CHAPTER II

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THE EIGENVALUE PROBLEMS

2.1 Introduction :

Let A be an $n \times n$ matrix. If we can find a scalar λ and an $n \times 1$ non-null vector x, such that

$$Ax = \lambda x \tag{1}$$

then λ is called as an eigenvalue and x is an eigenvector of A. The equation (1) can be written as

$$(A-\lambda I)x = 0 \tag{2}$$

The set of homogeneous equations (2) admits a nontrivial solution if and only if $\det(A-\lambda I) = 0$. The determinant $\det(A-\lambda I)$ is an n-th degree polynomial in λ and is called the characteristic polynomial in A. Thus, for an $n \times n$ matrix A we have n eigenvalues and n eigenvectors.

Thus the system in which a scalar λ occurs and we determine the values of λ for which the system has nontrivial solution,then such nontrivial solution is called an eigenfunction and the entire system is called as the Eigenvalue Problem.

2.2 General eigenvalue prooblem

Let R be the unitary space.If there exists a number λ and an element $u \neq 0$ such that,

$$\langle u,f \rangle = \lambda \langle u,f \rangle_j \tag{3}$$

holds for all $f \in R$, (Here $\langle u,f\rangle$ and $\langle u,f\rangle_j$ are scalar products) then λ is called an eigenvalue and u an eigenelement of λ with respect to the scalar products of the space R.

2.3 A specific eigenvalue problem

Let T be a linear self-adjoint operator in a unitary space R, it is possible then to take for $\langle u,f\rangle_j$ the scalar product $\langle u,f\rangle$ of that space and define

$$\langle u,f\rangle_j = \langle Tu,f\rangle \tag{4}$$

ie. $\langle Tu,f\rangle = \lambda\langle u,f\rangle$ (5)

$$\langle Tu - \lambda u,f\rangle = 0 \quad \text{for all } f \in R$$

If R is the Hilbert space, then in this case we have

$$Tu = \lambda u \tag{6}$$

This is a "special eigenvalue problem", λ is called an eigenvalue of the operator T. Every linear self-adjoint operator T on a Hilbert space R has only real eigenvalues They are positive in the case of positive definite operator T.

Such problems naturally occur in variety of disciplines such as differential equations, control theory, approximation theory, mathematical economics, theory of integreal equation, in the study of vibration of dynamical and structural systems etc.

Examples of the occurance of eigenvalue problems in a wide variety of areas of application are as follows:

2.4 Eigenvalue problem in integral equation

The homogeneous Fredholm integral equation gives us a classical eigenvalue problem as follows:

$$\varnothing(x) = \lambda \int_a^b K(x,t)\varnothing(t)dt \qquad (7)$$

Here λ is the eigenvalue and K is the Kernel. The kernel K is assumed to be symmetric in x and t.

Approximating the above integral by quadrature formula we get

$$\varnothing(x) = \lambda\ (b-a)[c_1K(x,t_1)\varnothing(t_1) + c_2K(x,t_2)\varnothing(t_2) + \ldots\ldots$$

$$+ c_nK(x,t_n)\varnothing(t_n)] \qquad (8)$$

where t_j, $j = 1(1)n$ are the subinterval points of the interval $[a,b]$ and the c_j, $j = 1(1)n$ are the weighting coefficients known from the quadrature formula used. Since equation (8) is valid for all values of $x = t_j$, $j = 1(1)n$.

Thus from equation (8) we obtain

$$\varnothing(t_j) = \lambda\ (b-a)[c_1K(t_j,t_1)\varnothing(t_1) + c_2K(t_j,t_2)\varnothing(t_2) + \ldots\ldots$$

$$+ c_nK(t_j,t_n)\varnothing(t_n)] \qquad (9)$$

Set $\varnothing(t_j) = \varnothing_j$

Then we obtain

$$\phi_1 = \lambda\ (b-a)[c_1K(t_1,t_1)\phi_1 + c_2K(t_1,t_2)\phi_2 + \ldots\ldots + c_nK(t_1,t_n)\phi_n$$

$$\phi_2 = \lambda\ (b-a)[c_1K(t_2,t_1)\phi_1 + c_2K(t_2,t_2)\phi_2 + \ldots\ldots + c_nK(t_2,t_n)\phi_n$$

$$\phi_3 = \lambda\ (b-a)[c_1K(t_3,t_1)\phi_1 + c_2K(t_3,t_2)\phi_2 + \ldots\ldots + c_nK(t_3,t_n)\phi_n$$

$$\vdots$$

$$\phi_n = \lambda\ (b-a)[c_1K(t_n,t_1)\phi_1 + c_2K(t_n,t_2)\phi_2 + \ldots\ldots + c_nK(t_n,t_n)\phi_n \tag{10}$$

From equation (10) we can obtain an eigenvalue problem of the form

$$D'\phi = \lambda\phi \tag{11}$$

where λ is an eigenvalue of D'.

And D' is given as

$$D' = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ d_{n1} & d_{n2} & \ldots & d_{nn} \end{bmatrix}$$

where d_{ij} is given as,

$$d_{ij} = (b-a)\ c_jK(t_i,t_j),\ i = 1(1)n,\ j = 1(1)n.$$

Since D' is known, we can compute the eigenvalues of D'. Once the eigenvalue λ is known, we solve the homogeneous equation

$(D'-\lambda I)\phi = 0$ and obtain a nontrivial solution for ϕ.

2.5 A geometrical example

The equation of an ellipsoid in n-dimensional space is given by

$$1/2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_i z_j + \sum_{j=1}^{n} b_j z_j + c = 0 ,$$

where $z = (z_1, z_2, \ldots, z_n)^T$

Using matrix notation this may be compactly written as

$1/2\ z^T A z + b^T z + c = 0$ where c is known constant, b is known vector and A is symmetric positive definite matrix.

By a suitable translation of axes

$$z = x - A^{-1} b \tag{13}$$

this equation is simplified to

$$1/2 x^T A x + c = 0 \tag{14}$$

If the position vector of a point x on this hyperellipsoid is the same as the gradient vector at x, then x is a principle axis of the hyperellipsoid.

Let us consider two-dimensional space. (Here the component of X are denoted by x & y)

The curve $ax^2 + 2bxy + cy^2 = d$ is drawn. At a general point (x,y), the gradient is in the direction (ax+by, bx+cy). At the particular point (x,y) the gradient is in the same direction as the position vector from the origin. At this point (x,y) there exists some scalar λ such that

$$ax + by = \lambda x \qquad bx + cy = \lambda y \tag{15}$$

or in matrix notation

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \tag{16}$$

From this equation we deduce that the principal axes are given by the eigenvectors of the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

In n-dimensional example the gradient vector is given by

$$g = Ax. \tag{17}$$

The principal axes are given by the n vectors satisfying

$$Ax = \lambda x,$$

that is by the n eigenvectors of A.

2.5 An eigenvalue problem in Mechanics

In Mechanics we are faced with eigenvalue problems of given square matrices A and B

$$Au = \lambda Bu \tag{18}$$

We have to find the eigenvalues λ for which the above equation admits eigenvector $u \neq 0$. The eigenvalues satisfy the characteristic equation

$$\varnothing(\lambda) = \det(A - \lambda B) = 0.$$

For Hermitean matrices A and B and positive definite B the following scalar product can be used

$$(f,g)_j = \bar{f}\ B\ g \tag{19}$$

$$\langle f,g\rangle = \bar{f}\ A\ g \tag{20}$$

and the problem is of the form

$$\bar{f}\ A\ g = \lambda(\bar{f}\ B\ g) \tag{21}$$

For B = I the problem reduces to a special eigenvalue problem

$$Au = \lambda u. \tag{22}$$

2.7 Eigenvalue problem in differential equation

Ordinary and partial differential equations often lead to eigenvalue problems of the type

$$Mu = \lambda Nu \qquad \text{..........(a)}$$

$$U_\mu u = 0 \qquad \text{..........(b)}$$

M and N are linear nonhomogeneous differential expression and $U_\mu u = 0$ expresses linear homogeneous boundary conditions.

Assuming the highest derivative in (a) is of order q, we require R to be a subspace of $C^q(B)$ specifically, the subset of functions which satisfy the boundary conditions (b).

Let us further assume that the eigenvalue problem is self adjoint, that is for any two functions u and v in R the relation holds

$$\int_B (\bar{u}\ Mv\ -\ v\ M\bar{u}\ )dx = \int_B (\bar{u}\ Nv\ -\ v\ N\bar{u}\ )dx = 0 \qquad (23)$$

The operator N is requried to be positive definite for any function $u \neq 0$ in R. We have

$$\int_B uN\bar{u}\ dx > 0$$

By using the following scalar products in R,

$$(f,g)_j = \int_B fNg\ \ dx$$

$$\langle f,g\rangle = \int_B fMg\ dx$$

We obtain the following eigenvalue problem.

$$\int_B fMg\ dx = \lambda \int_B fNg\ dx$$

2.8 <u>An example of mathematical economics.</u>

In the study of macroeconomics, one of the most useful tools available to the planner is input output analysis introduced by Leontief. The input output table links the individual industries to the overall working of the economy. Considering the sales and purchase of an industrial sector, we denote the sales of indstry i to the industry j by b_{ij}. The reatention of goods produced by the industry i by b_{ii}. The sales of goods prodced by industry i to outside user is denoted by y_i and the gross output by x_i

Thus

$$x_i = y_i + \sum_j b_{ij} \tag{25}$$

Assume that the sales of industry i to industry j are in constant proportion (a_{ij}) to the output of the indstry j, thus

$$b_{ij} = a_{ij}\, x_j \tag{26}$$

The quantities are defined to be the input coefficients. From equation (25) we see that in a static situation

$$x = y + Ax \tag{27}$$

where $x = (x_1, x_2, x_3, \ldots, x_n)^T$

$y = (y_1, y_2, y_3, \ldots, y_n)^T$

and A is the n×n matrix with the (i, j)th element a_{ij}. The matrix (I - A) is called as Leontief matrix. Equation (27) is used to determine the required gross outputs x of the industry sector to meet a preset final demand y. If supply and demand are not in equilibrium then we must replace equation (27) by dynamic model. Here the assumption is that the output in each industry changes at a rate which is propotional to the difference between the level of production and the level of sales. Thus our dynamic model takes the form

$$\frac{dx}{dt}(t) = D[(A - I)x(t) + y(t)] \tag{28}$$

where D is a diagonal matrix of the reaction coefficients of the industries. For this model the existence of eigenvalues with positive real part would indicate an instability in the system because the required gross output would grow exponentially with time. A similar use of the Leontief matrix and eigensystem analysis occures in a discrete dynamic system of the form

$$x(t+1) - x(t) = D[(A - I)x(t) + y(t)]$$

such models are of use in studying the stability of interindustry relations, multiple markets and intercontry trade.

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