CHAPTER I

CHAPTER I

INTRODUCTION AND BASIC CONCEPTS

1.1 Introduction:

Many problems of Physics, Engineering, Social sciences lead to the solution of an eigenvalue problem. Therefore the solution of such problem has wide applications in many branches of science.

Computation of eigenvalues and eigenvectors is one of the most important problem in numerical analysis. The computation of eigen values is nothing but the computation of zeros of a polynomial, called the charactaristic polynomial of a matrix.

The determination of the eigenvector is the problem of solving a set of homogeneous linear simultaneous equations.

The Standard eigenvalue problem is the determination of the nontrivial solutions of $Ax = \lambda x$. While solving this problem, we consider it as a problem of transformation of a given matrix into certain desired form i.e. diagonal, tridiagonal etc. For this, we have to use matrix transformations.

Before describing the eigenvalue problem in detail, here we describe notations used and give some preliminary definations, and some related theroms.

1.2 NOTATIONS & DEFINITIONS:

Definition 1 : Matrices will be denoted by capital letters. A matrix A, referred to as being $m \times n$ matrix, will have m rows and n columns, with the $(i, j)^{th}$ element a_{ij}, (a_{ij} may be real or complex) for i = 1,2,3,...m, and j = 1,2 ,3,....n.

Definition 2 : If the number of rows and columns of a matrix A is same then A is said to be a square matrix. The determinant of a square matrix A is denoted by det(A).

Definition 3 : If det(A) = 0, A is said to be singular, otherwise it is nonsingular.

Definition 4 : The identity matrix is denoted by I.

$$I = \delta_{ij} \text{ where } \delta_{ij} = 0 \text{ if } i \neq j$$
$$= 1 \text{ if } i = j$$

Definition 5 : An $n \times n$ matrix A is said to be diagonal, if $a_{ij} = 0$ whenever $i \neq j$.

Definition 6 : If the (i,j)th element is zero for all i,j such that $|i-j| > 1$, then the matrix A is said to be a tridiagonal matrix.

Definition 7 : The transpose of a matrix A, is the matirx with (i,j)th element equal to a_{ji}, and is denoted by A^T.

<u>Definition 8</u> : The complex conjugate of a matrix A, is the matrix with (i,j)th element equal to $\overline{a_{ij}}$, ($\overline{a_{ij}}$ denotes the comlpex conjugate of a_{ij}) which is denoted by $\overline{A}$.

<u>Definition 9</u> : The conjugate transpose of a matrix A is denoted by A^* which is the matrix with (i,j) th element = $\overline{a_{ji}}$.

<u>Definition 10</u> : Trace of a square matrix is the sum of all the diagonal elements i.e. $tr.(A) = \Sigma a_{ii}$.

<u>Definition 11</u> : Inverse of a matrix is denoted by A^{-1}, and it is a matrix such that $A A^{-1} = A^{-1}A = I$.

<u>Definition 12</u> : A matrix with large number of zero elements is said to be a sparse matrix. If there are few zero elements, the matrix is said to be dense.

1.3 <u>Special Matrices</u> :

Here we introduce some special matrices over the field of complex number whose elements are interrelated . These are as follows :

(1) Symmetric : $A = A^T$.

(2) Skew-symmetric : $A = -A^T$.

(3) Hermitian : $A = A^*$.

(4) Orthogonal : $A A^T = A^TA = I$.

(5) Unitary : $A^*A = A A^* = I$.

(6) Normal : $A^*A = A A^*$

(7) Positive Definite : $X^*AX > 0$ for all non null X

(8) Positive semidefinite : $X^*AX \geq 0$ for all X

$X^*AX = 0$ for some non null X.

(9) non-negative definite : $X^*AX \geq 0$ for all X.

1.4 Eigenvalues and eigenvectors of a matrix.

The basic problem with which we shall be concerned is the determination of the values of λ for which the n homogenous linear equations in n unknowns

$$Ax = \lambda x$$

have a nontrivial solution. In order to have a nontrivial solution we require

$$\det(A - \lambda I) = 0$$

which is a polynomial equation of degree n in λ. The n roots of this equation are called the eigenvalues of A. These values may or may not be distinct. Also they may be real or complex.

Corresponding to each value of λ satisfying the above polynomial equation there is a nontrivial vector x which is called as an eigenvector of A corresponding to that eigenvalue.

1.5 Similar matrices and similarity transformation:

An $n \times n$ matrix B is said to be similar to an $n \times n$ matrix A if there exists a non singular matrix S such that

$$B = S^{-1} A\, S \qquad \text{or}$$

$$B = S\, A\, S^{-1}$$

The transformation of A into B is called the similarity transformation.

1.6 Inner product Space:

Let X be a vector space over the field of real or complex numbers. A mapping denoted by $\langle f,g \rangle$, defined on $X \times X$ into the underlying field is called an inner product of any two elements f and g of X if the following conditions are satisfied.

(i) $\langle cf,g \rangle = c\langle f,g \rangle$ for $c \in F$; $f,g \in X$.

(ii) $\langle f+g,h \rangle = \langle f,h \rangle + \langle g,h \rangle$ for $f,g,h \in X$.

(iii) $\langle f,g \rangle = \overline{\langle g,f \rangle}$ (The bar indicates the complex conjugate number).

(iv) $\langle f,f \rangle > 0$ for $f \neq 0$.

If the inner product $\langle f,g \rangle$ is defined for every pair of elements $\langle f,g \rangle \in X \times X$, then the vector space X together with the inner product $\langle f,g \rangle$ is called an inner product space.

An inner product space will be called finite dimensional if the underlying vector space is finite dimensional.

A finite dimensional inner product space over the field of complex numbers is called a Unitary space.

1.7 Hilbert space : An inner product space X is called a Hilbert space if the normed space induced by the inner product is a complete normed space.

1.8 Some related Theorems

Theorem (1) : The eigenvalues of A and A^T are the same.

Proof: Let λ and μ be the eigenvalues of A and A^T respectively.

Then

$$|(A - \lambda I)| = 0 = |(A^T - \mu I)|$$
$$= |(A^T - \mu I)^T|$$
$$= |(A - \mu I)|$$

hence the theorem.

Teorem (2) : All the eigenvalues of a Hermitian matrix A are real

Proof : Let λ be the eigenvalues of an n×n matrix A. Then,

$$(Ax, x) = (\lambda x, x)$$
$$= \lambda(x, x)$$
$$(Ax, x) = (x, A^* x)$$
$$= (x, Ax)$$
$$= (x, \lambda x)$$
$$= \bar{\lambda}(x, x)$$

since $(x, x) > 0$, $\lambda = \bar{\lambda}$.

Hence the theorem.

Note : A real symmetric matrix is a particular case of the Hermitian matrix, therefore all its eigenvalues are real.

Theorem (3) : If A is nonsigular then the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A and the eigenvectors of A are the same as the eigenvectors of inverse of A.

Proof : If λ and x are eigenvalues and eigenvectors of A then

$$Ax = \lambda x$$

or

$$x = \lambda A^{-1} x$$

or

$$1/\lambda \; x = A^{-1} x$$

Hence the theorem.

Theorem (4) : If an n×n matirx A is real symmetric, then there exists a real orthogonal matrix S such that

$$S^{-1} A S = D$$

where D is a diagonal matrix.

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