CHAPTER III

ELEMENTS AND THEIR PROPERTIES

SECTION (I) : INTRODUCTION

The trial functions must be compatible with the corresponding element, to enable the coefficients a_i, in the trial function to be determined uniquely. It can be shown that the trial function, the location and number of the element nodes as well as the number of unknowns per node cannot all be specified independently. The type and order of the governing equation and the convergence requirements of the variational procedure must also be taken into account when selecting elements and their trial functions.

The general trial function representation over any element e is the linear form.

$$\hat{u} = N^e u^e \qquad \text{.....(3.1)}$$

where N is the shape function matrix and **u** is the element nodal vector.

For a Lagrangian element there is only one degree of freedom per node namely the value of the function and hence eq(3.1) can be written as

$$\hat{u} = \begin{bmatrix} N_1 & N_2 & \ldots N_k & \ldots N_s \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \cdot \\ \bar{u}_k \\ \cdot \\ \bar{u}_s \end{bmatrix} \qquad \text{.....} \quad (3.2)$$

where 1,2,....., k, ... s are the node identifiers and s is the total number of nodes of element e.

For a Hermitian element, each of the $\bar{u}_K$, must be regarded

as being a column matrix itself, since the derivatives of the function now appear as variables at the nodes also. If each of the s nodes of element e has q degrees of freedom, then the entries $\bar{u}_k$ in eq (3.2) are to be considered as column matrices.

$$\bar{u}_k = \begin{bmatrix} \bar{u}_{k1} \\ \bar{u}_{k2} \\ . \\ \bar{u}_{kq} \end{bmatrix}$$

and similarly the shape functions N_k are to be considered as row matrices.

$$N_k = [N_{k1}\ N_{k2}\ \ldots\ldots\ldots N_{kq}]$$

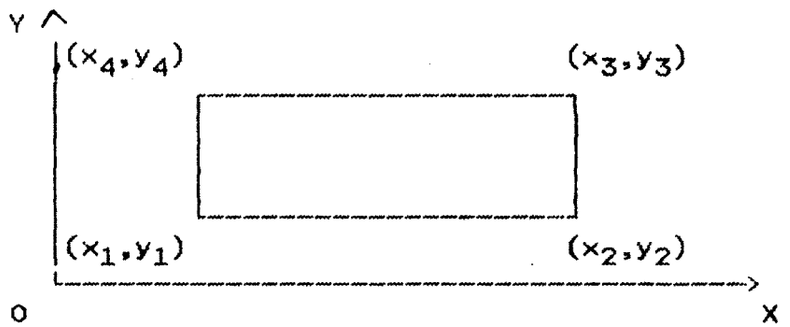
$$k = 1, 2, 3, 4, \ldots\ldots, s$$

The shape functions are functions of the independent variables and the nodal co-ordinates. To derive shape functions for any selected element, two methods are available.
The first which uses generalised co-ordinates and the second interpolation functions.

SECTION (II) : Deriving Shape Functions From Generalised Co-ordinates

This approach is particularly suitable for those simple elements based on low order polynomials which are complete.

To illustrate this procedure we use rectangular element e with sides parallel to the global system O_{xy}



The simplest oral function for the element contains only four unknown constants α_i corresponding to the elements four nodes.

$$\hat{u} = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \qquad \text{.... (3.3)}$$

which is sen to vary linearly along the element boundaries.

Applying (3.3) to the four nodes yields the equation

$$\begin{aligned} \bar{u}_1 &= \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 x_1 y_1 \\ \bar{u}_2 &= \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 x_2 y_2 \\ \bar{u}_3 &= \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 x_3 y_3 \\ \bar{u}_4 &= \alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 x_4 y_4 \end{aligned} \qquad \text{....(3.4)}$$

In the matrix notation this equation(3.4) can be written as

$$\mathbf{u} = \mathbf{A}\,\boldsymbol{\alpha} \qquad \text{....(3.5)}$$

where **u** is the element nodal vector

$$\mathbf{u} = [u_i] = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{bmatrix} \qquad \text{....(3.6)}$$

and

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}$$

$$\alpha = [\alpha_i] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

We can write equation (3.3) as

$$u = X \alpha \qquad \text{.... (3.7)}$$

where $X = [1 \quad x \quad y \quad xy]$

Matrix A can be inverted provided that its determinant does not vanish.

$$\det A = -\Delta^2 \qquad \text{.... (3.8)}$$

where Δ is the area of the rectangle, which never vanishes.

From eq (3.5) we have

$$\alpha = A^{-1} u \qquad \text{.... (3.9)}$$

Substituting equation (3.9) in equation (3.7) we have

$$\hat{u} = X A^{-1} u \qquad \text{.... (3.10)}$$

which can be written as

$$\hat{u} = N u \qquad \text{.... (3.11)}$$

where shape function matrix N can be written as

$$N = X A^{-1} \qquad \text{.... (3.12)}$$

SECTION (III) : Deriving Shape Functions from Interpolation Formula

Consider the rectangular element e shown in the previous figure, but suppose that the trial function representation of equation (3.3) is not known. The shape function relation for can be written in the general form i.e.,

$$\hat{u} = N u$$

$$\hat{u} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{bmatrix} \qquad \ldots\ldots (3.13)$$

Each shape function N_k, must have the value unity at node k and zero at every other node so that $\hat{u}$ will reduce to $\bar{u}_k$ when equation (3.13) is applied to node k. This property allows interpolating formulas to be used to derive the shape function.

Consider the approximation of the function u(x) by a p^{th} order polynomial where the values of u(x) are given as u_1, u_2, u_3,....., u_p at the (p+1) points x_1, x_2,....., x_p

$$u(x) = \sum_{i=1}^{p+1} L_i(x)\ u_i \qquad \ldots\ldots (3.14)$$

where the L_i (x) are the Lagrange polynomials defined by

$$L_i(x) = \prod_{j=1, j\neq i}^{p+1} \frac{(x - x_j)}{x_i - x_j} \qquad \ldots\ldots (3.15)$$

x_1, x_2, ..., x_p need not be spaced equally.

Applying equations (3.14) and (3.15) to side 1-2 of the rectangle e in the figure allows along this side to be written as

$$\hat{u}\Big|_{1-2} = L_1(x)\ \bar{u}_1 + L_2(x)\ \bar{u}_2 \qquad \ldots\ldots (3.16)$$

Where

$$L_1(x) = \frac{x - x_2}{x_1 - x_2}$$

$$L_2(x) = \frac{x - x_1}{x_2 - x_1} \qquad \ldots\ldots (3.17)$$

Similarly

$$\hat{u}\Big|_{4-3} = \bar{L}_1(x)\, u_3 + \bar{L}_2(x)\, u_4 \qquad \text{.... (3.18)}$$

Similarly in the y direction to give the interpolation over the element as

$$\hat{u} = L_1(y)\, \hat{u}\Big|_{1-2} + L_2(y)\, \hat{u}\Big|_{4-3} \qquad \text{.... (3.19)}$$

Where

$$L_1(y) = \frac{y - y_2}{y_1 - y_2}$$

$$L_2(x) = \frac{y - y_1}{y_2 - y_1} \qquad \text{.... (3.20)}$$

$$\hat{u} = \sum_{i=1}^{p+1} L_i(x)\, u_i \qquad \text{.... (3.21)}$$

$$\hat{u} = L_1(x)L_2(y)\, \bar{u}_1 + L_2(x)\, L_1(x)\, \bar{u}_2 + L_1(x)\, L_2(y)\, \bar{u}_3 + L_2(x)\, L_2(y)\, u_4$$

$$\hat{u} = \sum_{i=1}^{4} N_i\, \bar{u}_i$$

where

$$N_1 = L_1(x)\, L_2(y)$$

$$N_2 = L_2(x)\, L_1(y)$$

$$N_3 = L_1(x)\, L_1(y)$$

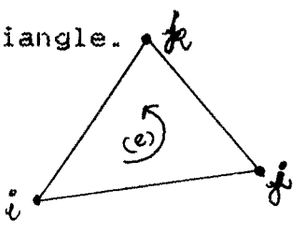
$$N_4 = L_2(x)\, L_2(y)$$

SECTION (IV): TYPES OF ELEMENTS

Any two dimensional region can be subdivided into triangular, and rectangular or quadrilateral elements.

1) Triangular Elements

The triangle is a particularly useful shape for any two dimensional analysis, as assemblies of triangles can easily be used to represent accurately regions enclosed by boundaries of quite complex shape. For a typical triangular element e, with nodes numbered anticlockwise as i, j, k and placed at the vertices of the triangle.



We shall look for an element shape function $N_i(x,y)$ such that N_i has the value unity at mode i and zero at nodes j and k. The global shape function N_i must be continuous across element boundaries and only nonzero on elements associated with node i. Assuming a linear form for N_i^e, we write

$$N_i^e = a + bx + cy \qquad \text{.....(3.22)}$$

where a, b, c are parameters defined for element e .

We evaluate (3.22) at each node and obtain

$$\begin{aligned} N_i &= a + bx_i + cy_i \\ N_j &= a + bx_j + cy_j \\ N_k &= a + bx_k + cy_k \end{aligned} \qquad \text{.....(3.23)}$$

$$\mathbf{a} = A^{-1} \begin{Bmatrix} u_i \\ u_j \\ u_k \end{Bmatrix}$$

where $\mathbf{a} = \{a\ b\ c\}^T$

and

$$A = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

The equation (3.22) may be written as

$$N^e = \begin{bmatrix} 1 & x & y \end{bmatrix} A^{-1} \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix} \qquad \ldots.(3.24)$$

where

$$A^{-1} = \frac{1}{2\Delta^e} \begin{bmatrix} x_j y_k - x_k y_j & x_k y_i - x_i y_k & x_i y_j - x_j y_i \\ y_j - y_k & y_k - y_i & y_i - y_j \\ x_k - x_j & x_i - x_k & x_j - x_i \end{bmatrix}$$

and

$$\Delta^e = \frac{1}{2} \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$

= 2 (area of element e)

Using the linear piecewise approximation solution over the element e may be written as

$$u^e = N_i u_i + N_j u_j + N_k u_k = N^e \phi^e$$

where $N^e = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix}$ and $\phi^e = \begin{bmatrix} u_i & u_j & u_k \end{bmatrix}$

$$N_i = \frac{1}{2\Delta^e} (a_i + b_i x + c_i y)$$

$$N_j = \frac{1}{2\Delta^e} (a_j + b_j x + c_j y)$$

$$N_k = \frac{1}{2\Delta^e} (a_k + b_k x + c_k y)$$

$$a_i = x_j y_k - x_k y_j \;,\; a_j = x_k y_i - x_i y_k \;,\; a_k = x_i y_j - x_j y_i$$

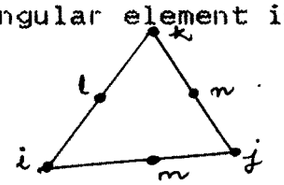
$$b_i = y_j - y_k \;,\; b_j = y_k - y_i \;,\; b_k = y_i - y_j$$

$$c_i = x_k - x_j \;,\; c_j = x_i - x_k \;,\; c_k = x_j - x_i$$

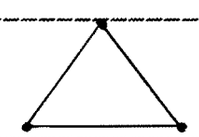
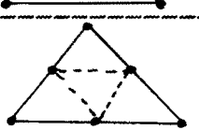
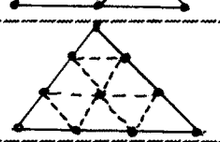
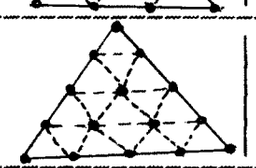
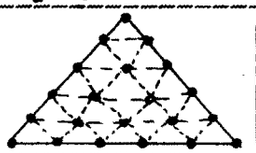
For a Quadratic triangular element , we use the quadratic approximation function of the form

$$u^e = N_i u_i + N_j u_j + N_k u_k + N_m u_m + N_n u_n + N_l u_l = N^e \phi^e$$

where the six node triangular element is as shown in the figure



THE LAGRANGIAN TRIANGULAR FAMILY

ELEMENT	TYPE	ORDER OF POLYNOMIAL USED AS TRIAL FUNCTION	NO. OF TERMS IN TRIAL FUNCTION
	LINEAR	1	3
	QUADRATIC	2	6
	CUBIC	3	10
	QUARTIC	4	15
	QUINTIC	5	21

Triangular elements of this family can be formed by selecting a sufficient number of nodes to allow a unique solution for the coefficients in the chosen polynomial trial function.

A complete polynomial of order n contains 1/2(n+1)(n+2) coefficients and an s-node Lagrangian triangular element, based on this polynomial, must contain the same number of nodes, hence

$$s = 1/2(n+1)(n+2)$$

2) RECTANGULAR AND QUADRILATERAL ELEMENTS

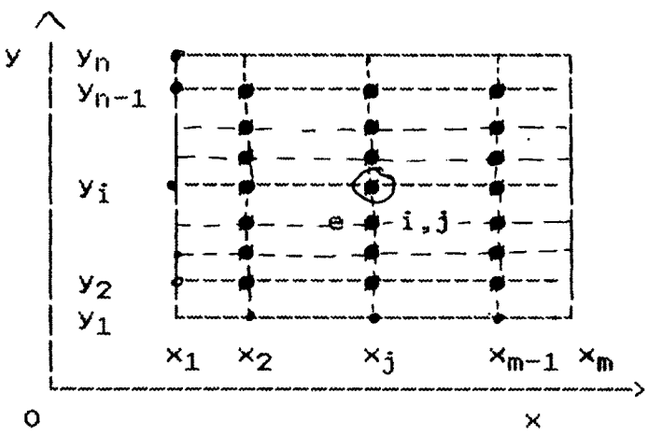
Rectangular elements are not well suited to irregular two dimensional regions, but are used in combination with the more extensively adopted triangular elements.

Lagrangian Rectangular Elements :-

Consider the two-dimensional element e in which there are m equally spaced nodes in each row and n equally spaced nodes in each column. For a node ij, the shape function is defined by the product of two Lagrangian polynomials as

$$N_{ij} = L_j^m(x) * L_i^n(y)$$

where $L_j(x)$, $L_i(y)$ are defined by equations (3.17) and (3.20) and the superscripts m and n are used to indicate the order of the polynomial.



Lagrangian Rectangular element e.

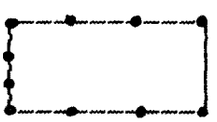
The Lagrange's interpolation relation $\hat{u}$ for this mn Lagrangian element can be written as

$$\hat{u} = N_{11}\,\bar{u}_{11} + N_{12}\,\bar{u}_{12} + \cdots\cdots + N_{1m}\,\bar{u}_{1m} + N_{21}\,\bar{u}_{21} + \cdots\cdots + N_{2m}\,\bar{u}_{2m} + \cdots\cdots + N_{n1}\,\bar{u}_{n1} + N_{n2}\,\bar{u}_{n2} + \cdots + N_{nm}\,\bar{u}_{nm}$$

$$\hat{u} = \sum_{i=1}^{n}\sum_{j=1}^{m} N_{ij}\,u_{ij} \qquad \cdots\cdots (3.23)$$

Lagrange elements of this family have interelement continuity in $\hat{u}$ only and have an incomplete interpolation polynomial. Except for the first member of this family, the bilinear element, the Lagrangian elements suffer from the disadvantages of having internal nodes and giving poor curve fitting especially for higher order polynomials. Therefore, other than the bilinear element, they are rarely used.

3) SERENDIPITY ELEMENTS

ELEMENT	TYPE	ORDER OF POLYNOMIAL USED AS TRIAL FUNCTION
	LINEAR	4
	QUADRATIC	8
	CUBIC	12

The Serendipity Family of Elements

These elements posses an equal number of nodes in the x and y directions. The description of these elements as linear,quadratic, cubic refers to the variation of the trial function in the x direction at constant y, or in the y direction at constant x.

The first three members of this family of elements are as shown above

The trial functions for these elements are incomplete quadratic, cubic and quartic polynomials respectively.The shape functions for the three elements in terms of the local co-ordinates can be obtained by using the following incomplete polynomial trial functions.

Linear :

$$\hat{u} = \alpha_1 + \alpha_2 \epsilon + \alpha_3 \eta + \alpha_4 \epsilon\eta$$

Shape function for Corner Nodes

$$N_i = 1/4 \, (1 + \epsilon \epsilon_i) \, (1 + \eta \eta_i)$$

Quadratic :

$$\hat{u} = \alpha_1 + \alpha_2 \epsilon + \alpha_3 \eta + \alpha_4 \epsilon^2 + \alpha_5 \epsilon\eta + \alpha_6 \eta^2 + \alpha_7 \epsilon^2\eta + \alpha_8 \epsilon\eta^2$$

Shape Function For Corner Nodes

$$N_i = 1/4 \, (1 + \epsilon \epsilon_i) \, (1 + \eta \eta_i) (\epsilon \epsilon_i + \eta \eta_i - 1)$$

For Side Nodes

(i) $\epsilon_i = 0 \Rightarrow N_i = 1/2 \, (1 - \epsilon^2) \, (1 + \eta \eta_i)$

(ii) $\eta_i = 0 \Rightarrow N_i = 1/2 \, (1 + \epsilon \epsilon_i) \, (1 - \eta^2)$

Cubic:

$$\hat{u} = \alpha_1 + \alpha_2 \epsilon + \alpha_3 \eta + \alpha_4 \epsilon^2 + \alpha_5 \epsilon\eta + \alpha_6 \eta^2 + \alpha_7 \epsilon^3 + \alpha_8 \epsilon^2\eta + \alpha_9 \epsilon\eta^2 + \alpha_{10} \eta^3 + \alpha_{11} \epsilon^2\eta + \alpha_{12} \epsilon\eta^3$$

Shape Functions For Corner Nodes

$$N_i = 1/32 (1 + \epsilon \epsilon_i) (1 + \eta \eta_i) \text{Æ } 9 (\epsilon^2 + \eta^2) - 10 \text{ Å}$$

Shape Functions For Side Nodes

(a) $\epsilon_i = \pm 1$

$\eta_i = \pm 1/3$

$$N_i = 9/32 (1 + \epsilon \epsilon_i) (1 - \eta^2) (1 + \eta \eta_i)$$

(b) $\epsilon_i = \pm 1/3$

$\eta_i = \pm 1$

$$N_i = 9/32 (1 - \epsilon^2) (1 + \eta \eta_i) (1 + 9 \epsilon \epsilon_i)$$

Along element boundaries, the trial function of a Serendipity element is a complete polynomial and hence there esists interelement continuity of the trial function.

Serendipity elements form a useful class of rectangular elements, which if combined with triangular elements, can be used quite effectively in regions with curved boundaries.

4) ISOPARAMETRIC ELEMENTS

In problems that have curved boundaries, it is necessary to use many straight sided (faced) elements along the boundaries in order to achieve a reasonable geometric representation of these boundaries. While other methods of creating curved elements exist, the only method used extensively involves mapping from regular (straight-edged or sided) elements. Since the shape functions of the regular parent element are known with respect to a local co-ordinate system, those of the generated curvilinear element can also be determined

The mapping from local co-ordinates ϵ, η to cartesian coordinates x, y is through the shape function relationships

$$x = \mathbf{N}_m \mathbf{x}$$
$$y = \mathbf{N}_m \mathbf{y} \qquad \ldots\ldots (3.24)$$

Shape functions $\mathbf{N}_m$ are functions of ϵ, η and the column matrices $\mathbf{x}$ and $\mathbf{y}$ are the nodal co-ordinates with respect to the global system. The trial function $\hat{u}$ can be written with respect to the local co-ordinate system as

$$\hat{u} = \mathbf{N}\,\mathbf{u} \qquad \ldots\ldots (3.25)$$

Where the elements of the shape function matrix N are functions of ϵ and η

If the shape function matrices $\mathbf{N}_m$ and $\mathbf{N}$ are identical in form, the generated element is termed as isoparametric. If the shape function matrix $\mathbf{N}_m$ is a lower order representation than the matrix $\mathbf{N}$, the generated curvilinear element is subparametric and if $\mathbf{N}_m$ is a higher order representation, the element is superparametric.

SECTION (V): TRANSFORMATION FROM LOCAL TO GLOBAL CO-ORDINATES

Many shape functions can most simply be expressed with reference to a particular local co-ordinate system. In such cases, the resulting element matrix equations will contain the unknown variables relative to the local system. These element matrix equations must be transformed to their corresponding equations with respect to the global system before assembly into the system matrix equation.

Let ϕ be a scalar-value function of x and y defined on an element Ω_e. Then we can convert ϕ to a function $\hat{\phi}$ of ϵ and η defined on $\hat{\Omega}$ by setting

$$\phi(x,y) = \phi(x(\epsilon, \eta) , y(\epsilon, \eta))$$
$$\phi(x,y) = \hat{\phi}(\epsilon, \eta) \qquad \text{.... (3.26)}$$

where $x(\epsilon, \eta)$ and $y(\epsilon, \eta)$ are given by

$$x = \sum_{j=1}^{N_e} x_j \hat{N}_j (\epsilon, \eta) ; \qquad y = \sum_{j=1}^{N_e} y_j \hat{N}_j (\epsilon, \eta) \qquad ...(3.27)$$

Thus element shape functions $N_j^e = N_j^e (x, y)$ for Ω are simply obtained from $\hat{N}_j (\epsilon, \eta)$ by

$$N_j^e (x, y) = \hat{N}_j (\epsilon(x, y) , \eta(x, y)) \qquad \text{.... (3.28)}$$
$$j = 1, 2, \ldots N_e$$

The derivatives of N_j^e are obtained by the chain rule

$$\frac{\partial N_j^e}{\partial x} = \frac{\partial \hat{N}_j}{\partial \epsilon}\frac{\partial \epsilon}{\partial x} + \frac{\partial \hat{N}_j}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_j^e}{\partial y} = \frac{\partial \hat{N}_j}{\partial \epsilon}\frac{\partial \epsilon}{\partial y} + \frac{\partial \hat{N}_j}{\partial \eta}\frac{\partial \eta}{\partial y} \qquad \text{.... (3.29)}$$

The derivatives of x and y can obtained as

$$\frac{\partial x}{\partial \epsilon} = \sum_{k=1}^{N_e} x_k \frac{\partial \hat{N}_k(\epsilon, \eta)}{\partial \epsilon}$$

$$\frac{\partial x}{\partial \eta} = \sum_{k=1}^{N_e} x_k \frac{\partial \hat{N}_k(\epsilon, \eta)}{\partial \eta}$$

$$\frac{\partial y}{\partial \epsilon} = \sum_{k=1}^{N_e} x_k \frac{\partial \hat{N}_k(\epsilon, \eta)}{\partial \epsilon}$$

$$\frac{\partial y}{\partial \eta} = \sum_{k=1}^{N_e} x_k \frac{\partial \hat{N}_k(\epsilon, \eta)}{\partial \eta}$$

Substituting equations (3.30) the above equation (3.29) we get

$$\frac{\partial \hat{N}_j}{\partial x} = \frac{1}{|J(\epsilon,\eta)|}\left\{\frac{\partial \hat{N}_j}{\partial \epsilon}\sum_{k=1}^{N_e} y_k \frac{\partial \hat{N}_k(\epsilon,\eta)}{\partial \eta} - \frac{\partial \hat{N}_j}{\partial \eta}\sum_{k=1}^{N_e} y_k \frac{\partial \hat{N}_k}{\partial \epsilon}\right\}$$

$$\frac{\partial \hat{N}_j}{\partial y} = \frac{1}{|J(\epsilon,\eta)|}\left\{\frac{\partial \hat{N}_j}{\partial \eta}\sum_{k=1}^{N_e} y_k \frac{\partial \hat{N}_k(\epsilon,\eta)}{\partial \epsilon} - \frac{\partial \hat{N}_j}{\partial \epsilon}\sum_{k=1}^{N_e} y_k \frac{\partial \hat{N}_k}{\partial \eta}\right\}$$

where $|J(\epsilon,\eta)|$ is the Jacobian of the transformation.

SECTION (VI) : **SELECTION OF AN ELEMENT**

For a particular problem, the question of which element to choose is a significant one. The complexity of programming, the total computation cost and effort and the accuracy of the solution are strongly affected by the element selected. The selection for choosing the best element depends on the type of problem, the geometry of the boundaries, the boundary conditiors, the accuracy required, the size of the available computer, the maximum available computing cost as well as other factors.

To aid in the selection of an element, the guidelines are as follows :-

(i) The trial function must be able to represent all the derivatives that occur in the functional

(ii) Elements satisfying both the completeness and conformity condition should be used only after a careful examination of their performance.

(iii) If the boundaries of the problem are regular, elements of simple geometry are usually selected, whereas for curved boundaries both regular and curved elements have to be used. To

match irregular boundaries, the choice is between many regular elements or few, more complex, isoparametric elements.

(4) Derivative elements should be used where the solution involves derivatives.

(5) It is advantageous to choose elements that have their nodal parameters concentrated at the vertices.