CHAPTER II

RIM DISTRIBUTION AND A GROUP OF CONFORMAL MOTIONS

1. PROPERTIES OF CONFORMAL MOTIONS

We recall the necessary conditions for conformal motions (Vide I, 3.2),

$$L_\xi g_{ab} = 2\psi g_{ab}, \qquad \ldots.(1.1)$$

i.e. $\nabla_b \xi_a + \nabla_a \xi_b = 2\psi g_{ab}$.

This implies that

$$\psi = \nabla_b \xi^b. \qquad \ldots.(1.2)$$

It is known that if a fluid spacetime admits a conformal Killing vector (CKV) ξ, then the following kinematical results hold (Maartens et al., 1986).

$$L_\xi u_a = \psi u_a + h_a, \qquad \ldots.(1.3)$$

$$h_a = 2 w_{ab} \xi^b + m\overset{*}{u}_a - m,_b h^b_a. \qquad \ldots.(1.4)$$

where $m = - \xi_a u^a$ and w_{ab} is the vorticity tensor and $h^b_a = -u^b u_a$. The contraction of (1.4) with h^a gives

$$h^2 = (\nabla_b m) h^b - 2 w_{ab} \xi^b h^a. \qquad \ldots.(1.5)$$

we observe from (1.3) that the fluid flow lines are mapped into fluid flow lines under the action of ξ if $h_a = 0$. Hence the integral curve of ξ^a is called as material curve if $h_a = 0$. Here the material curve in a fluid is a curve that moves with the fluid as the fluid evolves. We study the following two subcases of conditions (1.1).

i) <u>The choice $\xi^a = u^a$</u>

In this case (1.1) becomes

$$\nabla_b u_a + \nabla_a u_b = 2\psi g_{ab}. \qquad \ldots.(1.6)$$

The various contractions of this give

$$\dot{u}^a = 0 = \theta. \qquad \ldots.(1.7)$$

$$\psi = 0. \qquad \ldots.(1.8)$$

Thus if the fluid flow vector is CKV then fluid flow lines are expansionfree and geodesic. Moreover the CKV reduces to Killing vector (KV).

ii) <u>The choice $\xi^a = h^a$</u>

For this case, the equation (1.1) implies

$$\nabla_b h_a + \nabla_a h_b = 2\psi g_{ab}. \qquad \ldots.(1.9)$$

This equation with the help of Maxwell equations (5.1) generate the following results.

(i) $\theta = 0$,

(ii) $(\nabla_a h^2)u^a = 0 = (\nabla_a h^2)h^a$,

(iii) $\psi = 0$.

So we infer from these results that for RIM spacetime admitting conformal Killing magnetic field vector

(a) Fluid flow lines are expansionfree.

(b) The magnitude of magnetic field is preserved along the flow lines as well as along the magnetic lines.

(c) The conformal Killing vector ξ reduces to Killing vector along the magnetic field lines.

COMMENTS

According to above two cases the conformal motion along fluid flow lines as well as along magnetic lines degenerates into group of motions. Hence conformal motion along the fluid flow and magnetic lines are not permitted.

2. CONFORMAL MOTIONS AND FIELD EQUATIONS FOR THE RIM DISTRIBUTION

For CKV ξ, Coley and Tupper (1989) have proved that

$$L_\xi R_{ab} = -2(\nabla_{ab} \psi) - (\nabla_{cd} \psi) g^{cd} g_{ab}. \qquad \ldots(2.1)$$

But we have the expression for Ricci tensor corrosponding to RIM distribution as

$$R_{ab} = (\rho + \mu h^2) u_a u_b - \tfrac{1}{2}(\rho + \mu h^2) g_{ab} - \mu h_a h_b,$$

$$\text{i.e. } R_{ab} = A\, u_a u_b + B\, g_{ab} + C\, h_a h_b, \qquad \ldots(2.2)$$

where

$$\left.\begin{aligned} A &= (\rho + \mu h^2), \\ B &= -\tfrac{1}{2}(\rho + \mu h^2), \\ C &= -\mu. \end{aligned}\right] \qquad \ldots(2.3)$$

By taking Lie derivative of both sides of equation (2.2) with respect to ξ, we write

$$L_\xi R_{ab} = u_a u_b\,(L_\xi A) + A\,u_a(L_\xi u_b) + A\,u_b(L_\xi u_a) +$$
$$+ g_{ab}(L_\xi B) + B\,(L_\xi g_{ab}) + C(L_\xi h_a)h_b +$$
$$+ C\,h_a(L_\xi h_b) + (L_\xi C)h_a h_b,$$

i.e.

$$L_\xi R_{ab} = u_a u_b\,(L_\xi A) + A\,u_a(L_\xi u_b) + A\,u_b(L_\xi u_a) +$$
$$+ g_{ab}(L_\xi B) + B\,(L_\xi g_{ab}) + C(L_\xi h_a)h_b +$$
$$+ C\,h_a(L_\xi h_b), \qquad \text{since } L_\xi C = 0.$$

By utilising equations (1.3) in the above equation, we get

$$L_\xi R_{ab} = u_a u_b\,(L_\xi A) + A\,u_a(\psi u_b + h_b) +$$
$$+ A\,u_b(\psi u_a + h_a) + g_{ab}\,L_\xi B + B\,(2\psi g_{ab}) +$$
$$+ C(L_\xi h_a)h_b + C\,h_a(L_\xi h_b). \qquad \ldots.(2.4)$$

Hence equations (2.1) and (2.4) give

$$-2(\nabla_{ab}\psi) - (\nabla_{cd}\psi)g^{cd}\,g_{ab} =$$
$$= u_a u_b(L_\xi A) + A\,u_a(\psi u_b + h_b) +$$
$$+ A\,u_b(\psi u_a + h_a) + g_{ab}\,(L_\xi B) + B(2\psi g_{ab}) +$$
$$+ C\,(L_\xi h_a)h_b + C\,h_a(L_\xi h_b). \qquad \ldots.(2.5)$$

<u>Theorem</u> (2.1) : For the RIM distribution admitting a conformal groups of motions along vector ξ

(i) $L_\xi \rho \;=\; 6(\nabla_{cd}\psi)g^{cd} + R^a_b \nabla_a \xi^b.$

(ii) $L_\xi (\mu h^2) = -8(\nabla_{cd}\psi)g^{cd} + 4B\psi - (\nabla_{ab}\psi)u^a u^b - 2R^a_b \nabla_a \xi^b.$

<u>Proof</u> : By taking innermultiplication of equation (2.5) with u^a, we get

$$-2(\nabla_{ab}\psi)u^a - (\nabla_{cd}\psi)g^{cd}g_{ab}u^a =$$

$$= u_a u_b u^a (L_\xi A) + A\, u^a u_a(\psi u_b + h_b) +$$

$$+ A\, u_b u^a(\psi u_a + h_a) + u^a g_{ab}(L_\xi B) +$$

$$+ u^a(2\psi g_{ab}) B + u^a C(L_\xi h_a)h_b +$$

$$+ u^a h_a C(L_\xi h_b).$$

$u^a u_a = 1$

This can be simplified by using the values $u^a h_a = 0$, $u^a u_a = 0$ and $g^{ab}u_b = u^a$ as

$$-2(\nabla_{ab}\psi)u^a - (\nabla_{cd}\psi)g^{cd}u_b =$$

$$= u_b(L_\xi A) + A\psi u_b + Ah_b + \quad \mu q.$$

$$+ A\psi u_b + u_b(L_\xi B) + 2\psi B\mu u_b +$$

$$+ C\, u^a(L_\xi h_a)h_b, \qquad \ldots.(2.6)$$

i.e. $-2(\nabla_{ab}\psi)u^a - (\nabla_{cd}\psi)\, g^{cd}u_b =$

$$= \left[L_\xi(A + B)\right]u_b + 2u_b\,(A + B)\psi + + \left[A + Cu^a\,(L_\xi\, h_a)\right]h_b .$$

Further by contracting equation (2.5) with h^a, we get

$$-2(\nabla_{ab}\psi)h^a - (\nabla_{cd}\psi)\, g^{cd}g_{ab}h^a =$$

$$= h^a u_a u_b(L_\xi\, A) + Ah^a u_a(\psi\, u_b + h_b) + + Ah^a u_b(\psi\, u_a + h_a) + g_{ab}(L_\xi\, B)h^a . + Bh^a(2\psi\, g_{ab}) + Ch^a(L_\xi\, h_a)h_b . + Ch^a h_a(L_\xi\, h_b).$$

By knowing that $\bar{u}$ and $\bar{h}$ are orthogonal and magnitude of magnetic field is $(-h^2)$, this reduces to

$$-2(\nabla_{ab}\psi)h^a - (\nabla_{cd}\psi)g^{cd}h_b = Au_b h^2 + (L_\xi\, B)h_b + 2B\psi\, h_b + + C(L_\xi\, h_a)h^a h_b - Ch^2(L_\xi\, h_b),$$

i.e. $-2(\nabla_{ab}\psi)h^a - (\nabla_{cd}\psi)g^{cd}h_b =$

$$= \left[(L_\xi\, B) + 2B\psi + C(L_\xi\, h_a)h^a\right]h_b - - \left[Au_b + Ch^2(L_\xi\, h_b)\right]h^2. \qquad \text{....(2.7)}$$

Further the contraction of (2.5) with g^{ab} yields

$$-2(\nabla_{ab}\psi)g^{ab} - 4(\nabla_{cd}\psi)g^{cd} =$$

$$= \overbrace{u^b u_b}^{=1}(L_\xi A) + Au^b(\psi u_b + h_b) +$$

$$+ Au^a(\psi u_a + h_a) + 4(L_\xi B) +$$

$$+ 8B\psi + C(L_\xi h_a)h^a + Ch^b(L_\xi h_b),$$

i.e. $L_\xi (A + 4B) = - 6(\nabla_{cd}\psi)g^{cd} -2A\psi - 8B\psi - 2Ch^a(L_\xi h_a),$

....(2.8)

(since $u^a h_a = 0$ and $u^a u_a = 1$).

Further contraction of equation (2.6) with u^b and writting $u^a u_a = 1$, $u^a h_a = 0$ we get

$$- 2(\nabla_{ab}\psi)u^a u^b -(\nabla_{cd}\psi)g^{cd} = L_\xi A + 2A\psi + L_\xi B + 2B\psi$$

?

i.e. $L_\xi (A+B) = -(\nabla_{cd}\psi)g^{cd} -2(A+B)\psi - 2(\nabla_{ab}\psi)u^a u^b.$ ..(2.9)

Similarly, the contraction of (2.7) with h^b, after writting $u^a h_a = 0$, $h^a h_a = -h^2$ we get

$$-2(\nabla_{ab}\psi)h^a h^b + (\nabla_{ca}\psi)g^{cd}h^2 = -(L_\xi B)h^2 -2B\psi h^2 -$$

$$-2Ch^a(L_\xi h_a)h^2,$$

i.e. $(L_\xi B)h^2 = 2(\nabla_{ab}\psi)h^a h^b - (\nabla_{cd}\psi)g^{cd}h^2 -2B\psi h^2 -$

$- 2Ch^a(L_\xi h_a)h^2.$ ...(2.10)

Also on innermultiplying equation (2.6) with h^b and simplifying we obtain.

$$-2(\nabla_{ab}\psi)u^a h^b = A(-h^2) + C(L_\xi h_a)(-h^2)u^a,$$

i.e. $2(\nabla_{ab}\psi)u^a h^b = \left[A + C(L_\xi h_a)u^a\right]h^2.$ ....(2.11)

Since from the values of A and B we know that A = -2B so that equation (2.8) becomes

$$L_\xi(-2B + B) = -(\nabla_{cd}\psi)g^{cd} + 2B\psi - (\nabla_{ab}\psi)u^a u^b,$$

i.e.

$$L_\xi(\rho + \mu h^2) = 2(\nabla_{cd}\psi)g^{cd} - 4B\psi + 2(\nabla_{ab}\psi)u^a u^b. \quad (2.12)$$

We know that the CKV satisfying the conservation law generator equation

$$\nabla_a(R^a_b\,\xi^b) = -3(\nabla_{cd}\psi)g^{cd},$$

i.e. $(\nabla_a R^a_b)\xi^b + R^a_b(\nabla_a\xi^b) = -3(\nabla_{cd}\psi)g^{cd}.$ ....(2.13)

Since $R^a_b = T^a_b - \frac{1}{2}Tg^a_b$ (Via field equations),

Hence Equation (2.13) produces

$$\nabla_a R^a_b = \nabla_a T^a_b - \tfrac{1}{2}(\nabla_a T)g^a_b.$$

But by Ricci Identities, $\nabla_a T^a_b = 0$ and for the RIM distribution $T = \rho$, Hence equation (2.13) provides

$$-(\nabla_b\rho)\xi^b + 2R^a_b(\nabla_a\xi^b) = -6(\nabla_{cd}\psi)g^{cd},$$

i.e. $L_\xi \rho = 6(\nabla_{cd}\psi)g^{cd} + 2R^a_b(\nabla_a \xi^b)$ . ....(2.14)

This value by virtue of equation (2.12) produces

$$L_\xi (\mu h^2) = -8(\nabla_{cd}\psi)g^{cd} + 4B\psi - (\nabla_{ab}\psi)u^a u^b - R^a_b \nabla_a \xi^b. \qquad ....(2.15)$$

Thus we have derived the required expressions (a) and (b), [Vide equations (2.14) and (2.15)]. (i) (ii)

<u>Remark</u> : This theorem provides the rate of change of matter density and the magnitude of the magnetic field along CKV $\bar{\xi}$ admitted by RIM distribution.

3. SPECIAL CONFORMAL MOTION AND ASSOCIATED CONSEQUENCES

For the special conformal vector $\bar{\xi}$, we have the necessary condition

$$\nabla_{ab}\psi = 0.$$

Then equations (2.8), (2.9), (2.10) and (2.11) reduce to

$$L_\xi (A + 4B) = -2A\psi - 8B\psi - 2Ch^a(L_\xi h_a) \qquad ....(3.1)$$

$$L_\xi (A + B) = -2A\psi - B\psi , \qquad ....(3.2)$$

$$L_\xi B = -2B\psi - 2Ch^a(L_\xi h_a), \qquad ....(3.3)$$

$$0 = [A + C(L_\xi h_a)u^a]h^2,$$

i.e. $A + C(L_\xi h_a)u^a = 0.$ ....(3.4)

On substituting $A = -2B$ in (3.2)

$$L_\xi (-2B + B) = -2(-2B)\psi - B\psi$$

i.e. $L_\xi B = -3B\psi$ . ....(3.5)

By using equation (3.5) in (3.3) we get

$$B\psi = 2\, Ch^a(L_\xi h_a), \quad ....(3.6)$$

$$\text{i.e.} \quad \psi = \frac{2C}{B} h^a(L_\xi h_a), \quad ....(3.7)$$

$$\text{i.e.} \quad \psi = \frac{4\mu}{(\rho + \mu h^2)} h^a(L_\xi h_a).$$

This shows that the value of ψ depends mainly on the value of magnetic field.

<u>Note</u> : According to Holl and Decosta (1988), there is very limited scope for CKV in Relativity which is due to existance e of a covariently constant hypersurface, orthogonal and geodesic vector. In particular, Friedman Robertson Walkar (FRW) and perfect fluid model are excluded.