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CHAPTER II

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THE FINITE ELEMENT METHODS

SECTION (I): INTRODUCTION

The finite element method is a systematic and very powerful method of interpolation. The interpolation process is virtually independent of the geometry of the domain under consideration. Conditions on the given functions at points on the boundary can be matched in a systematic manner by the finite element interpolation The finite element concept leads to a very powerful method for the approximate solutions of the boundary value problem.

One advantage of the finite element method over the finite difference method is the relative case with which the boundary conditions of the problem are handled. Many physical problems have boundary conditions involving derivatives, and in general, the boundary of the region is irregularly shaped. Boundary conditions of this type are very difficult to handle using finite difference techniques, since each boundary condition involving a derivative must be approximated by a difference quotient at the grid points and irregular shaping of the boundary makes placing the grid points difficult. In the finite difference method we concentrate on defining the value of the unknown function $\phi(x)$ at a finite number of values of x. We make the process of function approximation more systematic and general.

The alternative method i. e., the finite element method approximates the solution by minimizing a certain functional involving integrals over a smaller class of functions determined by the problem. The first step in this procedure is to divide the region a finite number of sections of elements of regular shape,

either rectangles or triangles.

The set of functions used for approximation is generally a set of piecewise polynomials of fixed degree in x and y and the approximation requires that the polynomial be pieced together in such a manner that the resulting function will be continuous with an integrable or continuous first or second derivative on the entire region.

Suppose that we wish to approximate a given function in some region Ω bounded by a closed curve Γ. If any function θ can be found which takes on the same values as ϕ on Γ i.e. $\theta\big|_{\Gamma} = \phi\big|_{\Gamma}$ and if a set of independent trial functions is introduced such that $N_m\big|_{\Gamma} = 0$ then at all points in Ω, we can approximate to ϕ by

$$\phi \approx \hat{\phi} = \theta + \sum_{m=1}^{M} a_m N_m \qquad \text{.....} \qquad (2.1)$$

where a_m $m = 1,2,- - -$ M are some parameters which are computed so as to obtain a good fit. These type of trial functions are referred to as "shape" functions or "basis" functions. The manner in which θ and the trial function set are defined automatically ensures that this approximation has the property that $\hat{\phi}\big|_{\Gamma} = \phi\big|_{\Gamma}$ whatever the values of the parameter a_m. The trial function set should be so chosen so as to ensure that the improvement in the approximation occurs with the increase in the number M of trial functions used. In this approximation method, we assume that the trial functions N_m of the expansion (2.1) were defined by a single expression valid throughout the whole domain Ω. Alternatively we divide the region Ω into a number of nonoverlapping subdomains or elements Ω^e and then construct

approximation $\hat{\phi}$ in a piecewise manner over each subdomain. Then the trial functions are defined in a piecewise manner by using different expressions in the various subdomains Ω^e from which the total domain is developed.

SECTION (II): Weighted Residual Approximation Method

A general method for determining the constants in the approximation

$$\phi \approx \hat{\phi} = \theta + \sum_{m=1}^{M} a_m N_m$$

is the Weighted Residual Approximation Method.

We begin by introducing the error or residual R_Ω in the approximation which is defined by $R_\Omega = \phi - \hat{\phi}$ where R_Ω is a function of position in Ω .

To reduce this residual in an overall manner over the whole domain Ω , we require that an appropriate weighted integrals of the error over Ω , weighted in different ways be zero i.e.we attempt to make

$$\int_\Omega W_1 \, (\phi - \hat{\phi}) \, d\Omega = \int_\Omega W_1 \, R_\Omega \, d\Omega \qquad 1= 1,2,..,M \quad ...(2.2)$$

$\{ W_1 : 1 = 1, 2, 3, ..., \}$ is a set of independent "weighting" functions or " test " functions .

$$\int_\Omega W_1 \left[\phi - \theta - \sum_{m=1}^{M} a_m N_m \right] d\Omega = 0$$

$$\Rightarrow \int_\Omega W_1 \left[\phi - \theta \right] d_\Omega - \sum_{m=1}^{M} a_m \int_\Omega W_1 N_m \, d\Omega = 0$$

$$\Rightarrow \sum_{m=1}^{M} a_m \int_\Omega W_1 N_m \, d\Omega = \int_\Omega W_1 \left[\phi - \theta \right] d\Omega$$

Put $K_{lm} = \int_{\Omega} W_l \, N_m \, d\Omega \qquad 1 \le l, \quad m \le M$

and $f_l = \int_{\Omega} W_l \left[\phi - \theta \right] d\Omega \qquad 1 \le l \le M$

and $a^T = (a_1, a_2, \ldots\ldots\ldots\ldots, a_m)$

we write

$$\mathbf{K\,a = f} \qquad \ldots\ldots (2.3)$$

The weighted residual statement leads to a set of simultaneous linear equations for the unknown coeficients a_m. Thus when the function ϕ to be approximated is given, this equation (2.3) can be solved to obtain the coefficients in the approximation having first determined the function θ and chosing suitable trial and weighting function sets.

Taking $\sum_{e=1}^{E} \Omega^e = \Omega$ where E is the total number of subdivision of the region, then the definite integrals occuring in the approximating equations can be obtained by summing the contributions from each subdomain or element as

$$\int_{\Omega} W_l \, R_\Omega \, d\Omega = \sum_{e=1}^{E} \int_{\Omega^e} W_l \, R_\Omega \, d\Omega \qquad \ldots\ldots (2.4)$$

$$\therefore \quad K_{lm}^{e} = \int_{\Omega^e} W_l \, N_m \, d\Omega$$

$$f_l^{e} = \int_{\Omega^e} W_l \left[\phi - \theta \right] d\Omega$$

$$K_{lm}^{e} = \sum_{e=1}^{E} K_{lm}^{e}$$

$$f_l = \sum_{e=1}^{E} f_l^e$$

SECTION (III) : Galerkin Method.

Various form of weighting function sets can be used in practice , each leading to a different weighted residual approximation method. One of them is the Galerkin Method.

In this method, we take the trial functions themselves as the weighting functions $W_l = N_l$

$$\therefore K_{lm}^e = \int_{\Omega^e} N_l \, N_m \, d\Omega$$

$$\text{and } f_l^e = \int_{\Omega^e} N_l \left[\phi - \theta \right] d\Omega \qquad \ldots\ldots (2.5)$$

Advantages of the Galerkin Method

1) The Galerkin method leads to a symmetric stifness matrix thus reducing the computational effort in obtaining an approximate solution.

2) The space of trial function and test functions coincide. Hence only one set of basis functions N_m need to be constructed for such approximations. For piecewise linear elements, we can write

$$N_l \, N_m = 1 \quad \text{at} \quad l = m$$

$$= 0 \quad \text{at} \quad l \neq m$$

$$\Rightarrow \quad K_{lm} = 1 \quad \text{at} \quad l = m$$

$$= 0 \quad \text{at} \quad l \neq m$$

If nodes l and m do not belong to the same element then $K_{lm} = 0$ This implies that most of the matrix entries K_{lm} will be zero . Such matrices are said to be sparse. The non zero entries are clustered near the main diagonal of the matrix. Hence we get a banded matrix.

SECTION (IV) : Finite Element Method For Solving Differetial Equations .

Consider a differential equation in which boundary conditions are satisfied by the choice of trial functions.

Consider $A(\phi) = L\phi + P = 0$ in Ω

Where L is an appropriate linear differential operator and P is independent of ϕ and its associated boundary conditions.

$B(\phi) = M\phi + r = 0$ on Γ

where M is an appropriate linear differential operator and r is independent of ϕ .

We attempt to construct an approximation $\hat{\phi}$ to the actual solution of ϕ by using

$$\phi \approx \hat{\phi} = \theta + \sum_{m=1}^{M} a_m N_m$$

Choose θ and N_m such that $M\theta = -r$ and $MN_m = 0$, $m = 1,2,3...$ on $\hat{B}(\phi) = 0$

Assuming the shape functions to be continuously differentiable

$$\hat{A}(\phi) = \hat{L}\phi + p$$
$$R_\Omega = \hat{A}(\phi) \qquad \text{..... (2.6)}$$

To make $R_\Omega \approx 0$, we take

$$\int_{\Omega} W_1 \, R_{\Omega} \, d\Omega = 0$$

$$\Rightarrow \int_{\Omega} W_1 \, \hat{A}(\phi) \, d\Omega = 0 \qquad \ldots\ldots (2.7)$$

$$\Rightarrow \int_{\Omega} W_1 \, L \left[\theta + \sum_{m=1}^{M} a_m \, N_m\right] d\Omega = 0$$

$$\Rightarrow \int_{\Omega} W_1 \, L\theta \, d_{\Omega} - \sum_{m=1}^{M} a_m \int_{\Omega} W_1 \, LN_m \, d\Omega = 0$$

$$\Rightarrow \sum_{m=1}^{M} a_m \int_{\Omega} W_1 \, LN_m \, d\Omega = -\int_{\Omega} W_1 \, L \, \theta \, d\Omega \qquad \ldots\ldots (2.8)$$

Put $K_{lm} = \int_{\Omega} W_1 \, LN_m \, d\Omega \qquad 1 \leq l, \; m \leq M$

and $f_l = -\int_{\Omega} W_1 \, L\theta \, d\Omega \qquad 1 \leq l \leq M$

and $a^T = (a_1, a_2, \ldots\ldots\ldots\ldots, a_M)$

We produce a set of linear algebraic equations in the form

$$\mathbf{K\,a = f}$$

When the coefficients a_m have been computed, we can determine an approximate solution to the given differential equation.

Consider a differential equation in which the trial functions do not satisfy the boundary conditions.

The residual in the domain

$$\hat{A}(\phi) = L\hat{\phi} + p$$

$$R_\Omega = A(\hat{\phi})$$

is supplemented by a boundary residual

$$R_\Gamma = B(\hat{\phi}) = M\hat{\phi} + r$$

We attempt to reduce the residuals over the domain and boundary by writing

$$\int_\Omega W_l \, R_\Omega \, d\Omega + \int_\Gamma \bar{W}_l \, R_\Gamma \, d\Gamma = 0 \qquad \ldots\ldots (2.9)$$

$$\Rightarrow \int_\Omega W_l \, A(\hat{\phi}) \, d\Omega + \int_\Gamma \bar{W}_l \, B(\hat{\phi}) \, d\Gamma = 0$$

Where W_l and $\bar{W}_l$ are chosen independently.

$$\Rightarrow \int_\Omega W_l \, (L\hat{\phi} + p) \, d\Omega + \int_\Gamma \bar{W}_l \, (M\hat{\phi} + r) \, d\Gamma = 0$$

$$\Rightarrow \sum_{m=1}^{M} a_m \left[\int_\Omega W_l \, LN_m \, d\Omega + \int_\Gamma \bar{W}_l \, MN_m \, d\Gamma \right] = - \int_\Omega W_l \, p \, d\Omega - \int_\Gamma \bar{W}_l \, r \, d\Gamma$$

Put $K_{lm} = \int_\Omega W_l \, LN_m \, d\Omega + \int_\Gamma \bar{W}_l \, MN_m \, d\Gamma \qquad 1 \le l, \; m \le M$

and $f_l = -\int_\Omega W_l \, p \, d\Omega - \int_\Gamma \bar{W}_l \, r \, d\Gamma \qquad 1 \le l \le M$

and $a^T = (a_1, a_2, \ldots\ldots\ldots\ldots, a_M)$

We get a system of equations which can be solved in the form

$$\mathbf{K\,a = f}$$

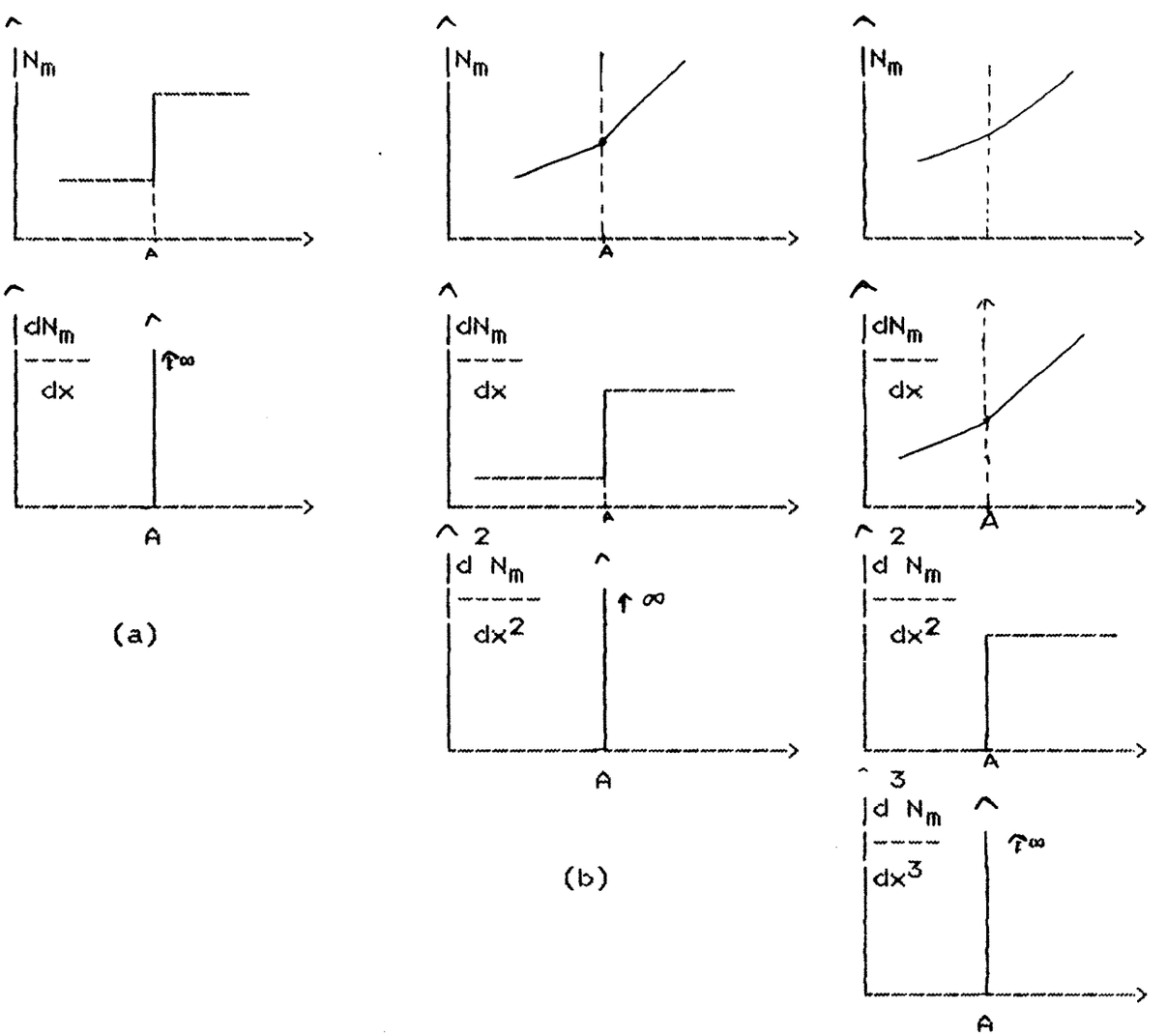
SECTION (V): C^S Continuity

In the finite element procedure it can happen that the shape function be discontinuous or continuous shape functions have

discontinuous derivatives.

Consider the behaviour of three-types of one-dimensional shape functions N_m near a function A of two elements.

The first function is discontinuous at point A, while the second shows a discontinuity in the slope dN_m / dx at the same point and third a discontinuity of the second derivative $d^2 Nm / dx$



Therefore the function gives infinite values for the first, second and third derivatives which should be avoided because at times such infinites may result in indeterminate integrals.

To avoid such difficulties it is necessary that if the integrals contain derivatives of order **s**, then we must ensure that the derivatives of order **s-1** are continuous in the trial functions N_m used in the approximation.

Therefore we shall require that the shape functions show C^{s-1} continuity.

If first derivatives occur then s= 1 and we have C^0 continuity. If s = 2, we have C^1 continuity. The continuity requirements imposed on the trial functions is also applicable to the weighting function W_1.

SECTION (VI): PATCHWORK APPROXIMATION

In the one-dimensional process, the finite element mesh is generated by partitioning an interval into line elements connected nodal points at their ends. For two-dimenional problems, having irregular domains, we use elements simple enough to minimize computational effort. Simple triangles and quadrilaterals are used.

The patchwork approximation characteristic of the finite element method is generated from basic functions each of which is nonvanishing only over a small subregion of the domain D.

In one dimension, let D be the real interval $[a,b]$ and let x_i be the prescribed point set on D.

$$a = x_1 < x_2 < x_3 < x_4 < \ldots\ldots < x_{n-1} < x_n = b$$

We define

$$\phi(x) = \phi(x_i) + \frac{x - x_i}{x_{i+1} - x_i} \left[\phi(x_{i+1}) - \phi(x_i) \right] \qquad \ldots\ldots(2.10)$$

$$x_i \leq x \leq x_{i+1} \quad \text{and} \quad \phi(x) \in C[a,b]$$

Alternatively the approximation may be expressed in terms of basis functions that are non-zero only over small subintervals of Œa,bÅ

$$\phi(x) = \sum_{i=1}^{n} \phi(x_i)\, N_i(x) \qquad \ldots.(2.11)$$

$$\phi(x) = \sum_{i=1}^{n} \phi_i\, N_i\,(x) \quad \text{where} \quad \phi_i = \phi(\, x_i\,)$$

and N_i is the hat function

$$N_i\,(x) = 0 \qquad x < x_{i-1} \text{ and } x > x_{i+1}$$

$$= (x - x_{i-1}) \,/\, (x_i - x_{i-1}) \qquad x_{i-1} \le x \le x_i$$

$$= (x - x_{i+1}) \,/\, (x_i - x_{i+1}) \qquad x_i \le x \le x_{i+1}$$

Two patchwork approximations are widely used for 2-D problems.

(i) Triangles

Domain D is partitioned into a network of nonoverlapping triangles and ϕ is a function that is continuous over D, linear within each triangle and is uniquely defined within a triangle by its value at the triangle vertices.

For the triangle

$$\phi\,(x,y) = \sum_{i=1}^{3} \phi(x_i,y_i)\, N_i(x,y)$$

Let $L_1(x,y)$ be the linear form (polynomial of degree one) which vanishes on side (2;3) of the triangle.

Then $N_1(x,y) = L_1(x,y)/L_1(x_1,y_1)$

This "wedge" basis function is associated with node 1 of the triangle.

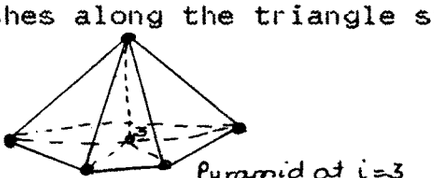
Similarly

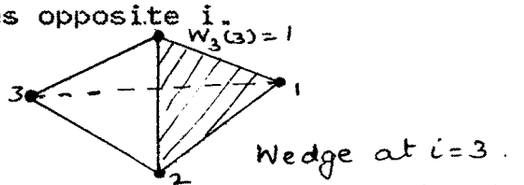
$$N_2(x,y) = L_2(x,y)/L_2(x_2,y_2)$$

and

$$N_3(x,y) = L_3(x,y)/L_3(x_3,y_3)$$

The wedges for the triangles which share vertex i, piece together to form a "pyramid" function with value unity at i. This function i continuous over the triangles which share vertex i and vanishes along the triangle sides opposite i.





Over the collection of triangles, the approximation is

$$\phi(x,y) = \sum_i \phi_i P_i(x,y)$$

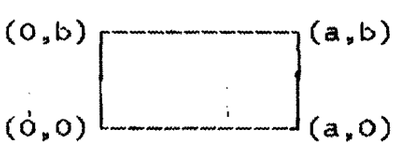
where $\phi(x,y)$ is continuous over the entire domain.

(ii) RECTANGLES

Domain D may be partitioned into a collection of nonoverlapping rectangles. The wedges are bilinear within each rectangle.

$$N_1(x,y) = (a - x)(b - y) / ab$$

$$N_2(x,y) = (a - x)y / ab$$



$$N_3 (x,y) = x\,y / ab$$

$$N_4 (x,y) = x(b - y) / ab$$

Each bilinear wedge is linear on each side of the rectangle so that the approximation

$$\phi(x,y) = \sum_{i=1}^{4} \phi_i N_i (x,y)$$

which is in general bilinear within the rectangle, is linear on each side. This bilinear approximation is adequate for the parallelogram.

Let $L_1(x,y) = 0$ on side (2;3)

and let $L_2(x,y) = 0$ on side (3;4)

Then $N_1(x,y) = L_1(x,y)\, L_2(x,y) / L_1(x_1,y_1)\, L_2(x_1, y_1)$

Similarly the other wedges can be defined,

$$N_2(x,y) = L_2(x,y)\, L_3(x,y) / L_2(x_2,y_2)\, L_3(x_2, y_2)$$

$$N_3(x,y) = L_3(x,y)\, L_4(x,y) / L_3(x_3,y_3)\, L_4(x_3, y_3)$$

$$N_4(x,y) = L_4(x,y)\, L_1(x,y) / L_4(x_4,y_4)\, L_1(x_4, y_4).$$

SECTION (VII): FINITE ELEMENT BASIS FUNCTIONS

There is no systematic way of constructing reasonable basis functions N_m for the approximation ϕ . These basis functions apart from being independent members are arbitrary. Therefore we are left with a number of possilities at our disposal. The quality of the approximate solution depends very strongly on the properties of the basis functions.

The basis functions can be defined piecewise over subregions of the domain called finite elements and that over any subdomain, N_m can be chosen to be very simple functions such as polynomials of low degree. These basis functions are also known as element shape functions. To construct a set of piecewise basis functions, we partition the domain Ω into a finite number of elements Ω^e. The length of each finite element is h_i. Within each element certain points are identified, called nodes or nodal points. As h_i becomes smaller, more elements must be introduced and therefore more basis functions are furnished.

We can construct a set of basis functions using the following fundamental criterion.

1) The basis functions are generated by simple functions defined piecewise element by element over the finite element mesh.
2) Each N_m (basis function) must have square integrable first derivatives and $N_m\Big|_\Gamma = 0$
3) Each basis function should have its value unity at one node and zero at all the other nodes.

$$N_m = 1 \text{ at node } m$$
$$= 0 \text{ at all other nodes.}$$

LAGRANGE POLYNOMIAL BASIS :-

The Lagrange finite element employs polynomial shape functions having the property

$$N_m = 1 \text{ at node } m$$
$$= 0 \text{ at all other nodes.}$$

which implies $\{N_m : m=1,2, \ldots\}$ are linearly independent.

Suppose we are given a function ϕ in the interval $0 \le x \le 1$

and that ø is smooth enough to be continuously differentiable k times and its derivative of order (k+1) is bounded (finite) on this interval. Any polynomial of degree k , or less can be represented uniquely in terms of Lagrange Polynomial Basis.

Lagrange shape functions are defined as

$$N_m(x) = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_{m-1})(x-x_{m+1})\cdots(x-x_{k+1})}{(x_m-x_1)(x_m-x_2)\cdots\cdots(x_m-x_{m-1})(x_m-x_{m+1})\cdots(x_m-x_{k+1})}$$

For k = 1 (linear shape functions) we have 2 nodes and hence

$$N_1(x) = \frac{x-x_2}{x_1-x_2} = \frac{1}{2}(1-x)$$

$$N_2(x) = \frac{x-x_1}{x_2-x_1} = \frac{1}{2}(1+x)$$

For k = 2 (quadratic shape functions) we have 3 nodes and hence

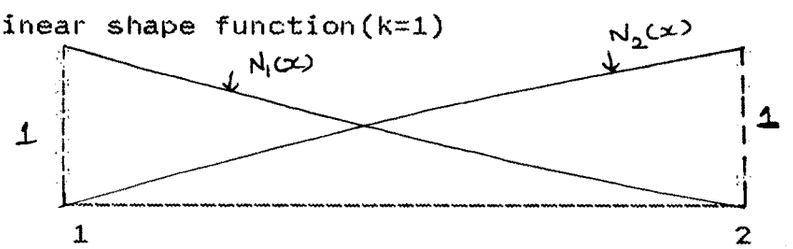
$$N_1(x) = \frac{1}{2}x(x-1)$$

$$N_2(x) = (1-x^2)$$

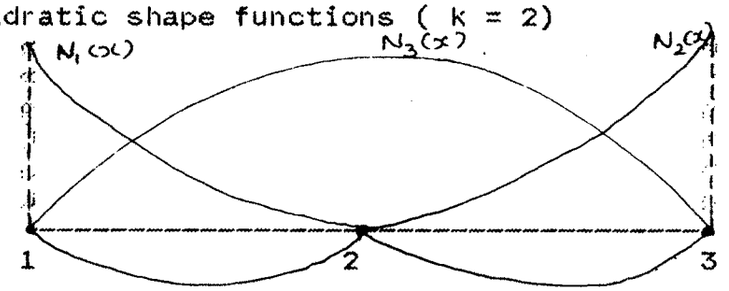
$$N_3(x) = \frac{1}{2}x(1+x)$$

Element with (k + 1) nodes

Linear shape function(k=1)



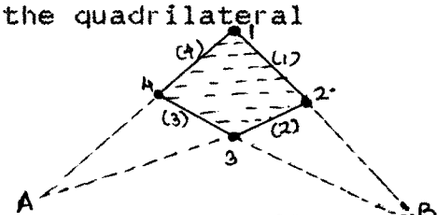
Quadratic shape functions (k = 2)



RATIONAL BASIS

The use of rational polynomials a a basis for finite element interpolation wa pioneered by Wachpress. Such rational functions may be utilized to develop general polygonal elements with straight or curved sides.

Consider the quadrilateral



An interpolant on the quadrilateral is of the form

$$\phi^e(x,y) = \sum_j^e \phi_j \, N_j^e(x,y)$$

Where the shape functions are rational functions i.e.

$$N_j^e(x,y) = p(x,y) / q(x,y)$$

Where p and q are polynomials. The rational functions must have the property

$$N_j^e(x_i, y_i) = \delta_{ij}$$

Each function $N_j^e(x_i, y_i)$ takes on a unit value at node i and zero on all sides except those adjacent to node i.

Let $f_i(x,y) = 0$ be the equation of side i. Then the product function

$$P_i(x,y) = f_{i+1}(x,y) * f_{i+2}(x,y) = 0$$

on the sides not adjacent to node i

e.g. $P_1(x,y) = f_2(x,y) * f_3(x,y) = 0$

on sides 2 and 3 opposite node 1.

If this quadratic is divided by the linear function f_{AB} defined by joining extrapolated sides and scaled with

$$C_1 = \frac{f_{AB}(x_1, y_1)}{f_2(x_1,y_1) * f_3(x_1,y_1)}$$

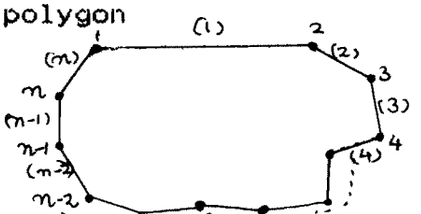
We obtain

$$N_1^e(x,y) = C_1 \frac{f_2(x,y) * f_3(x,y)}{f_{AB}(x,y)}$$

Similarly

$$N_i^e(x,y) = C_i \frac{f_{i+1}(x,y) * f_{i+2}(x,y)}{f_{AB}(x,y)}$$

Consider the polygon



We have $f_j(x,y) = 0$, $j = 1,2,3,\ldots\ldots,n$ and we define rational functions

$$N_i^e(x,y) = C_i \frac{f_{i+1}(x,y) * f_{i+2}(x,y)\ldots\ldots f_{i-2}(x,y)}{g_1(x,y) * g_2(x,y)\ldots\ldots\ldots\ldots g_{n-2}(x,y)}$$

$$C_i = \frac{g_1(x_i,y_i) * g_2(x_i,y_i)\ldots\ldots\ldots\ldots g_{n-2}(x_i,y_i)}{f_{i+1}(x_i,y_i) * f_{i+2}(x_i,y_i)\ldots\ldots f_{i-2}(x_i,y_i)}$$

and $g_s(x,y)$ are constructed such that f_{i+1}/g_s and f_{i+2}/g_s are constants on sides i and i-1 respectively.

SECTION (VIII): DEGREE OF APPROXIMATION

The basis is constructed of rational functions and to ascertain interpolation estimates we need to indicate the degree of the complete polynomial basis contained in a given rational

basis for an element. This degree is referred to as the "degree of approximation" and we say that a rational approximation is of "degree k" if the complete polynomial of greatest degree that can be represented in the basis has degree k.

OPPOSITE FACTOR, ADJACENT FACTOR AND ADJOINT POLYNOMIAL

Let the boundary curve of element Ω_e be denoted by Γ_m and consist of m continuous curve (sides).

Define the opposite factor associated with node i as

$$O_i(x,y) = \prod_{j \neq s(i)} f_j(x,y)$$

where $f_j(x,y)=0$ is the equation of the curved side j of Ω_e and $j \neq (i)$.

Hence $O_i(x,y) = 0$ on all sides of Ω_e except those containing node i.

The basis function should also be zero at the remaining nodes on the side or sides containing node i.

We define an adjacent factor $A_i(x,y)$ such that $A_i(x,y) = 0$ at this side or sides except node i.

Therefore we obtain

$$P_i(x,y) = O_i(x,y) * A_i(x,y)$$

which is the numerator expression for the rational basis.The denominator D(x,y) is the unique algebraic curve defined by the exterior intersection points of the extended sides which is called Adjoint Polynomial.

. . The rational basis function have the form

$$N_i^e(x,y) = C_i \frac{O_i(x,y) * A_i(x,y)}{D(x,y)}$$

where C_i is the normalizing constant chosen so that $N_i^e(x_i,y_i)=1$

SECTION (IX): PROPERTIES OF RATIONAL WEDGE BASIS FUNCTIONS

The properties required of the wedge basis functions to achieve continuous patchwork degree one approximation over a collection of well-set polycons.

1) There is a node at each vertex and on each conic side. For each node there is an associated wedge within each polycon containing the node.

2) Wedge $N_i(x,y)$ associated with node i is normalized to unity at node i.

3) Wedge N_i is linear on side adjacent to i.

4) Wedge N_i vanishes on sides opposite node i and at all nodes j for which $j \neq i$.

5) The wedges associated with a polycon form a basis for linear functions over the polycon. For a polycon with r conic and s linear sides, we must have at least 2r + s nodes

$$\sum_{i=1}^{2r+s} N_i(x,y) = 1 \quad ;$$

$$\sum_{i=1}^{2r+s} x_i N_i(x,y) = x;$$

$$\sum_{i=1}^{2r+s} y_i N_i(x,y) = y;$$

6) Each wedge function and all its derivatives are continuous within the polycon for which the wedge is a basis function. A function with this property is said to be linear.