{\cdot\cdot a\cdot}_{\cdot b\cdot\cdot})_{;k} u^k + A^{\cdot\cdot k\cdot}_{\cdot b\cdot\cdot} w^a_{\cdot k} + \ldots - A^{\cdot\cdot a\cdot}_{\cdot k\cdot\cdot} w^k_{\cdot c} - \ldots,$

$$w_{ab} = \frac{1}{2}(u_{a;b} - u_{b;a} - \dot{u}_a u_b + \dot{u}_b u_a) \quad \text{(rotation tensor)}$$

$$\dot{u}_a = u_{a;k} u^k, \quad u^a u_a = 1 \quad (a,b,c,k,\ldots = 1,2,3,4),$$

and a semicolon denotes covariant differentiation. The interpretation that a tensor is said to be stationary, if it is Jaumann transported implies that the gravitational potentials g_{ij} are stationary and that a geodesic flow is stationary (Jaumann transported flow). The 3-dimensional projection operator h_{ab} is stationary when and only when the first curvature of the streamline vanishes (the flow is geodesic). The necessary and sufficient condition for the Jaumann-transport of the 2-dimensional projection operator $P_{ab} \equiv g_{ab} - u_a u_b + P_a P_b$ are found to be

$$k_2 + \frac{1}{2}(\gamma_{123} - \gamma_{132}) = 0$$

$$\gamma_{124} = \gamma_{142}$$

The nonstationary character of the relativistic Serret-Frenet tetrad is also established.

Chapter II is titled "Fermi Transport". After noting that gravitational potentials, the flow vector and consequently the 3-dimensional projection operator h_{ab} are Fermi transported, three theorems are proved :

i) The torsion-free ($k_2 = 0$) worldline characterizes the Fermi transport of the 2-dimensional projection operator

$$g_{ab} - u_a u_b + P_a P_b .$$

ii) The bitorsion free ($k_3 = 0$) worldline corresponds to

$$F_u(g_{ab} - u_a u_b + R_a R_b) = 0 .$$

The necessary and sufficient condition for the Fermi transport of relativistic Serret-Frenet tetrad is found to be $k_2 = 0$.

The most popular transport in relativistic continuum mechanics namely the <u>Lie transport</u> is the subject matter of the Chapter III. In Section 1, a historical account of Lie transport of the 3-dimensional projection operator besides higher order transports is attempted. While the Lie derivative of a covariant material vector (1-form) is always a material vector (vector field orthogonal to the flow u^a), it is found that the Lie derivative of a contravariant material vector ($£_u V^a$, say) does not retain the material character. It is shown that only when the contravariant material vector V^a is orthogonal to the acceleration vector $\dot{u}^a$ the flow orthogonality of $£_u V^a$ can be achieved. In Section 3, it is noted that for a vector field x^a the expression

$(£_u F_u - F_u £_u) x^a$ is independent of $\dot{x}^a$ and $\ddot{x}^a$ and this leads to an interesting characterization, of the conditions for the commutativity of Lie and Fermi operators. The next section deals with the necessary and sufficient conditions for the Lie transport of the relativistic Serret-Frenet tetrad. In Section 5 advantage of the recent spectacular breakthrough in causal (relativistic) thermodynamics by Carter (1988) incorporating the finite speed of heat propagation is taken and the Lie-transport of the gravitational potentials with respect to the augmented flow λu^a (which may represent particle current or entropy current or chemical momentum or thermal momentum) are considered. The necessary as well as sufficient criteria for the 'augmented transport' of g_{ij} are couched in terms of the Ricci co-efficients of rotation of the relativistic Serret-Frenet frame.

We introduce the concept of CQ–transport i.e., Carter-Quintana's transport of a tensor field in Chapter IV. The Carter-Quintana's derivative of a material tensor is inevitably a material tensor, which is a remarkable characteristic of the CQ-operator, essential for the constitutive equation of matter (Oldroyd, 1970). Later, we consider the CQ-transport of the metric tensor leading to the characterization of rigid time-like flow congruence. In other words the gravitational potentials are CQ-transported iff the flow of the continuum is expansion free and shear free. The criteria for CQ-transport of the relativistic Serret-Frenet frame is investigated in Section 3.

Section 4 contains the proof of a theorem on the equivalence of the following statements :

(i) the 2-dimensional projection operator $(g_{ab} - u_a u_b + P_a P_b)$ is CQ-transported.

(ii) $k_2 + \gamma_{132} = \gamma_{134} + \gamma_{143} = \gamma_{133} = \gamma_{144} = 0.$

In the last section the noncommutativity of the CQ-operator with raising or lowering of the index of a tensor field is established.

In the Chapter V a <u>new transport operator</u> in relativistic continuum mechanics is introduced and it is denoted here by D_u. In Section 1 it is argued how D_u is a generalization of the Jaumann transport, the Lie transport, the Fermi transport, the Carter-Quintana transport and also the Truesdell transport. Section 2 delineates the formulae for the D-transport of a contravariant (or co-variant) vector field by imposing the condition that the D-transport of a scalar field must agree with the usual material transport of the scalar field (which is referred here as special Leibnitz property). The next section records the accomplishment of the formula for the $D_u A_b{}^a$. Finally the question of producing a new material vector $D_u x^a$ from a given material vector x^a is settled in the form of a condition on a scalar appearing in the operator D_u. It is found that D_u does not commute with raising/lowering of suffixes of tensor fields. In the last section, the criteria for the D_u-transported relativistic Serret-Frenet frame $Z^a_{(i)} = \{ u^a, P^a, Q^a, R^a \}$ are determined through the relation

$$D_u Z^a_{(i)} = f\dot{Z}^a_{(i)} + (\alpha\sigma_c^{\ a} + \beta\omega_c^{\ a} + \gamma_1 \delta_c^{\ a}\theta + \gamma_2 u^a u_c \theta + \chi \dot{u}^a u_c + \psi u^a \dot{u}_c) Z^c_{(i)} = 0 .$$

where $f, \alpha, \beta, \gamma_1, \gamma_2, \chi, \psi$ are scalars and θ, σ_{ab}, ω_{ab} are the expansion, shear and rotation of u^a.

The transport of mass as the equation of continuity, the transport of momentum yielding the equation of motion, the transport of energy resulting into equation of energy treated in classical continuum mechanics (vide Fredrickson, 1964) are not extended to relativistic domain in this dissertation, since the above transports pertain to the dynamical aspects of continuum mechanics, while this dissertation restricts to kinematical aspects.

	NOTATION		MEANING
1.	u^a	-	Flow velocity vector field.
2.	P^a	-	Unit acceleration field.
3.	Q^a	-	Unit vector orthogonal to velocity and acceleration.
4.	R^a	-	Unit vector orthogonal to u^a, P^a, Q^a.
5.	$Z^a_{(i)}$	–	$\{u^a, P^a, Q^a, R^a\}$, Relativistic Serret-Frenet Tetrad frame.
6.	k_1	-	First curvature of the world line.
7.	k_2	-	Second curvature of the world line, (Torsion).
8.	k_3	-	Third curvature of the world line, (Bitorsion).
9.	σ_{ab}	-	Shear tensor of u_a.
10.	ω_{ab}	-	Rotation tensor of u_a.
11.	θ	-	Expansion of u_a.
12.	h_{ab}	-	Three dimensional projection tensor.
13.	C_u	-	Convective derivative with respect to u^a.
14.	D_u	-	A new derivative with respect to u^a defined in Chapter V.
15.	F_u	-	Fermi derivative with respect to u^a.
16.	J_u	-	Jaumann derivative with respect to u^a.
17.	$£_u$	-	Lie derivative with respect to u^a.

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CONVENTIONS

1.	$(-, -, -, +)$	: Signature of the metric tensor g_{ab}.
2.	γ_{ABC}	: Ricci coefficients of rotation.
3.	' ; '	: Covariant derivative.
4.	,	: Partial derivative.
5.	$(\ )^{\cdot}$	: Covariant derivative of ( ) with respect to the unit time-like vector u^a.
6.	η^{abcd}	: Levi-Civita Permutation tensor.