CHAPTER - I

JAUMANN TRANSPORT IN RELATIVISTIC CONTINUUM MECHANICS

1. INTRODUCTION :

a) RELATIVISTIC CONTINUUM MECHANICS :

A continuum is a structureless substance and to each point of the 3-dimensional continuum we can assign kinematical or dynamical variables which are continuous functions of the three co-ordinates of the point. In continuum mechanics, we study the behaviour of systems under the action of forces ignoring the molecular structure of matter. The study of continuum mechanics has two aspects : (i) Kinematics, which is concerned with the motion of particles and bodies, (ii) Dynamics, which deals with forces causing the motion of particles and bodies. The exploration of gravitational fields, electromagnetic fields, hydrodynamics, aerodynamics, elastic/plastic materials constitute typical instances of continuum mechanics.

Strong and rapidly changing gravitational fields are found in neutron stars, wherein high pressure ($p \sim c^2$), high speeds ($v \sim c$) of supermassive bodies ($M \sim \frac{Rc^2}{2G}$) are prevalent. Here ρ is the density, M is the mass, R is the radius of the star, G is the constant of gravitation. It is in such astrophysical circumstances the relevance of general relativistic continuum mechanics appears. In this dissertation we confine to the kinematical aspects of the relativistic continuum mechanics.

Time has no absolute status in the theory of relativity. Hence the time rate of kinematical variables in classical continuum mechanics has to be suitably changed for applying to relativistic domain. This is achieved by replacing the time rate by covariant differentiation along TIME-LIKE flow vector, since time-like vectors have absolute character in special as well as general relativity (Oldroyd, 1970).

b) RELATIVISTIC SERRET FRENET TETRAD :

The unit time-like velocity field u^a.

If s is the arc-length parameter then the natural equation for the world-line of a particle in a continuum is

$$Z^a = Z^a(s), \quad (a = 1, 2, 3, 4).$$

If u^a is the tangent vector to this world-line (4-dimensional space curve) then

$$u^a = \frac{dZ^a}{ds}. \qquad \text{... (1.1)}$$

This means that u^a is the velocity vector field of the particle.
The metric relation $ds^2 = dZ^a dZ_a$ implies

$$\frac{dZ^a}{ds} \cdot \frac{dZ_a}{ds} = 1 \quad \text{or} \quad u^a u_a = 1. \qquad \text{... (1.2)}$$

Thus, we have, u^a as the unit tangent vector field on the world-line, due to the choice of the signature (-, -, -, +) of the metric tensor. Since $u^a u_a > 0$ we infer that u^a is a time-like unit vector.

The unit space-like acceleration field P^a.

The acceleration field (which is not equal to $\frac{du^a}{ds}$ but equal to) $\frac{\delta u^a}{\delta s}$ is denoted by $\dot{u}^a$ and is defined by

$$\dot{u}^a = u^a_{;b}u^b \quad \ldots \quad (1.3)$$

where a semicolon denotes covariant differentiation. It should be noted that $\dot{u}^a$ is not a unit vector field. It is a space-like vector field since it is orthogonal to the time-like vector u^a, for, from (1.2)

$$(u^a u_a)^{\cdot} = 0$$

which shows that

$$\dot{u}^a u_a = 0 \quad \ldots \quad (1.4)$$

This one orthogonal relation helps us to examine whether an orthogonal tetrad can be constructed, by introducing two more space-like vector fields.

Let P^a represent the unit vector field along the acceleration, then, we have

$$P^a = \frac{\dot{u}^a}{|\dot{u}|}$$

or $P^a = \dot{u}^a \,/\, (-\dot{u}^k\dot{u}_k)^{\frac{1}{2}}$

The negative sign before $\dot{u}^k\dot{u}_k$ is due to the fact that $\dot{u}^a$ is space-like and $\dot{u}^a\dot{u}_a < 0$. We introduce a scalar field through

$$K_1 = \sqrt{-\dot{u}^a\dot{u}_a} \quad \ldots \quad (1.5)$$

and observe that

$$P^a = \dot{u}^a/K_1 \,. \quad \ldots \quad (1.6)$$

Here K_1 is called as the <u>first curvature</u> of the world-line. We note that

$$P^a P_a = -1 .$$

<u>The two unit space-like orthogonal vector fields which are orthogonal to both velocity and acceleration vector fields : Q^a, R^a.</u>

Suppose Q^a, R^a represent the two unit vector fields which are orthogonal to both u^a and P^a. Then we have the algebraic relations

$$P^a Q_a = 0, \quad P^a R_a = 0, \quad u^a Q_a = 0, \quad u^a R_a = 0 \qquad \text{... (1.7)}$$

$$R^a R_a = Q^a Q_a = -1 \qquad \text{... (1.8)}$$

and the differential relations can be described as (Davis 1970)

$$\begin{bmatrix} \dot{u}^a \\ \dot{P}^a \\ \dot{Q}^a \\ \dot{R}^a \end{bmatrix} = \begin{bmatrix} 0 & K_1 & 0 & 0 \\ K_1 & 0 & K_2 & 0 \\ 0 & -K_2 & 0 & K_3 \\ 0 & 0 & -K_3 & 0 \end{bmatrix} \begin{bmatrix} u^a \\ P^a \\ Q^a \\ R^a \end{bmatrix} \qquad \text{... (1.9)}$$

where an overhead dot represents the covariant differentiation along the flow vector field u^a. They are referred as <u>Relativistic Serret-Frenet formulae</u> and we list them for use in this dissertation.

$$\dot{u}^a = K_1 P^a \qquad \text{..................... (RSF-1)}$$

$$\dot{P}^a = K_1 u^a + K_2 Q^a \qquad \text{..................... (RSF-2)}$$

$$\dot{Q}^a = -K_2 P^a + K_3 R^a \qquad \text{..................... (RSF-3)}$$

$$\dot{R}^a = -K_3 Q^a \qquad \text{..................... (RSF-4)}$$

where K_2, K_3 are called the second and the third curvatures of the stream-line in the 4-dimensional space-time .

The explicit expressions for Q^a, R^a, K_2, K_3 are given by (Magdum, 1988).

$$Q^a = \frac{1}{K_2}\left(\frac{\ddot{u}^a}{K_1} + \frac{\dot{K}_1}{K_1^2}\dot{u}^a - K_1 u^a\right),$$

$$R^a = \frac{-1}{K_1^2 K_2}\eta^{abcd}u_b\dot{u}_c\ddot{u}_d ,$$

$$K_2 = \left[-\frac{\ddot{u}^a\ddot{u}_a}{K_1^2} - \left(\frac{\dot{K}_1}{K_1}\right)^2 + K_1^2\right]^{\frac{1}{2}} ,$$

$$K_3 = \frac{1}{K_1^3 K_2^2}\eta^{abcd}u_b\dot{u}_c\ddot{u}_d\dddot{u}_a ,$$

where, η^{abcd} is the Levi-Civita permutation tensor, $\ddot{u}^d = (\dot{u}^d)_{;k}u^k$.

Definition : The set $\{u^a, P^a, Q^a, R^a\}$ satisfying the condition (RSF-1 to 4) is called the Relativistic Serret-Frenet Tetrad.

Remark 1 : It may be interesting to note that the co-efficient matrix in RSF-formulae (1.9) is neither symmetric nor skew symmetric; the corresponding co-efficient matrix in classical Serret-Frenet formulae is skew symmetric (vide O' Neill, 1966),

$$\begin{bmatrix}\dot{T}\\ \dot{N}\\ \dot{B}\end{bmatrix} = \begin{bmatrix} 0 & K_1 & 0\\ -K_1 & 0 & K_2\\ 0 & -K_2 & 0\end{bmatrix}\begin{bmatrix}T\\ N\\ B\end{bmatrix}$$

Here T, N, B are the unit tangent, principal normal and binormal vector fields respectively.

Remark 2 : The Serret-Frenet tetrad is a moving frame on the world line of the particle in the space-time continuum. In this dissertation the RSF-tetrad is exploited to derive several elegant results on the different types of transports in relativistic continuum mechanics.

c) THE DECOMPOSITION OF THE FLOW GRADIENT :

Since the Relativistic Serret-Frenet tetrad $\{ u^a, P^a, Q^a, R^a \}$ is a set of linearly independent vectors, we can express any second rank tensor say $u^\ell_{;k}$ as a linear combination of the 16 outer products of the basis vectors. Explicitly the decomposition of $u^\ell_{;k}$ in terms of the Relativistic Serret-Frenet tetrad is

$$
\begin{aligned}
u^\ell_{;k} = \; & (u^a_{;b} u_a u^b) u_k u^\ell + (-u^a_{;b} u^b Q_a) u_k Q^\ell + (-u^a_{;b} u^b P_a) u_k P^\ell \\
& + (-u^a_{;b} u^b R_a) u_k R^\ell + (-u^a_{;b} P^b u_a) P_k u^\ell + (u^a_{;b} P^b P_a) P_k P^\ell \\
& + (u^a_{;b} P^b Q_a) P_k Q^\ell + (u^a_{;b} P^b R_a) P_k R^\ell + (-u^a_{;b} Q^b u_a) Q_k u^\ell \\
& + (u^a_{;b} Q^b Q_a) Q_k Q^\ell + (u^a_{;b} Q^b P_a) Q_k P^\ell + (u^a_{;b} Q^b R_a) Q_k R^\ell \\
& + (-u^a_{;b} R^b u_a) R_k u^\ell + (u^a_{;b} R^b Q_a) R_k Q^\ell + (u^a_{;b} R^b P_a) R_k P^\ell \\
& + (u^a_{;b} R^b R_a) R_k R^\ell . \\
= \; & K_1 u_k P^\ell + (u^a_{;b} P^b P_a) P_k P^\ell + (u^a_{;b} P^b Q_a) P_k Q^\ell + (u^a_{;b} P^b R_a) P_k R^\ell \\
& + (u^a_{;b} Q^b Q_a) Q_k Q^\ell + (u^a_{;b} Q^b P_a) Q_k P^\ell + (u^a_{;b} Q^b R_a) Q_k R^\ell \\
& + (u^a_{;b} R^b Q_a) R_k Q^\ell + (u^a_{;b} R^b P_a) R_k P^\ell + (u^a_{;b} R^b R_a) R_k R^\ell .
\end{aligned}
$$

The decomposition of the tensor gradient of flow field in terms of the Relativistic Serret-Frenet frame is accomplished below in terms of Ricci co-efficients of rotation. The concomitant expressions for the symmetric shear field, skew symmetric rotation field and the expansion scalar field associated with the time like flow vector fields are evaluated in terms of the Ricci rotation co-efficients as computational aids for utilization in later investigations.

$$u^{\ell}_{;k} = + K_1 u_k P^{\ell} + \gamma_{122} P_k P^{\ell} + \gamma_{132} P_k Q^{\ell} + \gamma_{142} P_k R^{\ell} + \gamma_{133} Q_k Q^{\ell}$$
$$+ \gamma_{123} Q_k P^{\ell} + \gamma_{143} Q_k R^{\ell} + \gamma_{134} R_k Q^{\ell} + \gamma_{124} R_k P^{\ell} + \gamma_{144} R_k R^{\ell}$$

Here γ_{ABC} represent the Ricci rotation co-eficients defined through (Weatherburn 1963)

$$\gamma_{ABC} = \lambda^{\ell}{}_{A;m} \lambda_{\ell B} \lambda^{m}{}_{C}$$

where $\lambda^{\ell}{}_1 = u^{\ell}$, $\lambda^{\ell}{}_2 = P^{\ell}$, $\lambda^{\ell}{}_3 = Q^{\ell}$, $\lambda^{\ell}{}_4 = R^{\ell}$.

The capital Latin subscripts refer to tetrad indices and the small Latin scripts refer to tensor indices.

d) <u>COMPUTATIONAL AIDS</u> :

The following results will be needed in this and later chapters.

I) $u^{\ell}_{;k} u_{\ell} = 0$ (identically). We note that $\gamma_{ABC} = -\gamma_{BAC}$

II) $u^{\ell}_{;k} P_{\ell} = -(K_1 u_k + \gamma_{122} P_k + \gamma_{123} Q_k + \gamma_{124} R_k)$

III) $u^{\ell}_{;k} Q_{\ell} = -(\gamma_{132} P_k + \gamma_{133} Q_k + \gamma_{134} R_k)$

iv) $u^{\ell}_{;k} R_{\ell} = -(\gamma_{142} P_k + \gamma_{143} Q_k + \gamma_{144} R_k)$

v) $u^{\ell}_{;k} u^k = K_1 P^{\ell}$, (RSF - 1)

vi) $u^{\ell}_{;k} P^k = -(\gamma_{122} P^{\ell} + \gamma_{132} Q^{\ell} + \gamma_{142} R^{\ell})$

vii) $u^{\ell}_{;k} Q^k = -(\gamma_{133} Q^{\ell} + \gamma_{123} P^{\ell} + \gamma_{143} R^{\ell})$

viii) $u^{\ell}_{;k} R^k = -(\gamma_{143} Q^{\ell} + \gamma_{124} P^{\ell} + \gamma_{144} R^{\ell})$

ix) $u_{\ell;k} + u_{k;\ell} = K_1(u_k P_{\ell} + u_{\ell} P_k) + 2(\gamma_{122} P_k P_{\ell} + \gamma_{133} Q_k Q_{\ell} + \gamma_{144} R_k R_{\ell})$
$+ (\gamma_{123} + \gamma_{132})(P_k Q_{\ell} + P_{\ell} Q_k) + (\gamma_{124} + \gamma_{142})(P_k R_{\ell} + R_k P_{\ell})$
$+ (\gamma_{134} + \gamma_{143})(Q_k R_{\ell} + Q_{\ell} R_k)$.

x) $$u_{\ell;k} - U_{k;\ell} = K_1(u_k P_\ell - u_\ell P_k) + (\gamma_{132} - \gamma_{123})P_k Q_\ell + (\gamma_{123} - \gamma_{132})Q_k P_\ell$$
$$+ (\gamma_{142} - \gamma_{124})P_k R_\ell + (\gamma_{124} - \gamma_{142}) R_k P_\ell$$
$$+ (\gamma_{143} - \gamma_{134})Q_k R_\ell + (\gamma_{134} - \gamma_{143})R_k Q_\ell \quad \text{...(a)}$$

$$P^\ell (u_{\ell;k} - u_{k;\ell}) = -[K_1 u_k + (\gamma_{123} - \gamma_{132})Q_k + (\gamma_{124} - \gamma_{142})R_k] \quad \text{...(b)}$$

$$u^k (u_{\ell;k} - u_{k;\ell}) = K_1 P_\ell \quad \text{... (c)}$$

$$P^\ell u^k (u_{\ell;k} - u_{k;\ell}) = -K_1 \quad \text{... (d)}$$

$$R^k P^\ell (u_{\ell;k} - u_{k;\ell}) = \gamma_{124} - \gamma_{142} \quad \text{... (e)}$$

xi) Expansion $= \theta = u^\ell_{;\ell} = -(\gamma_{122} + \gamma_{133} + \gamma_{144})$.

xii) Shear tensor $= \sigma_{k\ell} = \frac{1}{2}(u_{\ell;k} + u_{k;\ell} - \dot{u}_\ell u_k - \dot{u}_k u_\ell) - \frac{1}{3}\theta h_{k\ell}$

$$2\sigma_{k\ell} = 2(\gamma_{122}P_k P_\ell + \gamma_{133}Q_k Q_\ell + \gamma_{144}R_k R_\ell) + \gamma_{132}(P_k Q_\ell + P_\ell Q_k)$$
$$+ \gamma_{142}(P_k R_\ell + (P_\ell R_k) + \gamma_{123}(Q_k P_\ell + Q_\ell P_k) + \gamma_{143}(Q_k R_\ell + Q_\ell R_k)$$
$$+ \gamma_{134}(R_k Q_\ell + R_\ell Q_k) + \gamma_{124}(P_k R_\ell + P_\ell R_k)$$
$$- \tfrac{2}{3}(\gamma_{122} + \gamma_{133} + \gamma_{144})(P_k P_\ell + Q_k Q_\ell + R_k R_\ell) \quad \text{...(a)}$$

$$\sigma_{k\ell}P^\ell = -\tfrac{1}{2}\left[\tfrac{4}{3}\gamma_{122} - \tfrac{2}{3}(\gamma_{133} + \gamma_{144})\right]P_k + (\gamma_{132} + \gamma_{123})Q_k$$
$$+ (\gamma_{142} + \gamma_{124})]R_k \quad \text{... (b)}$$

$$\sigma_{k\ell} u^\ell = = 0 \quad \text{... (c)}$$

$$\sigma_{K\ell} Q^\ell = -\tfrac{1}{2}(\gamma_{132} + \gamma_{123})P_k - \tfrac{1}{6}(\gamma_{122} + \gamma_{144} + 8\gamma_{133})Q_k$$
$$- \tfrac{1}{2}(\gamma_{143} + \gamma_{134})R_k \quad \text{... (d)}$$

$$\sigma_{K\ell} R^\ell = -\tfrac{1}{2}[(\gamma_{142} + \gamma_{124})P_k + (\gamma_{143} + \gamma_{134})Q_k]$$
$$- \tfrac{1}{6}(6\gamma_{142} + \gamma_{122} + \gamma_{133} + \gamma_{144})R_k \quad \text{... (e)}$$

(XIII) Rotation $= \omega_{\ell k} = \frac{1}{2}(u_{\ell;k} - u_{k;\ell} + u_\ell \dot{u}_k - u_k \dot{u}_\ell)$

$$2\omega_{\ell k} = (\gamma_{132} - \gamma_{123})P_k Q_\ell + (\gamma_{123} - \gamma_{132})Q_k P_\ell + (\gamma_{142} - \gamma_{124})P_k R_\ell$$
$$+ (\gamma_{124} - \gamma_{142})R_k P_\ell + (\gamma_{143} - \gamma_{134})Q_k R_\ell + (\gamma_{134} - \gamma_{143})R_k Q_\ell \quad \text{...(a)}$$

$$2\omega_{\ell k}P^\ell = (\gamma_{132} - \gamma_{123})Q_k + (\gamma_{142} - \gamma_{124})R_k \quad \text{...(b)}$$

$$2\omega_{\ell k}P^\ell Q^k = \gamma_{123} - \gamma_{132} \quad \text{...(c)}$$

$$2\omega_{\ell k}P^\ell R^k = \gamma_{124} - \gamma_{142} \quad \text{...(d)}$$

$$2\omega_{\ell k}R^k = (\gamma_{142} - \gamma_{124})P_\ell + (\gamma_{143} - \gamma_{134})Q_\ell \quad \text{...(e)}$$

$$2\omega_{\ell k}Q^k = (\gamma_{132} - \gamma_{123})P_\ell + (\gamma_{134} - \gamma_{143})R_\ell \,. \quad \text{...(f)}$$

2. THE JAUMANN TRANSPORT :

History of classical continuum mechanics tells us that the operator J_u was invented by Jaumann in 1911, in the form

$$J_u X_a = \dot{X}_a - X_k \partial_{[k} u_{a]}\,, \quad a, k = 1,2,3\,.$$
$$u^a = \frac{dx^a}{dt}, \quad \dot{X}^a = \frac{dx^a}{dt}$$

The superiority of this Jaumann operator over other operators like, Lie, Fermi has been described by Prager (1961). The formal extension of this operator to relativistic continuum mechanics was made by Radhakrishna et.al. (1981).

Definition : A tensor field $A^{..a.}_{.b..}$ is said to be Jaumann transported if and only if

$$J_u A^{..a.}_{.b..} = 0$$

where $J_u A^{..a.}_{.b..} = (A^{..a.}_{.b..})_{;k} u^k + A^{..k.}_{.b..} \omega^a_{.k} + \ldots$

$$- A^{..a.}_{.k..} \omega^k_{.c} \quad - \quad \ldots$$

$$\omega_{ab} = \tfrac{1}{2}(u_{a;b} - u_{b;a} - \dot{u}_a u_b + \dot{u}_b u_a) \text{ (rotation tensor)}$$

$$\dot{u}_a = u_{a;k} u^k, \quad u^a u_a = 1 \quad (a,b,c,k, \ldots = 1,2,3,4)$$

and a semicolon denotes covariant differentiation.

2 a) THE JAUMANN TRANSPORT OF THE 3-DIMENSIONAL PROJECTION OPERATOR h_{ab} :

The aim of this section is to prove two theorems on the transport of certain projection operators. In order to establish them, the stationary character of a general tensor field is introduced. We need the following properties of Jaumann transport for assisting in the proof.

(1) The metric tensor field is Jaumann transported always (identically). We have, (Radhakrishna et.al. 1981)

$J_u g_{ab} = 0$ identically,

because $J_u g_{ab} = g_{ab;c} u^c + g_{cb} \omega^c_{.a} + g_{ac} \omega^c_{.b}$

$= (\omega_{ba} + \omega_{ab})$, since g_{ab} are covariant constant.

$J_u g_{ab} = 0$, since ω_{ab} is skew symmetric.

<u>Definition</u> : A tensor is said to be stationary, if it is Jaumann transported.

Thus, the gravitational potentials g_{ij} are stationary.

(2) We observe that the flow vector is not Jaumann transported identically, for, we have,

$J_u u_a = \dot{u}_a + u_k\, \omega^k_{.a}$, by definition.

$= K_1 P_a$, by (RSF-1) and $\omega^k_{.a}$ is a material tensor.

This shows that, ingeneral, the flow is not stationary.

Remark : If the flow is geodesic, we have,

$\dot{u}_a = 0$ (or $P^a = 0$)

and so, obviously a geodesic flow is stationary,(Jaumann transported flow).

(3) We now examine the Jaumann tranpsort of the 3-dimensional projection operator. We consider,

$J_u h_{ab} = J_u g_{ab} - J_u u_a u_b$.

$= - u_a J_u u_b - u_b J_u u_a$, since $J_u g_{ab} = 0$

$= - u_a \dot{u}_b - \dot{u}_a u_b$, since $J_u u_b = \dot{u}_b$

$J_u h_{ab} = - K_1 (P_a u_b + P_b u_a)$, by (RSF-1) ...(2.1)

Thus, $J_u h_{ab}$ does not vanish ingeneral.

We find that

$P^a J_u h_{ab} = - K_1 (-u_b)$

$= K_1 u_b$

$P^a u^b J_u h_{ab} = k_1$ . ... (2.2)

Theorem 1 : $J_u h_{ab} = 0$ iff $K_1 = 0$

Proof : From (2.1), we have,

$$J_u h_{ab} = -K_1 (u_a P_b + u_b P_a) .$$

Hence $K_1 = 0$ implies $J_u h_{ab} = 0$.

conversely, when $J_u h_{ab} = 0$, we get, from (2.2) that

$$K_1 = 0 .$$

Remark : The 3-dimensional projection operator is stationary when and only when the first curvature of the streamline vanishes, (the flow is geodesic).

2 b) THE JAUMANN TRANSPORT OF P_{ab} :

We find that the 2-dimensional projection tensor P_{ab} is not Jaumann transported always. The case of the Jaumann transport of P_{ab} is given in the following theorem. As usual the acronym TFAE denotes 'The Following (statements) Are Equivalent'

Theorem 2 : TFAE : (I) $J_u P_{ab} = 0$

: (II) $K_2 + \frac{1}{2} (\gamma_{123} - \gamma_{132}) = 0$

$$\gamma_{124} = \gamma_{142} .$$

Proof : We have,

$$J_u P_{ab} = J_u (g_{ab} - u_a u_b + P_a P_b)$$

$$= J_u g_{ab} - u_a J_u u_b - u_b J_u u_a + P_a J_u P_b + P_b J_u P_a, \text{ by linearity.}$$

$$= -u_a \dot{u}_b - \dot{u}_a u_b + P_a (\dot{P}_b - P_k \omega^k_{\cdot b}) + P_b (\dot{P}_a - P_k \omega^k_{\cdot a}), \text{ Since } J_u g_{ab} = 0, \; J_u u_b = \dot{u}_b$$

$$= -K_1 (u_a P_b + P_a u_b) + P_a (K_1 u_b + K_2 Q_b) - P_a P_k \omega^k_{\cdot b} + P_b (K_1 u_a + K_2 Q_a) - P_b P_k \omega^k_{\cdot a} \quad \text{by (RSF-1,2)}$$

$$= -K_1 (u_a P_b + P_a u_b - P_a u_b - P_b u_a) + K_2 (P_a Q_b + P_b Q_a) - P_a P_k \omega^k_{\cdot b} - P_b P_k \omega^k_{\cdot a} , \quad \text{on rearrangement.}$$

$J_u P_{ab} = K_2(P_a Q_b + P_b Q_a) - P_a P_k \omega^k_{\cdot b} - P_b P_k \omega^k_{\cdot a}$, on simplification.

$$J_u P_{ab} = K_2(P_a Q_b + P_b Q_a) - P_a[\tfrac{1}{2}(\gamma_{132} - \gamma_{123})Q_b + (\gamma_{142} - \gamma_{124})R_b]$$
$$- P_b[(\gamma_{132} - \gamma_{123})Q_a + (\gamma_{142} - \gamma_{124})R_a] \quad \ldots (2.3)$$

Now, we find the circumstances when all the 10 inner products of $J_u P_{ab}$ with the tetrad (u^a, P^a, Q^a, R^a) vanish. The eight double inner products which vanish identically are

$$u^a u^b J_u P_{ab} = 0, \quad P^a P^b J_u P_{ab} = 0, \quad Q^a Q^b J_u P_{ab} = 0, \quad R^a R^b J_u P_{ab} = 0,$$

$$u^a Q^b J_u P_{ab} = 0, \quad u^a R^b J_u P_{ab} = 0, \quad Q^a R^b J_u P_{ab} = 0, \quad u^a P^b J_u P_{ab} = 0 .$$

I) $J_u P_{ab}$ when transvected with Q^b yields

$$Q^b J_u P_{ab} = Q^b [K_2(P_a Q_b + P_b Q_a) - P_a P_k \omega^k_{\cdot b} - P_b P_k \omega^k_{\cdot a}]$$
$$= - K_2 P_a - P_a P_k Q^b \omega^k_{\cdot b}$$
$$Q^b J_u P_{ab} = - P_a (K_2 + P_k Q^b \omega^k_b) .$$

Consequently, we get

$$P^a Q^b J_u P_{ab} = - P^a P_a (K_2 + P_k Q^b \omega^k_{\cdot b})$$

$$P^a Q^b J_u P_{ab} = K_2 + Q^b P_k \omega^k_{\cdot b}$$

$$P^a Q^b J_u P_{ab} = K_2 + \tfrac{1}{2}(\gamma_{123} - \gamma_{132}) \quad \text{(by computational aid)}\ldots(2.4)$$

(II) Contracting $J_u \bar{P}_{ab}$ with P^b and R^a, we have,

$$P^b J_u P_{ab} = P^b[K_2(P_a Q_b + P_b Q_a) - \tfrac{1}{2} K_1 (P_a u_b + P_b u_a)$$
$$- \tfrac{1}{2} P_a P^r (u_{r;b} - u_{b;r}) - \tfrac{1}{2} P_b P^r (u_{r;a} - u_{a;r})]$$

$$= - K_2Q_a + \tfrac{1}{2} K_1u_a - \tfrac{1}{2} P_aP^bP^r (u_{r;b} - U_{b;r}) + \tfrac{1}{2} P^r(u_{r;a} - u_{a;r})$$

since $P^bP_b = -1, \quad P^bQ_b = P^bu_b = 0$

$$P^bJ_uP_{ab} = - K_2Q_a + \tfrac{1}{2} K_1u_a + \tfrac{1}{2} P^r (u_{r;a} - u_{a;r})$$

$$R^aP^bJ_uP_{ab} = R^a [- K_2Q_a + \tfrac{1}{2} K_1u_a + \tfrac{1}{2} P^r (u_{r;a} - u_{a;r})]$$

$$R^aP^bJ_uP_{ab} = \tfrac{1}{2} R^aP^r (u_{r;a} - u_{a;r})$$

$$R^aP^bJ_uP_{ab} = \tfrac{1}{2} (R^bP^a\omega_{ab})$$

$$= \tfrac{1}{2} (\gamma_{124} - \gamma_{142}) \quad \ldots \quad (2.5)$$

Now, (I) implies (II).

$J_uP_{ab} = 0$ implies $K_2 = - \tfrac{1}{2} (\gamma_{123} - \gamma_{132})$, by (2.4)

$\gamma_{124} = \gamma_{142}$ by (2.5)

Again, (II) implies (I), on simplification of (2.3).

This completes the proof.

3. SECOND ORDER JAUMANN TRANSPORTS :

Second order Lie transports have been exploited by Carter and Quintana (1977) and third order Lie transports have been used by Katkar (1989) for gravitational radiation. We now study the second order Jaumann transport of the 3-dimensional projection operator. Our aim is to prove:

Theorem 3 : TFAE

1) $J_uJ_uh_{ab} = 0$
2) $K_1 = 0, \quad \gamma_{124} = \gamma_{142}$

<u>Proof</u> : $J_u h_{ab} = - u_a \dot{u}_b - \dot{u}_a u_b$

$$J_u h_{ab} = - (u_a u_b)^{\cdot}$$

we have,

$$J_u h_{ab} = - K_1 (u_a P_b + u_b P_a)$$

$$J_u J_u h_{ab} = J_u [- K_1 (u_a P_b + u_b P_a)]$$

$$= - \dot{K}_1 (u_a P_b + u_b P_a) - K_1 [K_1 P_a P_b + u_a (K_1 u_b + K_2 Q_b)$$

$$- u_a P_k \omega_{\cdot b}^{k} + K_1 P_b P_a + u_b (K_1 u_a + k_2 Q_a) - u_b P_k \omega_{\cdot a}^{k}]$$

by (RSF - I, II).

$$J_u J_u h_{ab} = -\dot{K}_1 (u_a P_b + u_b P_a) - 2K^2_1 (P_a P_b + u_a u_b) - K_1 K_2 (u_a Q_b + u_b Q_a)$$

$$+ K_1 P_k (u_a \omega_{\cdot b}^{k} + u_b \omega_{\cdot a}^{k}), \quad \text{by rearrangement} \quad ...(2.6)$$

We observe that,

$$u_a \omega_{\cdot b}^{k} + u_b \omega_{\cdot a}^{k} = \tfrac{1}{2} u_a g^{kr} (u_{r;b} - u_{b;r} - \dot{u}_r u_b + u_r \dot{u}_b)$$

$$+ \tfrac{1}{2} u_b g^{kr} (u_{r;a} - u_{a;r} - \dot{u}_r u_a + u_r \dot{u}_a) \quad ... (2.7)$$

$$J_u J_u h_{ab} = - \dot{K}_1 (u_a P_b + u_b P_a) - 2k^2_1 (P_a P_b + u_a u_b) - K_1 K_2 (u_a Q_b + u_b Q_a)$$

$$+ K_1 P_k \tfrac{1}{2} u_a g^{kr} (u_{r;b} - u_{b;r} - \dot{u}_r u_b + u_r \dot{u}_b)$$

$$+ K_1 P_k \tfrac{1}{2} u_b g^{kr} (u_{r;a} - u_{a;r} - \dot{u}_r u_a + u_r \dot{u}_a), \quad \text{since from (2.7)}$$

$$= -\dot{K}_1 (u_a p_b + u_b P_a) - 2k^2_1 (P_a P_b + u_a u_b) - K_1 K_2 (u_a Q_b + u_b Q_a)$$

$$+ \tfrac{1}{2} K_1 [u_a P^r (u_{r;b} - u_{b;r} - K_1 P_r u_b) + u_b P^r (u_{r;a} - u_{a;r} - K_1 P_r u_a)$$

since $P^r u_r = 0$ by (RSF-1).

$$= - \dot{K}_1(u_aP_b+u_bP_a) - 2K^2{}_1\ (P_aP_b+u_au_b) - K_1K_2\ (u_aQ_b + u_bQ_a)$$
$$+ \tfrac{1}{2}\ K_1u_aP^r(u_{r;b} - u_{b;r}) + \tfrac{1}{2}\ K^2{}_1u_au_b + \tfrac{1}{2}\ K_1u_bP^r\ (u_{r;a}-u_{a;r})$$
$$+ \tfrac{1}{2}\ K^2{}_1\ u_bu_a \qquad \text{since} \quad P^rP_{r.} = -1.$$

Hence, we obtain ,

$$J_uJ_uh_{ab} = - \dot{k}_1\ (u_aP_b+u_bP_a) - 2K^2{}_1\ (P_aP_b+u_au_b) - K_1K_2(u_aQ_b+u_bQ_a)$$
$$+ \tfrac{1}{2}\ K_1P^ru_a(u_{r;b}-u_{b;r}) + \tfrac{1}{2}\ K_1P^ru_b\ (u_{r;a} - u_{a;r}\ )$$
$$+ K_1^2\ u_au_b\ .$$

We note that K_1 appears in every term.

Obviously, when $K_1 = 0$, we get, $J_uJ_uh_{ab} = 0$.

We now find that following 8 inner products of $J_uJ_uh_{ab}$ with the tetrad (u^a, P^a, Q^a, R^a) vanish.

$P^aQ^bJ_uJ_uh_{ab} = 0 \qquad P^aR^bJ_uJ_uh_{ab} = 0$

$Q^aP^bJ_uJ_uh_{ab} = 0 \qquad Q^aQ^bJ_uJ_uh_{ab} = 0$

$R^bQ^aJ_uJ_uh_{ab} = 0 \qquad R^aP^bJ_uJ_uh_{ab} = 0$

$R^aQ^bJ_uJ_uh_{ab} = 0 \qquad R^aR^bJ_uJ_uh_{ab} = 0$

<u>Non-vanishing inner products</u> :

(1) Contracting $J_uJ_uh_{ab}$ with u^b .

$$u^bJ_uJ_uh_{ab} = - \dot{K}_1P_a - 2K^2{}_1u_a \qquad - K_1K_2Q_a + \tfrac{1}{2}\ K_1P^ru^bu_a(u_{r;b}-u_{b;r})$$
$$+ \tfrac{1}{2}\ K_1P^r\ (u_{r;a} - u_{a;r}) + K_1{}^2u_a$$

Now we infer that

$$u^a u^b J_u J_u h_{ab} = u^a [-\dot{K}_1 P_a - 2K_1{}^2 u_a - K_1 K_2 Q_a + \tfrac{1}{2} K_1 P^r u^b u_a (u_{r;b} - u_{b;r})$$
$$- \tfrac{1}{2} K_1 P^r (u_{r;a} - u_{a;r}) + K_1{}^2 u_a]$$
$$= - 2K_1{}^2 + \tfrac{1}{2} K_1 P^r u^b (u_{r;b} - u_{b;r}) - \tfrac{1}{2} K_1 P^r u^a (u_{r;a} - u_{a;r})$$
$$+ K_1{}^2 \quad \text{since } u^a P_a = 0,\ u^a u_a = 1,\ u^a Q_a = 0 .$$

$$u^a u^b J_u J_u h_{ab} = -K_1{}^2 + \tfrac{1}{2} K_1 P^r u^b (u_{r;b} - u_{b;r}) - \tfrac{1}{2} K_1 P^r u^a (u_{r;a} - u_{a;r})$$
$$= - K_1{}^2 + \frac{K_1}{2} u^b [-\{ K_1 u_b + (\gamma_{123} - \gamma_{132}) Q_b + (\gamma_{124} - \gamma_{142}) R_b]$$
$$- \tfrac{1}{2} K_1 u^a [-\{ K_1 u_a + (\gamma_{123} - \gamma_{132}) Q_a + (\gamma_{124} - \gamma_{142}) R_a\}]$$
$$= - K_1{}^2 + \frac{K_1}{2} (-K_1) + \tfrac{1}{2} K_1{}^2$$
$$= - K_1{}^2 - \frac{K_1^2}{2} + \frac{K_1^2}{2}$$

$$u^a u^b J_u J_u h_{ab} = - K_1{}^2$$

(2) $J_u J_u h_{ab}$ when transvected with P^b, yields.

$$P^b J_u J_u h_{ab} = \dot{K}_1 u_a + 2K_1{}^2 P_a + \tfrac{1}{2} K_1 P^r u_a P^b (u_{r;b} - u_{b;r}) ,$$
$$\text{since } P^b P_b = -1,\ P^b u_b = 0,\ P^b Q_b = 0$$

consequently, we get,

$$u^a P^b J_u J_u h_{ab} = u^a [\dot{K}_1 u_a + 2K_1{}^2 P_a + \tfrac{1}{2} K_1 P^r u_a P^b (u_{r;b} - u_{b;r})]$$
$$u^a P^b J_u J_u h_{ab} = \dot{k}_1 + \tfrac{1}{2} K_1 P^r P^b (u_{r;b} - u_{b;r})$$
$$u^a P^b J_u J_u h_{ab} = \dot{K}_1 ,$$

since double inner product of symmetric with skew symmetric always vanishes.

(3) Again, contracting $J_u J_u h_{ab}$ with R^b and u^a

$$R^b J_u J_u h_{ab} = R^b[-\dot{K}_1(u_a P_b + u_b P_a) - 2K_1^2 (P_a P_b + u_a u_b) - K_1 K_2 (u_a Q_b + u_b Q_a) + \tfrac{1}{2} K_1 P^r u_a (u_{r;b} - u_{b;r}) + \tfrac{1}{2} K_1 P^r u_b (u_{r;a} - u_{a;r}) + \hat{K}_1^2 u_a u_b]$$

$$= \tfrac{1}{2} K_1 P^r u_a R^b (u_{r;b} - u_{b;r}), \quad \text{since } R^b P_b = 0, \; R^b u_b = 0, \; R^b Q_b = 0.$$

$$u^a R^b J_u J_u h_{ab} = u^a [\tfrac{1}{2} K_1 P^r u_a R^b (u_{r;b} - u_{b;r})]$$

$$u^a R^b J_u J_u h_{ab} = \tfrac{1}{2} K_1 P^r R^b (u_{r;b} - u_{b;r})$$

$$= \tfrac{1}{2} K_1 (\gamma_{124} - \gamma_{142})$$

(4) $J_u J_u h_{ab}$ when transvected with Q^b, yields.

$$Q^b J_u J_u h_{ab} = K_1 K_2 u_a + \tfrac{1}{2} K_1 P^r u_a Q^b (u_{r;b} - u_{b;r})$$

Consequently, we get,

$$u^a Q^b J_u J_u h_{ab} = K_1 K_2 + \tfrac{1}{2} K_1 P^r Q^b (U_{r;b} - U_{b;r})$$

$$u^a Q^b J_u J_u h_{ab} = K_1 K_2 + \tfrac{1}{2} K_1 (\gamma_{123} - \gamma_{132})$$

(5) Again contracting $J_u J_u h_{ab}$ with P^b and P^a

$$P^b J_u J_u h_{ab} = \dot{K}_1 u_a + 2 K_1^2 P_a + \tfrac{1}{2} K_1 P^r u_a P^b (u_{r;b} - u_{b;r})$$

$$P^a P^b J_u J_u h_{ab} = -2 K_1^2$$

This shows that $J_u J_u h_{ab} = 0$ implies that $K_1 = 0$.

4. ON THE NONSTATIONARY CHARACTER OF RELATIVISTIC SERRET-FRENET TETRAD :

We examine in this section the Jaumann transport of the whole RSF-tetrad and show that it is not feasible.

i) We have already shown that

$$J_u u^a = k_1 P^a .$$

Since $k_1 \neq 0$ for the existence of the RSF-tetrad, we infer that $J_u u^a \neq 0$.

ii) $$J_u P^a = P^a_{;k} u^k - P^k \omega_k^{\cdot a}$$
$$= k_1 u^a + [k_2 + \frac{1}{2}(\gamma_{123} - \gamma_{132})]\, Q^a + \frac{1}{2}(\gamma_{142} - \gamma_{124}) R^a ,$$

by (RSF-2) and computational aids : [XIII(b)] .

$J_u P^a = 0$ implies and implied by $k_1 = 0$

$$k_2 + \frac{1}{2}(\gamma_{123} - \gamma_{132}) = 0$$
$$\gamma_{142} = \gamma_{124}$$

iii) $$J_u Q^a = Q^a_{;k} u^k - Q^k \omega_k^{\cdot a}$$
$$= [-k_2 + \frac{1}{2}(\gamma_{132} - \gamma_{123})]P^a + [k_3 + \frac{1}{2}(\gamma_{134} - \gamma_{143})]R^a ,$$

by (RSF-3) and computational aids [XIII(f)]

$J_u Q^a = 0$ iff $k_2 = \frac{1}{2}(\gamma_{132} - \gamma_{123})$

$$k_3 = \frac{1}{2}(\gamma_{143} - \gamma_{134})$$

iv) $J_u R^a = R^a_{;k} u^k - R^k \omega_k^{\cdot a}$

$$= - k_3 Q^a + \frac{1}{2} \left[(\gamma_{142} - \gamma_{124}) P^a + (\gamma_{143} - \gamma_{134}) Q^a \right]$$

by (RSF-4) and computational aids [XIII(e)].

$$= \frac{1}{2} (\gamma_{142} - \gamma_{124}) P^a + [-k_3 + \frac{1}{2} (\gamma_{143} - \gamma_{134})] Q^a$$

$J_u R^a = 0$ implies and implied by $\gamma_{142} = \gamma_{124}$.

$$k_3 = \frac{1}{2} (\gamma_{143} - \gamma_{134}) .$$

It follows that (A is the tetrad index)

$J_u \lambda^a_A \neq 0$, for $\lambda^a_A = (u^a, P^a, Q^a, R^a)$ and $k_1 \neq 0$ and

so the RSF-tetrad is not stationary.

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