CHAPTER - II

FERMI - TRANSPORT

1. INTRODUCTION :

An elegant exposition of the Fermi derivative D_F/D_s of a vector field $\bar{x}$ with respect to a time-like vector field $\bar{u}$ has been given in the famous book "The large scale structure of space-time" by Hawking and Ellis (1973) in index-free notation as

$$\frac{D_F \bar{x}}{ds} = \frac{D\bar{x}}{ds} - g\left(\bar{x}, \frac{d\bar{x}}{ds}\right)\bar{u} + g(\bar{x},\bar{u})\frac{D\bar{u}}{ds}$$

Later Radhakrishna and Bhosle (1975) considered due corrections for signature of the metric tensor (-, -, -, +) on the form of the Fermi derivative and gave the following notation for the Fermi derivative of x^a_b along $\bar{u}$.

$$F_u\, X^a_b = x^a_{b;k}\, u^k + x^k_b(\dot{u}_k u^a - u_k \dot{u}^a) + x^a_k(\dot{u}^k u_b - u^k \dot{u}_b) .$$

For geodesic motion $\dot{u}^a = 0$ and then

$$F_u\, x^a{}_b = \dot{x}^a{}_b$$

i.e., the material derivative and Fermi derivative are identical. Thus, Fermi derivative is a generalization of the material or intrinsic derivative.

Material tensors : If $x^a u_a = 0$, then, we get,

$$F_u x^a = \perp \dot{x}^a$$

where $\perp$ is the 3-dimensional projection operator defined by

$$\perp A^a = \gamma^a_b A^b = (\delta^a_b - u^a u_b) A^b.$$

(Note : $\gamma^a_a = 3, \gamma^b_a \gamma^c_b = \gamma^c_a$)

We comment that the Fermi derivative of a material tensor is again a material tensor. In other words Fermi derivative of a material tensor measures the spatial growth of the tensor (Bressan, 1978).

We observe that

i) $F_u g_{ab} = 0$

ii) $F_u u^a = 0$

iii) $F_u u_a = 0$

iv) $F_u \gamma_{ab} = F_u (g_{ab} - u_a u_b) = 0$

Thus the gravitational potentials, the flow vector and the three-dimensional projection operator are Fermi transported, identically.

Physical Interpretation of the Fermi Transported Tetrad :

Under Fermi transport the magnitude of a vector is unchanged and the scalar product of two vectors is unchanged. Hence, if we define any orthonormal tetrad $\lambda^k_{(a)}$ at any point on a curve γ of a congruence, it remains orthonormal tetrad at any other point on the curve γ of the congruence if the tetrad is Fermi transported. This property is shared by parallel transport.

Fermi Transport and Parallel Transport :

The parallel transport is defined by a simple equation and it is the natural transport for the comparison of two vectors at two different points in the space. But the Fermi transport is more important in some physical situations for the formation of physical laws (Stephani, 1982). Both transports conserve an orthonormal tetrad under advance along the curve of a time-like congruence but still a great difference is between these two transports. If we define an orthonormal tetrad $\lambda^{k}_{(a)}$ on a curve γ of a congruence at some points so that $\lambda^{k}_{(4)} = u^{k}$, it is observed that $\lambda^{k}_{(4)}$ at any another point on the curve does not coincide with u^{k} under parallel transport unless the congruence is geodesic. But under Fermi transport $\lambda^{k}_{(4)}$ remains tangent to the curve γ of a congruence. This provides us the physical interpretation of the Fermi transported tetrad (Synge, 1960).

Remark :

The Fermi transport not only conserves the orthonormal tetrad along the time-like congruence, but it also provides an orthonormal triad of congruences orthogonal to the time-like congruence. "This Fermi transported tetrad system is the best approximation to the co-ordinated system of an observer who employs locally a non-rotating inertial system in the sense of Newtonian mechanics" (Stephani, 1982).

2. FERMI TRANSPORT OF $\perp_{ab}$:

We now propose to study the Fermi transport of the 2-dimensional projection operator $\perp_{ab} = g_{ab} - u_a u_b + P_a P_b$. Obviously $\perp_{ab}$ is not Fermi transported identically, since

$$F_u P_a \neq 0$$

The exact expression for $F_u \perp_{ab}$ is derived in the following theorem. In fact we determine the necessary and sufficient conditions for the vanishing of $F_u \perp_{ab}$.

There are three types of 2-dimensional projection operators, namely

i) $\perp_{ab} = g_{ab} - u_a u_b + P_a P_b$

ii) $\perp'_{ab} = g_{ab} - u_a u_b + Q_a Q_b$

iii) $\perp''_{ab} = g_{ab} - u_a u_b + R_a R_b$

We study their Fermi transports in the following discussions.

<u>Theorem - 1</u> : The following are equivalent (TFAE).

1) $\perp_{ab}$ is Fermi transported.

2) $k_2 = 0$ (Torsion of the stream line vanishes).

<u>Proof</u> : (1) implies (2).

We consider,

$F_u \perp_{ab} = F_u (g_{ab} - u_a u_b + P_a p_b)$, by definition of $\perp_{ab}$.

$= F_u g_{ab} - F_u u_a u_b + F_u P_a P_b$, due to linearity of F.

$= P_a F_u P_b + P_b F_u P_a$, since $F_u u_b = 0$, $F_u g_{ab} = 0$.

$= P_a[\dot{P}_b + P_k(\dot{u}^k u_b - \dot{u}_b u^k)] + P_b[\dot{P}_a + P_k(\dot{u}^k u_a - \dot{u}_a u^k)]$

by definition of $F_u P_a$

$= P_a(\dot{P}_b + P_k \dot{u}^k u_b) + P_b(\dot{p}_a + p_k \dot{u}^k u_a)$, since $p_k u^k = 0$

$= P_a (k_1 u_b + k_2 Q_b) + P_b(k_1 u_a + k_2 Q_a) + P_k \dot{u}^k (P_a u_b + P_b u_a)$

by (RSF-2).

$= (P_a u_b + P_b u_a)(k_1 + P_k \dot{u}^k) + P_a Q_b(k_2) + P_b Q_a(k_2)$

on rearrangement .

$= k_2 (P_a Q_b + P_b Q_a)$, since $k_1 + P_k \dot{u}^k = 0$.

We have the relation

$$F_u \perp_{ab} = k_2 (P_a Q_b + P_b Q_a) \qquad \dots (2.1)$$

On contracting $F_u \perp_{ab}$ with p^a, we obtain,

$$P^a F_u \perp_{ab} = k_2 (P^a P_a Q_b + P_b P^a Q_a)$$
$$= - k_2 Q_b, \text{ since } P^a P_a = -1, \; P^a Q_a = 0$$

Transvecting with Q^b, we get .

$$Q^b P^a F_u \perp_{ab} = -k_2 Q^b Q_b$$

$$Q^b P^a F_u \perp_{ab} = k_2 \quad , \quad \text{since } Q^b Q_b = -1 \qquad \dots (2.2)$$

It follows that

when, $F_u \perp_{ab} = 0$,

we get $k_2 = 0$, from (2.2)

(2) implies (1) :

Put $k_2 = 0$ in (2.1), to get

$$F_u \perp_{ab} = 0$$

Hence,

$$F_u \perp_{ab} = 0 \quad \text{iff } k_2 = 0$$

This completes the proof.

We now investigate the Fermi transport of the 2-dimensional operator

$$\perp'_{ab} = g_{ab} - u_a u_b + Q_a Q_b$$

Theorem 2 : The following are equivalent (TFAE) .

1) $\perp'_{ab}$ is Fermi transported

2) $k_2 = k_3 = 0$

Proof : (1) implies (2)

We consider,

$$F_u \perp'_{ab} = F_u\,(g_{ab} - u_a u_b + Q_a Q_b)$$

$$= Q_a F_u Q_b + Q_b F_u Q_a\,, \text{ since } F_u g_{ab} = 0,\ F_u u_a = 0,$$

$$= Q_a[\dot{Q}_b + Q_k(\dot{u}_k u_b - \dot{u}_b u k)] + Q_b[\ \dot{Q}_a + Q_k(\dot{u}^k u_a - \dot{u}_a u^k)\ ]$$

$$= Q_a\dot{Q}_b + \dot{Q}_b Q_a + Q_a u_b Q_k \dot{u}^k + Q_b Q_k \dot{u}^k u_a, \text{ since } Q_k u^k = 0.$$

$$F_u \perp'_{ab} = Q_a\dot{Q}_b + Q_b\dot{Q}_a + Q_k\dot{u}^k\,(Q_a u_b + Q_b u_a) \qquad \text{... (2.3)}$$

$$F_u \perp'_{ab} = Q_a(-k_2 P_b + k_3 P_b) + Q_b(-k_2 P_a + K_3 P_a) + Q_k k_1 P^k (Q_a u_b + Q_b u_a)$$

(by RSF – 2)

$$= -k_2\,(Q_a P_b + Q_b P_a) + K_3(Q_a R_b + Q_b R_a) + k_1 Q_k P^k (Q_a u_b + Q_b u_a).$$

$$F_u \perp'_{ab} = -k_2(Q_a P_b + Q_b P_a) + K_3(Q_a R_b + Q_b R_a)\ , \qquad \text{... (2.4)}$$

since $Q_k P^k = 0$

contracting $F_u \perp'_{ab}$ with Q^a and R^b successively, we get .

$$Q^a F_u \perp'_{ab} = Q^a\ [\ -k_2(Q_a P_b + Q_b P_a) + k_3(Q_a R_b + Q_b R_a)\ ]$$

$$= k_2 P_b - k_3 R_b\ , \text{ since } Q^a Q_a = -1,\ Q^a P_a = 0,\ Q^a R_a = 0\ .$$

$$R^b Q^a F_u \perp'_{ab} = R^b(k_2 P_b - k_3 R_b)$$

$$R^b Q^a\ (F_u \perp'_{ab}) = k_3 \qquad \text{... (2.5)}$$

Now, transvecting (2.4) with P^a and Q^b, we have .

$$P^a F_u \perp'_{ab} = P^a[-k_2(Q_a P_b + Q_b P_a) + k_3[Q_a R_b + Q_b R_a)]$$

$$= k_2 Q_b + 0, \text{ since } P^a Q_a = 0,\ P^a P_a = -1,\ P^a R_a = 0\ .$$

$$P^a F_u \perp'_{ab} = k_2 Q_b\ .$$

$Q^b P^a F_u \perp'_{ab} = Q^b . k_2 Q_b$, since $Q^b Q_b = -1$

$Q^b P^a \, (F_u \perp'_{ab}) = -k_2$ .... (2.6)

Hence, from (2.5) and (2.6), we have,

$F_u \perp'_{ab} = 0$ implies $k_2 = k_3 = 0$

Conversely, if, $k_2 = 0, \; k_3 = 0$. implies $F_u \perp'_{ab} = 0$ by (2.4)

This completes the proof.

Note :

(a) The two non-vanishing double inner products are

i) $R^b Q^a \, (F_u \perp'_{ab}) = k_3$.

ii) $Q^b P^a \, (F_u \perp'_{ab}) = -k_2$.

(b) Other 14 double inner products with $F_u \perp'_{ab}$ vanish, identically.

(c) $k_2 = 0$ implies $k_3 = 0$, by definition of k_3.

Theorem - 3 : The following are equivalent (TFAE).

1) $\perp''_{ab}$ is Fermi transported.

2) $k_3 = 0$, (Bitorsion or third curvature vanishes).

Proof : (1) implies (2)

We consider,

$F_u \perp''_{ab} = F_u (g_{ab} - u_a u_b + R_a R_b)$

$= F_u g_{ab} - u_a F_u u_b - u_b F_u u_a + R_a F_u R_b + R_b F_u R_a$, since linearity of F.

$= R_a F_u R_b + R_b F_u R_a$, since $F_u g_{ab} = 0$, $F_u u_b = 0$.

$= R_a [\, \dot{R}_b + R_k (\dot{u}^k u_b - u^k \dot{u}_b)] + R_b [\dot{R}_a + R_k (\dot{u}^k u_a - u^k \dot{u}_a)]$, by definition.

$F_{u}\perp''_{ab} = R_a\dot{R}_b + R_b\dot{R}_a + R_k\dot{u}^k(R_a u^b + R_b u_a)$, since $R_k u^k = 0$.

$= R_a(-k_3 Q_b) + R_b(-k_3 Q_a) + R_k K_1 P^k(R_a u_b + R_b u_a)$,

by (RSF-I and II).

$F_{u}\perp''_{ab} = -k_3(R_a Q_b + R_b Q_a)$, since $R_k P^k = 0$ ...(2.7)

Contracting (2.7) with Q^b

$Q^b F_u \perp''_{ab} = Q^b \, [-k_3(R_a Q_b + R_b Q_a)\,]$

$Q^b F_u \perp''_{ab} = -\,k_3$, since $R^a R_a = Q^b Q_b = -1$, $Q^b R_b = 0$ ... (2.8)

Other double inner products of $F_{u}\perp''_{ab}$ vanish identically

Hence, $F_u \perp''_{ab} = F_u(g_{ab} - u_a u_b + R_a R_b) = -\,k_3 \; (R_a Q_b + R_b Q_a)$

It follows that, when, $F_u \perp''_{ab} = 0$,

we get, $k_3 = 0$, from (2.8)

(2) implies (1),

Put $k_3 = 0$ in (2.7), to get, $F_u \perp''_{ab} = 0$

Hence,

$F_u \perp''_{ab} = 0$ iff $k_3 = 0$

This completes the proof.

3. THE FERMI TRANSPORT OF THE RELATIVISTIC SERRET-FRENET TETRAD :

i) $F_u u^a = 0$, by property.

ii) $F_u P^a = \dot{P}^a + P^k (\dot{u}_k u^a - u_k \dot{u}^a)$

$= k_1 u^a + k_2 Q^a + P^k k_1 P_k u^a$, by(RSF-1,2)and $P^k u_k = 0$.

$F_u P^a = k_2 Q^a$

iii) $F_u Q^a = \dot{Q}^a + Q^k (\dot{u}_k u^a - u_k \dot{u}^a)$

$= -k_2 P^a + k_3 R^a$, since $Q^k u_k = 0$ $Q^k P_k = 0$.

iv) $F_u R^a = \dot{R}^a + R^k (\dot{u}_k u^a - u_k \dot{u}^a)$

$= - k_3 Q^a + R^k k_1 P_k u^a$, since $R^k u_k = 0$.

$= - k_3 Q^a$.

Serret Frenet tetrad satisfies

$$F_u u^a = 0$$
$$F_u P^a = k_2 Q^a$$
$$F_u Q^a = -k_2 P^a + k_3 R^a$$
$$F_u R^a = - k_3 Q^a .$$

Since $k_2 = 0$ implies $k_3 = 0$, we infer that the necessary and sufficient condition for the Fermi transport of relativistic Serret-Frenet tetrad is that $k_2 = 0$.

REFERENCES

BRESSAN,A. (1978). Relativistic Theories of Materials. Springer-Verlag, Berlin Heidelberg, New York.

GUMASTE,S.P. (1984). Congruences in General Relativity. Ph.D. Thesis, Chapt. IV. Shivaji University, Kolhapur, p. 121-155.

HAWKING,S.W. and ELLIS,H.R. (1973). The Large Scale Structure of Space-time Physics. Cambridge University Press, Cambridge, p.161.

KATKAR,L.N. (1981). On the Application of Newman-Penrose Formalism in Relativistic Electromagnetic Fields. Unpublished Ph.D. thesis, submitted to Shivaji University, Kolhapur.

LIGHTMAN,A.P., PRESS,W.H., PRICE,R.H. and TEUKOLSKY,S.A. (1975). Problem book in Relativity and Gravitatiion, Princeton University, Press.

RADHAKRISHNA,L. and BHOSALE,D.R. (1975-76). On the Fermi Propagation and Lie Propagation of Tensors in General Relativity, Proc. Int. Symp. Rel. and Unified Field Theory, Calcutta, pp.265-268.

STEPHANI,H. (1982). General Relativity, An Introduction to the Theory of the Gravitational Field. Cambridge University Press, New York, 166 ·

SYNGE, J.L. (1960). Relativity : The General Theory. North-Holland Publishing Company, Amsterdam.