# CHAPTER 3

# CHAPTER3 FUZZY RELATION EQUATIONS ON FUZZY SETS

In this chapter we study a theory of fuzzy relation equations, when the fuzzy relations are defined on fuzzy sets instead of crisp sets.

## 3.1 FUZZY RELATIONS AND THEIR COMPOSITIONS:

This section deals with fuzzy relations defined on fuzzy sets and four types of compositions along with their properties.

**Definition 3.1.1** [F]: Let A and B be two fuzzy sets in X. The cartesian product of A and B, is a fuzzy set, $A \times B$, on $X \times X$ defined as follows:

$A \times B(x, y) = \min\{A(x), B(y)\}$, for all $x, y \in X$.

**Definition 3.1.2** [F]: Let A and B be two fuzzy sets in X. Then the fuzzy set $P: X \times X \rightarrow I$ is called a fuzzy relation from A to B, if

$P(x, y) \leq \min\{A(x), B(y)\}$, for all $x, y \in X$.

We shall denote the set of all fuzzy relations from fuzzy set A to fuzzy set B by $F(A \times B)$.

**Definition 3.1.3**: Let $P \in F(A \times B)$ be a fuzzy relation from a fuzzy set A to a fuzzy B. Then the fuzzy set $P^{-1}$ defined as follows:

$$P^{-1}(y, x) = P(x, y)$$

is a fuzzy relation from a fuzzy set B to a fuzzy set A. i. e. $P^{-1} \in F(B \times A)$.

It is obvious that $(P^{-1})^{-1} = P$.

**Theorem 3.1.4:** Let $P \in F(A \times B)$ and $Q \in F(B \times C)$ be two fuzzy relations and T be any continuous t-norm. Define a fuzzy set $P \circ_T Q: X \times X \to I$ by

$$P \circ_T Q(x, y) = \sup_{z \in X}\{T(P(x, z), Q(z, y))\}, \text{ for all } x, y \in X.$$

Then $P \circ_T Q$ is a fuzzy relation from fuzzy set A to fuzzy set C.

Proof: Since $P \in F(A \times B)$ and $Q \in F(B \times C)$,

$P(x, z) \leq \min\{A(x), B(z)\}$, for all $x, z \in X$ and

$Q(z, y) \leq \min\{B(z), C(y)\}$, for all $z, y \in X$.

$$P \circ_T Q(x, y) = \sup_{z \in X}\{T(P(x, z), Q(x, y))\}$$

$$\leq \sup_{z \in X}\{T(\min(A(x), B(z), \min(B(z), C(y))\}$$

$$\leq \sup_{z \in X}\{\min(\min(A(x), B(z)), \min(B(z), C(y)))\}$$

$$\leq \sup_{z \in X}\{\min(A(x), C(y))\}$$

Thus, $P \circ_T Q(x, y) \leq \min(A(x), C(y))$ for all $x, y \in X$.

**Definition 3.1.5** [F]: Let $P \in F(A \times B)$ and $Q \in F(B \times C)$ be two fuzzy relations and T be any continuous t-norm. The fuzzy set $P \circ_T Q$ is called sup-T composition of P and Q.

**Theorem 3.1.6**: Let $P \in F(A \times B)$ and $Q \in F(B \times C)$ be two fuzzy relations. Then

$(P \circ_T Q)^{-1} = Q^{-1} \circ_T P^{-1}$

Following theorem is obvious

**Theorem 3.1.7** [F]: Let $P \in F(A \times B)$ and $Q \in F(B \times C)$ be two fuzzy relations and S be any continuous t-conorm. Define a fuzzy set $P\ _{S}o\ Q: X \times X \to I$ by

$$P\ _{S}o\ Q(x, y) = \min\{ \inf_{z \in X} \{S(P(x, z), Q(z, y))\}, (A \times C)(x, y)\}, \forall\ x, y \in X.$$

Then $P\ _{S}o\ Q$ is a fuzzy relation from a fuzzy set A to a fuzzy set C.

**Definition 3.1.8** [F]: Let $P \in F(A \times B)$ and $Q \in F(B \times C)$ be two fuzzy relations and S be any continuous t-conorm. The fuzzy set $P\ _{S}o\ Q$ is called inf-S composition of P and Q.

**Theorem 3.1.9** [F]: Let $P, P_1, P_2 \in F(A \times B)$ and $Q, Q_1, Q_2 \in F(B \times C)$

(i) $P\ _{S}o\ (Q_1 \cap Q_2) = (P\ _{S}o\ Q_1) \cap (P\ _{S}o\ Q_2)$

(ii) If $P_1 \subseteq P_2$, then $P_1\ _{S}o\ Q \subseteq P_2\ _{S}o\ Q$

(iii) $(P\ _{S}o\ Q)^{-1} = Q^{-1}\ _{S}o\ P^{-1}$

Proof: i) Let $x, y \in X$.

$[P\ _{S}o\ (Q_1 \cap Q_2)]\ (x, y)$

$$= \min\{ \inf_{z \in X} \{S(P(x, z), Q_1 \cap Q_2 (z, y))\}, A \times C(x, y)\}$$

$$= \min\{ \inf_{z \in X} \{S(P(x, z), \min\{Q_1(z, y), Q_2(z, y)\})\}, A \times C(x, y)\}$$

$$= \min\{ \inf_{z \in X} \{\min\{S(P(x, z), Q_1(z, y)),\ S(P(x, z), Q_2(z, y))\}, A \times C(x, y)\}$$

$$= \min\{\min\{ \inf_{z \in X} \{S(P(x, z), Q_1(z, y))\},\ \inf_{z \in X} \{S(P(x, z), Q_2(z, y))\}\}, A \times C\ (x, y)\}\}$$

$$= \min\{\min\{ \inf_{z \in X} \{S(P(x, z), Q_1(z, y))\}, A \times C(x, y)\}, \min\{ \inf_{z \in X} \{S(P(x, z), Q_2(z, y))\}, A \times C(x, y)\}\}$$

$$= \min\{P\ _{S}o\ Q_1(x, y), P\ _{S}o\ Q_2(x, y)\}$$

$$= (P\ _{S}o\ Q_1) \cap (P\ _{S}o\ Q_2)(x, y)$$

ii) Let $P_1 \subseteq P_2$

$$P_1 \, {}_S o \, Q(x, y) = \min\{ \inf_{z \in X} \{S(P_1(x, z), Q(z, y))\}, A \times C(x, y)\}$$

$$\leq \min\{ \inf_{z \in X} \{S(P_2(x, z), Q(z, y))\}, A \times C(x, y)\}$$

$$= P_2 \, {}_S o \, Q(x, y).$$

iii) $(P \, {}_S o \, Q)^{-1} (x, y) = (P \, {}_S o \, Q)(y, x)$

$$= \min\{ \inf_{z \in X} \{S(P(y, z), Q(z, x))\}, A \times C(y, x)\}$$

$$= \min\{ \inf_{z \in X} \{S(P^{-1}(z, y), Q^{-1} (x, z))\}, C \times A(x, y)\}$$

$$= \min\{ \inf_{z \in X} \{ S(Q^{-1}(x, z), P^{-1}(z, y))\}, C \times A(x, y)\}$$

$$= (Q^{-1} \, {}_S o \, P^{-1}) (x, y)$$

**Definition 3.1.10** [F]: Let A, B, C be fuzzy sets in X and $P \in F(A \times B)$, $Q \in F(B \times C)$ be fuzzy relations. The inf-$w_T$ composition of P and Q is a fuzzy relation $P \overset{o}{w_T} Q \in F(A \times C)$ defined as follows:

$$P \overset{o}{w_T} Q(x, y) = \min \{ \inf_{z \in X} \{w_T(P(x, z), Q(z, y))\}, A \times C(x, y)\}, \text{ for all } x, y \in X$$

If $P \in F(A \times B)$ and $Q \in F(B \times C)$ be fuzzy relations, then $(P \overset{o}{w_T} Q)^{-1}$ is a fuzzy relation from a fuzzy set C to A.

**Theorem 3.1.11** [F]: Let $P, P_1, P_2 \in F(A \times B)$ and $Q, Q_1, Q_2 \in F(B \times C)$ be such that $P_1 \subseteq P_2$ and $Q_1 \subseteq Q_2$. Then

i) $P \overset{o}{w_T} Q_1 \subseteq P \overset{o}{w_T} Q_2$

ii) $P_1 \overset{o}{w_T} Q \supseteq P_2 \overset{o}{w_T} Q$.

Proof i) $P \circ_{w_T} Q_1(x, y) = \min\{ \inf_{z \in X} \{w_T(P(x, z), Q_1(z, y))\}, A \times C(x, y)\}$

$\leq \min\{ \inf_{z\in X} \{w_T(P(x, z), Q_2(z, y))\}, A \times C(x, y)\}$, by Theorem 1.2.8(iv)

$\leq P \circ_{w_T} Q_2(x, y)$.

ii) $P_1 \circ_{w_T} Q(x, y)$

$= \min\{ \inf_{z \in X} \{w_T (P_1(x, z), Q(z, y))\}, A \times C(x, y)\}$

$\geq \min\{ \inf_{z \in X} \{w_T (P_2(x, z), Q(z, y))\}, A \times C (x, y)\}$, by Theorem 1.2.8(iv)

$\geq P_2 \circ_{w_T} Q(x, y)$.

**Theorem 3.1.12** [F]: Let $P \in F(A \times B)$, $Q \in F(B \times C)$ and $R \in F(A \times C)$ be fuzzy relations. Then

i) $P \subseteq (P \circ_{w_T} Q) \circ_{w_T} Q^{-1}$

ii) $R \subseteq (R \circ_{w_T} Q^{-1}) \circ_{w_T} Q$.

Proof: i) $(P \circ_{w_T} Q) \circ_{w_T} Q^{-1}(x, y) = \min\{\inf_{z \in X}\{w_T(P \circ_{w_T} Q(x, z), Q(y, z))\}, A \times B(x, y)\}$

$= \min\{\inf_{z \in X} \{w_T(\min\{ \inf_{t \in X} \{w_T(P(x, t), Q(t, z))\}, A \times C(x, z)\}), Q(z, y))\}, A \times B(x, y)\}$.

$\geq \min\{ \inf_{z \in X} \{w_T ( \inf_{t \in X} \{w_T(P(x, t), Q(t, z))\}, Q(y, z))\}, A \times B(x, y)\}$,

$\geq \min\{ \inf_{z \in X} \{w_T(w_T(P(x, y), Q(y, z)), Q(y, z))\}, A \times B(x, y)\}$

$\geq \min\{ \inf_{z \in X} \{P(x, y)\}, A \times B(x, y)\}$, By Theorem 1.2.8(ii)

$= P(x, y)$, Since $P \in F(A \times B)$

ii) Follows similarly.

**Theorem 3.1.13** [F]: Let $P \in F(A \times B)$, $Q \in F(B \times C)$ and $R \in F(A \times C)$ be fuzzy relations. Then

i) $Q \subseteq P^{-1} \circ_{w_T} (P \circ_T Q)$

ii) $R \supseteq P \circ_T (P^{-1} \circ_{w_T} R)$

iii) $P \subseteq [Q \circ_{w_T} (P \circ_T Q)^{-1}]^{-1}$

iv) $R \supseteq (Q \circ_{w_T} R^{-1})^{-1} \circ_T Q$

Proof: i) $P^{-1} \circ_{w_T} (P \circ_T Q)(x, y)$

$= \min\{ \inf_{z \in X} \{w_T(P^{-1}(x, z), P \circ_T Q(z, y))\}, B \times C(x, y)\}$

$= \min\{ \inf_{z \in X} \{w_T(P^{-1}(x, z), \sup_{t \in X} \{T(P(z, t), Q(t, y))\})\}, B \times C(x, y)\}$

$= \min\{ \inf_{z \in X} \{w_T(P^{-1}(x, z), \max[T(P(z, x), Q(x, y)), \sup_{t \neq x} \{T(P(z, t), Q(t, y))\}]\},$

$(B \times C)(x, y)\}$

$\geq \min\{ \inf_{z \in X} \{w_T(P(z, x), T(P(z, x), Q(x, y)))\}, B \times C(x, y)\}$

$\geq \min\{ \inf_{z \in X} \{Q(x, y), B \times C(x, y)\},$

$\geq Q(x, y)$

Hence, $Q \subseteq P^{-1} \circ_{w_T} (P \circ_T Q)$

ii) $P \circ_T (P^{-1} \circ_{w_T} R)(x, y)$

$= \sup_{z \in X} \{T(P(x, z), P^{-1} \circ_{w_T} R(z, y))\}$

$= \sup_{z \in X} \{T(P(x, z), \min\{ \inf_{t \in X} \{w_T(P(t, z), R(t, y))\}, B \times C(x, y)\})\}$

$\leq \sup_{z \in X} \{T(P(x, z), \inf_{t \in X} \{w_T(P(t, z), R(t, y))\})\}$, Since $\min x_i \leq x_i$

$\leq \sup_{z \in X} \{T(P(x, z), w_T(P(x, z), R(x, y)))\}$, By monotonicity of T.

$\leq R(x, y)$,

Hence, $R \supseteq P \circ_T (P^{-1} \circ_{w_T} R)$.

iii) $[Q \overset{o}{w}_T (P \, o_T \, Q)^{-1}]^{-1} (x, y)$

$= [Q \overset{o}{w}_T (P \, o_T \, Q)^{-1}] (y, x)$

$= \min\{ \inf_{z \in X} \{w_T(Q(y, z), (P \, o_T \, Q)^{-1}(z, x))\}, \; B \times A(y, x)\}$

$= \min\{ \inf_{z \in X} \{w_T(Q(y, z), (P \, o_T \, Q)(x, z))\}, A \times B(x, y)\}$

$= \min\{ \inf_{z \in X} \{w_T(Q(y, z), \sup_{t \in X} \{T(P(x, t), Q(t, z))\})\}, A \times B(x, y)\}$

$= \min\{ \inf_{z \in X} \{w_T(Q(y, z), \; \max\{T(P(x, y), Q(y, z)), \; \sup_{t \neq y} \{T(P(x, t), \; Q(t, z))\}\})\}, \; A \times B(x, y)\}$

$\geq \min\{ \inf_{z \in X} \{w_T(Q(y, z), \; T(P(x, y), Q(y, z)))\}, A \times B(x, y)\}$

$\geq \min \{P(x, y), \; A \times B(x, y)\}$

$\geq P(x, y)$, Since $P \in F(A \times B)$

Hence, $P \subseteq [Q \overset{o}{w}_T (P o_T Q)^{-1}]^{-1}$

iv) $(Q \overset{o}{w}_T R^{-1})^{-1} o_T Q (x, y)$

$= \sup_{z \in X} \{T((Q \overset{o}{w}_T \; R^{-1})^{-1} (x, z), Q(z, y))\}$

$= \sup_{z \in X} \{T((Q \overset{o}{w}_T R^{-1}) (z, x), Q(z, y))\}$

$= \sup_{z \in X} \{T(\min\{ \inf_{t \in X} \{w_T(Q(z, t), R(x, t))\}, B \times A(x, y)\}, Q(z, y))\}$

$\leq \sup_{z \in X} \{T( \inf_{t \in X} \{w_T(Q(z, t), R(x, t))\}, Q(z, y))\}$

$\leq \sup_{z \in X} \{T(w_T(Q(z, y), R(x, y)), Q(z, y))\}$

$\leq R(x, y)$, By Theorem 1.2.8(x)

Hence, $R \supseteq (Q \overset{o}{w}_T R^{-1})^{-1} o_T Q$

**Definition 3.1.14** [F]: Let A, B, C be fuzzy sets in X and $P \in F(A \times B)$, $Q \in F(B \times C)$ be fuzzy relations. The sup-$\omega_S$ composition of P and Q is a fuzzy relation $P \,^{o}\omega_S\, Q \in F(A \times C)$ defined as follows:

$$P \,^{o}\omega_S\, Q\,(x, y) = \min\{ \sup_{z \in X} \{\omega_S (P(x, z), Q(z, y))\}, A \times C(x, y)\}, \text{ for all } x, y \in X$$

**Theorem 3.1.15** [F]: Let $P, P_1, P_2 \in F(A \times B)$ and $Q, Q_1, Q_2 \in F(B \times C)$ be such that $P_1 \subseteq P_2$ and $Q_1 \subseteq Q_2$. Then

i) $P \,^{o}\omega_S\, Q_1 \subseteq P \,^{o}\omega_S\, Q_2$.

ii) $P_1 \,^{o}\omega_S\, Q \supseteq P_2 \,^{o}\omega_S\, Q$.

Proof: i) Since $Q_1 \subseteq Q_2$, $Q_1(z, y) \leq Q_2(z, y)$, for all $z, y \in X$.

$\Rightarrow \omega_S (P(x, z), Q_1(z, y)) \leq \omega_S (P(x, z), Q_2(z, y))$, by Theorem 1.2.10(iv)

Thus, $P \,^{o}\omega_S\, Q_1(x, y) = \min\{ \sup_{z \in X} \{ \omega_S (P(x, z), Q_1(z, y))\}, A \times C (x, y) \}$

$\leq \min\{ \sup_{z \in X} \{ \omega_S (P(x, z), Q_2(z, y))\}, A \times C(x, y) \}$

$\leq P \,^{o}\omega_S\, Q_2(x, y)$

Hence, $P \,^{o}\omega_S\, Q_1 \subseteq P \,^{o}\omega_S\, Q_2$.

ii) Since $P_1 \subseteq P_2$, $P_1(x, z) \leq P_2(x, z)$, $\forall x, z \in X$.

$\Rightarrow \omega_S (P_1(x, z), Q(z,y) \geq \omega_S (P_2(x, z), Q(z, y))$, By Theorem 1.2.10(iv)

Thus, $P_1 \,^{o}\omega_S\, Q(x, y) = \min\{ \sup_{z \in X} \{ \omega_S (P_1(x, z), Q(z, y))\}, A \times C(x, y)\}$

$\geq \min\{ \sup_{z \in X} \{\omega_S (P_2(x, z), Q(z, y))\}, A \times C(x, y)\}$

$\geq P_2 \,^{o}\omega_S\, Q(x, y)$

Hence, $P_1 \,^{o}\omega_S\, Q \supseteq P_2 \,^{o}\omega_S\, Q$.

**Theorem 3.1.16** [F]: Let $P \in F(A \times B)$, $Q \in F(B \times C)$ and $R \in F(A \times C)$ be fuzzy relations. Then

i) $Q \supseteq P^{-1} \circ \omega_S (P \; _S o \, Q)$

i) $P \supseteq (Q \circ \omega_S (P \; _S o \, Q)^{-1})^{-1}$

iii) $Q \supseteq P^{-1} o_T (P \circ w_T \; Q)$

iv) $R \subseteq P \circ w_T \; (P^{-1} o_T R)$

v) $R \supseteq P \circ \omega_S (P^{-1} \; _S o \, R)$

Proof: i) $P^{-1} \circ \omega_S \; (P \; _S o \, Q)(x, y)$

$$= \min\{ \sup_{z \in X} \{\omega_S (P^{-1}(x, z), P \; _S o Q(z, y))\}, B \times C(x, y)\}$$

$$= \min\{ \sup_{z \in X} \{\omega_S(P(z, x), \min[ \inf_{t \in X} \{S(P(z, t), Q(t, y))\}, A \times C(z, y)]), (B \times C)(x, y)\}$$

$$\leq \min\{ \sup_{t \in X} \{\omega_S (P(z, x), \; \inf \; (S(P(z, t), Q(t, y)))), \; B \times C(x, y)\}, z \in X$$

$$\leq \min\{ \sup_{z \in X} \{\omega_S(P(z, x), \min[(S(P(z, x), Q(x, y)), \; \inf_{t \neq x} \{S(P(z, t), Q(t, y))\}], B \times C(x, y)\}.$$

$$\leq \min\{ \sup_{z \in X} \{\omega_S(P(z, x), S(P(z, x), Q(x, y)))\}, \; B \times C(x, y)\}$$

$\leq \min\{Q(x, y), B \times C(x, y)\}$,

$\leq Q(x, y)$, since $Q \in F(B \times C)$

Hence, $Q \supseteq P^{-1} \circ \omega_S (P \; _S o \, Q)$

ii) $(Q \circ \omega_S (P \; _S o \, Q)^{-1})^{-1}(x, y) = (Q \circ \omega_S (P \; _S o \, Q)^{-1})(y, x)$

$$= \min\{ \sup_{z \in X} \{\omega_S (Q(y, z), (P \; _S o \, Q)^{-1}(z, x))\}, B \times A (y, x)\}$$

$$= \min\{ \sup_{z \in X} \{\omega_S (Q(y, z), P\, {}_S o\, Q(x, z))\}, A \times B(x, y)\}$$

$$= \min\{ \sup_{z \in X} \{\omega_S (Q(y, z), \min [ \inf_{t \in X} \{S(P(x, t), Q(t, z))\}, A \times C(x, z)] , A \times B(x, y)\}$$

$$= \min\{ \sup_{z \in X} \{\omega_S (Q(y, z), \min[\min[ S(P(x, y), Q(y, z)), \inf_{t \neq y} S(P(x, t), Q(t, z))], A \times B(x, y)\} A \times C(x, z)],$$

$$\leq \min\{ \sup_{z \in X} \{\omega_S (Q(y, z), S(P(x, y), Q(y, z)))\}, A \times B(x, y)\}$$

$$\leq \min\{P(x, y), A \times B(x, y)\}$$

$= P(x, y)$, Since $P \in F(A \times B)$ .

iii) $P^{-1} o_T (P \overset{o}{w}_T Q)(x, y)$

$$= \sup_{z \in X} \{T(P^{-1}(x, z), P \overset{o}{w}_T Q(z, y)\}$$

$$= \sup_{z \in X} \{T(P(z, x), \min [ \inf_{t \in X} \{w_T(P(z, t), Q(t, y))\}, A \times C(z, y)])\}$$

$\leq \sup_{z \in X} \{T(P(z, x), \inf_{t \in X} \{w_T(P(z, t), Q(t, y))\})\}$, By Monotonicity of T.

$\leq \sup_{z \in X} \{T(P(z, x), w_T (P(z, x), Q(x, y)))\}$, By Monotonicity of T.

$\leq Q(x, y)$

Hence, $Q \supseteq P^{-1} o_T (P \overset{o}{w}_T Q)$.

iv) $P \overset{o}{w}_T (P^{-1} o_T R) (x, y)$

$$= \min \{ \inf_{z \in X} \{w_T(P(x, z), P^{-1} o_T R(z, y))\}, A \times C(x, y)\}$$

$$= \min \{ \inf_{z \in X} \{ w_T (P(x, z), \sup_{t \in X} \{T(P^{-1}(z, t), R(t, y))\})\}, A \times C(x, y)\}$$

$$= \min \{ \inf_{z \in X} \{ w_T (P(x, z), \max \{T(P^{-1}(z, x), R(x, y)), \sup_{t \neq x} \{T(P^{-1}(z, t), R(t, y))\}\})\}, A \times C(x, y)\}.$$

$$\geq \min \{ \inf_{z \in X} \{ w_T (P(x, z), T(P(x, z), R(x, y)))\}, A \times C(x, y)\}$$

$$\geq \min \{R(x, y), A \times C(x, y)\}$$

$= R(x, y)$

v) $P \,{}^{o}\omega_S\, (P^{-1} so\, R)(x, y) = \min\{ \sup_{z \in X} \{\omega_S (P(x, z), P^{-1} so\, R(z, y))\}, A \times C(x, y)\}$

$= \min\{ \sup_{z \in X} \{\omega_S (P(x, z), \min\{ \inf_{t \in X}\{S(P^{-1}(z, t), R(t, y))\}, B \times C(z, y)\})\}, A \times C(x, y)\}.$

$\leq \min\{ \sup_{z \in X} \{\omega_S (P(x, z), \inf_{t \in X} \{S(P(t, z), R(t, y))\})\}, A \times C(x, y)\}$

$\leq \min\{ \sup_{z \in X} \{\omega_S(P(x, z), S(P(x, z), R(x, y)))\}, A \times C(x, y)\}$

$\leq \min \{R(x, y), A \times C(x, y)\},$

$= R(x, y).$

**Theorem 3.1.17** [F]: Let $P \in F(A \times B)$ and $R \in F(A \times C)$

If $\sup_{z \in X} A(z) \leq \min\{B(x), C(y)\}$, for all $x, y \in X$, then $P^{-1\,o}\omega_S R \in F(B \times C)$.

Proof: We claim that for $x, y \in X$, $\sup_{z \in X} \{\max(P(z, x), R(z, y))\} \leq B \times C(x, y)$.

We have,

$\sup_{z \in X} \{\max(P(z, x), R(z, y))\} \leq \sup_{z \in X} \{\max\{\min\{A(z), B(x)\}, \min\{A(z), C(y)\}\}\}$

$= \sup_{z \in X} \{\max\{A(z), A(z)\}\}$

$= \sup_{z \in X} \{A(z)\}$

$\leq \min\{B(x), C(y)\}$

$= B \times C(x, y)$

Now $P^{-1\,o}\omega_S\, R(x, y) = \min\{ \sup_{z \in X} \{\omega_S (P(z, x), R(z, y))\}, B \times C(x, y)\}$

$\leq \min\{ \sup_{z \in X} \{\max(P(z, x), R(z, y))\}, B \times C(x, y)\}$

$= \sup_{z \in X} \{\max(P(z, x), R(z, y))\}$

$= B \times C(x, y).$

Therefore, $P^{-1} \circ \omega_S R \in F(B \times C)$.

**Theorem 3.1.18** [F]: Let $P \in F(A \times B)$, $R \in F(A \times C)$ and

$\sup_{z \in X} A(z) \leq \min\{B(x), C(y)\}$, for all $x, y \in X$, then $R \subseteq P\ _So\ (P^{-1} \circ \omega_S R)$

Proof: $P\ _So\ (P^{-1} \circ \omega_S R)(x, y)$

$$= \min\{ \inf_{z \in X} \{S(P(x, z), P^{-1} \circ \omega_S R(z, y))\}, A \times C(x, y)\}$$

$$= \min\{ \inf_{z \in X}\{S(P(x, z), \min\{ \sup_{t \in X} \{\omega_S (P^{-1}(z, t), R(t, y))\}, B \times C(x, y)\}\})\}, A \times C(x, y)\}$$

$$= \min\{ \inf_{z \in X} \{S(P(x, z), \sup_{t \in X} \{\omega_S (P^{-1}(z, t), R(t, y))\}\})\}, A \times C(x, y)\}$$

$$= \min\{ \inf_{z \in X} \{S(P(x, z), \max[\omega_S (P(x, z), R(x, y)), \sup_{t \neq x} \{\omega_S (P(t, z), R(t, y))\}])\},$$
$$A \times C(x, y)\}$$

$$\geq \min\{ \inf_{z \in X} \{S(P(x, z), \omega_S (P(x, z), R(x, y)))\}, A \times C(x, y)\}.$$

$$\geq \min\{ \inf_{z \in X} \{R(x, y)\}, A \times C(x, y)\},$$

$$= \min\{R(x, y), A \times C(x, y)\}$$

$= R(x, y)$, Since $R \in F(A \times C)$

Therefore, $R \subseteq P\ _So\ (P^{-1} \circ \omega_S R)$

**Theorem 3.1.19** [F]: Let $Q \in F(B \times C)$, $R \in F(A \times C)$ and

$\sup_{z \in X} C(z) \leq \min\{B(x), A(y)\}$, for all $x, y \in X$. Then $Q \circ \omega_S R^{-1} \in F(B \times A)$

Proof: We claim that $\sup_{z \in X} \{\max(Q(x, z), R^{-1}(z, y))\} \leq B \times A(x, y)$, for all $x, y \in X$.

We have,

$$\sup_{z \in X} \{\max(Q(x, z), R^{-1}(z, y))\} = \sup_{z \in X} \{\max(Q(x, z), R(y, z))\}$$

$$\leq \sup_{z \in X} \{\max\{\min\{B(x), C(z)\}, \min\{A(y), C(z)\}\}\}$$

$$= \sup_{z \in X} \{\max\{C(z), C(z)\}$$

$$= \sup_{z \in X} \{C(z)\}$$

$$\leq B \times A(x, y)$$

Now $Q\ {}^{o}\omega_S\ R^{-1}(x, y) = \min\{ \sup_{z \in X} \{\omega_S (Q(x, z), R^{-1}(z, y))\}, B \times A(x, y)\}$

$$\leq \min\{ \sup_{z \in X} \{\max(Q(x, z), R^{-1}(z, y))\}, B \times A(x, y)\}$$

$$= \sup_{z \in X} \{\max(Q(x, z), R^{-1}(z, y))\}$$

$$\leq B \times A(x, y)$$

Therefore, $Q\ {}^{o}\omega_S\ R^{-1} \in F(B \times A)$.

**Theorem 3.1.20** [F]: Let $Q \in F(B \times C)$, $R \in F(A \times C)$ and

$\sup_{z \in X} C(z) \leq \min\{B(x), A(y)\}$, for all $x, y \in X$. Then $R \subseteq (Q\ {}^{o}\omega_S\ R^{-1})^{-1}\ so\ Q$.

Proof: $(Q\ {}^{o}\omega_S\ R^{-1})^{-1}\ so\ Q\ (x, y)$

$$= \min\{ \inf_{z \in X} \{S(Q\ {}^{o}\omega_S\ R^{-1})(z, x), Q(z, y))\}, A \times C(x, y)\}$$

$$= \min\{ \inf_{z \in X}\{S(\min\{ \sup_{t \in X} \{\omega_S (Q(z, t), R(x, t))\}, B \times A(z, x)\}), Q(z, y))\}, A \times C(x, y)\}$$

$$= \min\{ \inf_{z \in X} \{S( \sup_{t \in X} \{\omega_S (Q(z, t), R(x, t))\}, Q(z, y))\}, A \times C(x, y)\}$$

$$= \min\{ \inf_{z \in X} \{S(\max[\omega_S (Q(z, y), R(x, y)), \operatorname{Sup}_{t \neq x} \{\omega_S (Q(z, t), R(x, t))]\}, Q(z, y))\},$$

$$A \times C(x, y).$$

$$\geq \min \{ \inf_{z \in X} \{S(\omega_S (Q(z, y), R(x, y)), Q(z, y))\}, A \times C(x, y)\}$$

$$\geq \min \{ \inf_{z \in X} \{R(x, y)\}, A \times C(x, y)\},$$

$$= \min\{R(x, y), A \times C(x, y)\}$$

$$= R(x, y)$$

**Theorem 3.1.21** [F]: Let $P \in F(A \times B)$ and $Q \in F(B \times C)$

If $\sup_{z \in X} B(z) \leq \min (A(x), C(y)), \forall\ x, y \in X$, then $Q \subseteq P^{-1} so (P^{o}\omega_S Q)$.

Proof: $P^{-1} so (P\ ^{o}\omega_S Q)(x, y)$

$$= \min\{ \inf_{z \in X} \{S(P^{-1}(x, z), \min \{ \operatorname{Sup}_{t \in X} \{\omega_S (P(z, t), Q(t, y))\}, A \times C(z, y)\}), B \times C (x, y)\}$$

$$= \min\{ \inf_{z \in X} \{S(P(z, x), \sup_{t \in X} \{\omega_S (P(z, t), Q(t, y))\})\}, B \times C(x, y)\}$$

$$= \min\{ \inf_{z \in X} \{S(P(z, x), \max [\omega_S (P(z, x), Q(x, y)), \operatorname{Sup}_{t \neq x} \{\omega_S (P(z, t), Q(t, y))\}]\},$$

$$B \times C(x, y)\}$$

$$\geq \min\{ \inf_{z \in X} \{S(P(z, x), \omega_S (P(z, x), Q(x, y)))\}, B \times C(x, y)\}$$

$$\geq \min\{ \inf_{z \in X} \{Q(x, y)\}, B \times C(x, y)\},$$

$$= Q(x, y).$$

**Theorem 3.1.22** [F]: Let $P \in F(A \times B)$ and $Q \in F(B \times C)$

i) If $\operatorname{Sup} B(z) \leq \min \{A(x), C(y)\}$, for $x, y \in X$, then $P \supseteq (P\ ^{o}\omega_S Q)\ ^{o}\omega_S Q^{-1}$.

$z \in X$

ii) If $\sup_{z \in X} C(z) \leq \min\{A(x), B(y)\}$, for $x, y \in X$, then $R \supseteq (R\,{}^{o}\omega_S\, Q^{-1})\,{}^{o}\omega_S\, Q$.

Proof: i) $[(P\,{}^{o}\omega_S\, Q)\,{}^{o}\omega_S\, Q^{-1}]\,(x, y)$

$$= \min\{\sup_{z \in X} \{\omega_S\,(P\,{}^{o}\omega_S\, Q)(x, z), Q^{-1}(z, y))\}, A \times B(x, y)\}$$

$$= \min\{\sup_{z \in X} \{\omega_S\,(\min\{\sup_{t \in X} \{\omega_S\,(P(x, t), Q(t, z))\}, A \times C(x, z))\}, Q(y, z))\},$$

$$A \times B(x, y)\}$$

$$= \min\{\sup_{z \in X} \{\omega_S\,(\sup_{t \in X} \{\omega_S\,(P(x, t), Q(t, z))\}, Q(y, z))\}, A \times B(x, y)\}$$

$$= \min\{\sup_{z \in X} \{\omega_S\,(\max[\omega_S\,(P(x, y), Q(y, z)), \sup_{t \neq y} \{\omega_S\,(P(x, t), Q(t, z))\}], Q(y, z)\},$$

$$A \times B(x, y)\}$$

$$\leq \min\{\sup_{z \in X} \{\omega_S\,(\omega_S\,(P(x, y), Q(y, z)), Q(y, z))\}, A \times B(x, y)\}$$

$$\leq \min\{\sup_{z \in X} \{P(x, y)\}, A \times B(x, y)\}$$

$$\leq P(x, y).$$

ii) $[(R\,{}^{o}\omega_S\, Q^{-1})\,{}^{o}\omega_S\, Q]\,(x, y)$

$$= \min\{\sup_{z \in X} \{\omega_S\,(R\,{}^{o}\omega_S\,(Q^{-1}(x, z), Q(z, y))\}, A \times C(x, y)\}$$

$$= \min\{\sup_{z \in X}\{\omega_S\,(\min\{\sup_{t \in X}\{\omega_S\,(R(x, t), Q(z, t))\}, A \times B(x, z)\}, Q(z, y))\}, A \times C(x, y)\}$$

$$= \min\{\sup_{z \in X} \{\omega_S(\sup_{t \in X} \{\omega_S\,(R(x, t), Q(z, t))\}, Q(z, y))\}, A \times C(x, y)\},$$

$$= \min\{ \sup_{z \in X} \{\omega_S ( \max [\omega_S (R(x, y), Q(z, y)), \ \sup_{t \neq y} \{\omega_S (R(x, t), Q(z, t))], Q(z, y))\}, \ A \times C(x, y)\}$$

$$\leq \min\{ \sup_{z \in X} \{\omega_S (\omega_S (R(x, y), Q(z, y)), Q(z, y))\}, A \times C(x, y)\}.$$

$$\leq \min\{ \sup_{z \in X} \{R(x, y)\}, A \times C(x, y)\}$$

$$\leq R(x, y).$$

### 3.2 FUZZY RELATION EQUATIONS

Throughout this section A, B and C denote fuzzy sets in the universal set X.

**Definition 3.2.1**: Let $P \in F(A \times B)$, $Q \in F(B \times C)$ and $R \in F(A \times C)$ be three fuzzy relations. Then the equation $P \text{ o } Q = R$ is called a fuzzy relation equation, where o denote composition of fuzzy relations.

If $\text{o} = \text{o}_T$($\text{o} = {}_S\text{o}$ , $\text{o} = {}^{\text{o}}w_T$ and $\text{o} = {}^{\text{o}}\omega_S$ respectively), then the fuzzy relation equation $P \text{ o } Q = R$ is called sup-T ( inf-S, inf-$w_T$ and sup-$\omega_S$ respectively) fuzzy relation equation.

### 3.2(a) SUP-T FUZZY RELATION EQUATIONS

In this section we will discuss solution of a given sup-T fuzzy relation equation $P \text{ o}_T Q = R$, when any two of the fuzzy relations are given. Followings are the cases:

I) Given P and Q, R can be obtained by using the definition of sup-T composition. In this case R is unique.

II) If P and R are given, then the set $S(P, R) = \{Q \in F(B \times C) \mid P \circ_T Q = R\}$ is called the solution set of $P \circ_T Q = R$ for Q.

**Theorem 3.2.2** [F]: If $S(P, R) \neq \phi$, then the fuzzy relation $P^{-1} \circ_{w_T} R \in F(B \times C)$ is the maximum solution of the fuzzy relation equation $P \circ_T Q = R$ for Q.

Proof: Let $Q \in S(P, R)$. Then $P \circ_T Q = R$.

Therefore, $Q \subseteq P^{-1} \circ_{w_T} (P \circ_T Q)$, By Theorem 3.1.13(i)

Hence, $Q \subseteq P^{-1} \circ_{w_T} R$

Thus, $P\circ_T Q \subseteq P\circ_T (P^{-1} \circ_{w_T} R)$

Therefore, $R \subseteq P \circ_T (P^{-1} \circ_{w_T} R)$

But then $R \subseteq P \circ_T (P^{-1} \circ_{w_T} R) \subseteq R$

Hence, $P \circ_T (P^{-1} \circ_{w_T} R) = R$

III) If Q and R are given, then the set $S(Q, R) = \{P \in F(A \times B) \mid P \circ_T Q = R\}$ is called the solution set of $P \circ_T Q = R$ for P. The solution of this type of equation can be obtained by using above type and by using inverse relation. For example we want to find a solution of $P \circ_T Q = R$ for P. Consider the equation $Q^{-1} \circ_T P^{-1} = R^{-1}$, the solution of this equation, $P^{-1}$, can be obtained by using above type (II). The fuzzy relation $P = (P^{-1})^{-1}$ will be required solution of $P \circ_T Q = R$ for P.

**Theorem 3.2.3** [F]: If $S(Q, R) \neq \phi$ , then the fuzzy relation $(Q \overset{o}{w}_T R^{-1})^{-1} \in F(A \times B)$ is the maximum solution of the fuzzy relation equation $P \; o_T \; Q = R$ for P.

Proof: Let $P \in F(A \times B)$ such that $P \; o_T \; Q = R$

Then by Theorem 3.1.13 (iii), $P \subseteq [Q \overset{o}{w}_T (P \; o_T \; Q)^{-1}]^{-1}$

Thus, $P \subseteq (Q \overset{o}{w}_T R^{-1})^{-1}$

Therefore, $P o_T Q \subseteq (Q \overset{o}{w}_T R^{-1})^{-1} o_T Q$

Thus, $R = P o_T Q \subseteq (Q \overset{o}{w}_T R^{-1})^{-1} o_T Q$

Hence, $R \subseteq (Q \overset{o}{w}_T R^{-1})^{-1} o_T Q \subseteq R$, by Theorem 3.1.13(iv)

i. e. $(Q \overset{o}{w}_T R^{-1})^{-1} o_T Q = R$

**3.2(b) INS-S FUZZY RELATION EQUATIONS**

In this section we will discuss the solutions of the inf-S fuzzy relation equations $P \; _So \; Q = R$.

I) Given P and Q, R can be obtained by using the definition of inf-S composition. In this case R is unique.

II) If P and R are given, then the set $S(P, R) = \{Q \in F(B \times C) \mid P \; _So \; Q = R\}$ is called the solution set of $P \; _So \; Q = R$ for Q.

**Theorem 3.2.4** [F]: Let $P \in F(A \times B)$, $R \in F(A \times C)$ such that

$$\sup_{z \in X} A(z) \leq \min (B(x), C(y)) \text{ for all } x, y \in X.$$

If $S(P, R) \neq \phi$, then the fuzzy relation $P^{-1}\ {}^{o}\omega_S\ R \in F(B \times C)$ is the minimum solution of the fuzzy relation equation $P\ _{S}o\ Q = R$ for Q.

Proof: Let $Q \in F(B \times C)$ be such that $P\ _{S}o\ Q = R$.

Then by Theorem 3.1.16(i), $Q \supseteq P^{-1}\ {}^{o}\omega_S\ (P\ _{S}o\ Q)$

Thus, $Q \supseteq P^{-1}\ {}^{o}\omega_S\ R$

Therefore, $P\ _{S}o\ Q \supseteq P\ _{S}o\ (P^{1}\ {}^{o}\omega_S R)$

Thus, $R = P\ _{S}oQ \supseteq Q\ _{S}o\ (Q^{1}\ {}^{o}\omega_S\ R)$

Therefore, $R \supseteq Q\ _{S}o\ (Q^{1}\ {}^{o}\omega_S\ R) \supseteq R$, by Theorem 3.1.16(iv)

Hence, $P\ _{S}o\ (P^{1}\ {}^{o}\omega_S\ R) = R$

III) If Q and R are given, then the set $S(Q, R) = \{P \in F(A \times B) \mid P\ _{S}o\ Q = R\}$ is called the solution set of $P\ _{S}o\ Q = R$ for P. The solution of this type of equation can be obtained by using above type and by using inverse relation. For example we want to find a solution of $P\ _{S}o\ Q = R$ for P. Consider the equation $Q^{-1}\ _{S}o\ P^{-1} = R^{-1}$, the solution of this equation, $P^{-1}$, can be obtained by using above type (II). The fuzzy relation $P = (P^{-1})^{-1}$ will be required solution of $P\ _{S}o\ Q = R$ for P.

**Theorem 3.2.5** [F]: Let $Q \in F(B \times C)$ and $R \in F(A \times C)$ be such that

$$\sup_{z \in X} C(z) \leq \min(B(x), A(y)), \text{ for all } x, y \in X.$$

If $S(Q, R) \neq \phi$, then the fuzzy relation $(Q\ {}^{o}\omega_S\ R^{-1})^{-1} \in F(A \times B)$ is the minimum solution of the fuzzy relation equation $P\ _{S}o\ Q = R$

Proof: Let $P \in S(Q, R)$ be such that, $P \; so \; Q = R$.

Then $(Q \,{}^{o}\omega_S\, (P \; so \; Q)^{-1})^{-1} \subseteq P$, by Theorem 3.1.16(ii)

Therefore, $(Q \,{}^{o}\omega_S\, R^{-1})^{-1} \subseteq P$

But then $(Q \,{}^{o}\omega_S\, R^{-1})^{-1} \; so \; Q \subseteq P \; so \; Q$

Thus, $(Q \,{}^{o}\omega_S\, R^{-1})^{-1} \; so \; Q \subseteq P \; so \; Q = R$

But by Theorem 3.1.20, $R \subseteq (Q \,{}^{o}\omega_S\, R^{-1})^{-1} \; so \; Q$

Hence, $(Q \,{}^{o}\omega_S\, R^{-1})^{-1} \; so \; Q = R$

### 3.2(c) INF-$W_T$ FUZZY RELATION EQUATIONS

In this section we will discuss solution of a given inf-$w_T$ fuzzy relation equation $P \,{}^{o}w_T\, Q = R$, when any two of the fuzzy relations are given. Followings are the cases:

I) Given P and Q, R can be obtained by using the definition of inf-$w_T$ composition. In this case R is unique.

II) If P and R are given, then the set $S(P, R) = \{Q \in F(B \times C) \mid P \,{}^{o}w_T\, Q = R\}$ is called the solution set of $P \,{}^{o}w_T\, Q = R$ for Q.

Following theorem gives the minimum element of S(P, R).

**Theorem 3. 2. 6** [F]: Let $P \in F(A \times B)$, $R \in F(A \times C)$ be two fuzzy relations. If $S(P, R) \neq \phi$, then the fuzzy relation $P^{-1} o_T R \in F(B \times C)$ is the minimum solution of the fuzzy relation equation $P \overset{o}{w}_T Q = R$ for Q.

Proof: Let $Q \in S(P, R)$ be a solution of $P \overset{o}{w}_T Q = R$.

Then by Theorem 3.1.16(iii), $P^{-1} o_T (P \overset{o}{w}_T Q) \subseteq Q$

Thus, $P^{-1} o_T R \subseteq Q$.

Therefore, $P \overset{o}{w}_T (P^{-1} o_T R) \subseteq P \overset{o}{w}_T Q$

i. e. $P \overset{o}{w}_T (P^{-1} o_T R) \subseteq P \overset{o}{w}_T Q = R$

But by Theorem 3.1.16(iv), $R \subseteq P \overset{o}{w}_T (P^{-1} o_T R)$

Hence, $P \overset{o}{w}_T (P^{-1} o_T R) = R$

III) If Q and R are given, then the set $S(Q, R) = \{P \in F(A \times B) \mid P \overset{o}{w}_T Q = R\}$ is called the solution set of $P \overset{o}{w}_T Q = R$ for P.

Following theorem gives the maximum element of S(Q, R).

**Theorem 3.2.7** [F]: Let $Q \in F(B \times C)$, $R \in F(A \times C)$ be two fuzzy relations. If $S(Q, R) \neq \phi$, then the fuzzy relation $R \overset{o}{w}_T Q^{-1} \in F(A \times B)$ is the maximum solution of the fuzzy relation equation $P \overset{o}{w}_T Q = R$ for P.

Proof: Let $P \in S(Q, R)$ be such that $P \overset{o}{w}_T Q = R$.

Then $P \subseteq (P \overset{o}{w}_T Q) \overset{o}{w}_T Q^{-1}$

Thus, $P \subseteq R \ {}^{o}w_T \ Q^{-1}$

Therefore, $P \ {}^{o}w_T \ Q \supseteq (R \ {}^{o}w_T \ Q^{-1}) \ {}^{o}w_T \ Q$

i. e. $R \supseteq (R \ {}^{o}w_T \ Q^{-1}) \ {}^{o}w_T \ Q$

But by Theorem 3.1.12(ii), $((R \ {}^{o}w_T \ Q^{-1}) \ {}^{o}w_T \ Q) \supseteq R$

Hence, $(R \ {}^{o}w_T \ Q^{-1}) \ {}^{o}w_T \ Q = R$.

## 3.2(d) SUP -$\omega_S$ FUZZY RELATION EQUATIONS

In this section we will discuss solution of a given sup-$\omega_S$ fuzzy relation equation $P \ {}^{o}\omega_S \ Q = R$, when any two of the fuzzy relations are given. Followings are the cases:

I) Given P and Q, R can be obtained by using the definition of sup-$\omega_S$ composition. In this case R is unique.

II) If P and R are given, then the set $S(P, R) = \{Q \in F(B \times C) \mid P \ {}^{o}\omega_S \ Q = R\}$ is called the solution set of $P \ {}^{o}\omega_S \ Q = R$ for Q.

Following theorem gives the maximum element of S(P, R).

**Theorem 3.2.8** [F]: Let $P \in F(A \times B)$, $R \in F(A \times C)$ be such that

$$\sup_{z \in X} B(z) \leq \min(A(x), C(y)), \text{ for all } x, y \in X$$

If $S(P, R) \neq \phi$, then the fuzzy relation $P^{-1}$ so $R \in F(B \times C)$ is the maximum solution of the fuzzy relation equation $P \ {}^{o}\omega_S \ Q = R$ for Q.

Proof - Let $Q \in F(B \times C)$ such that $P\, {}^{o}\omega_S\, Q = R$. Then $Q \subseteq P^{-1}$ so $(P\, {}^{o}\omega_S\, Q)$

Thus, $Q \subseteq P^{-1}$ so $R$

Therefore, $P\, {}^{o}\omega_S\, Q \subseteq P\, {}^{o}\omega_S\, (P^{-1}$ so $R)$

i. e. $R \subseteq P\, {}^{o}\omega_S\, (P^{-1}$ so $R)$

But by Theorem 3.1.16(iv), $P\, {}^{o}\omega_S\, (P^{-1}$ so $R) \subseteq R$

Hence, $P\, {}^{o}\omega_S\, (P^{-1}$ so $R) = R$

III) If Q and R are given, then the set $S(Q, R) = \{P \in F(A \times B) \mid P\, {}^{o}\omega_S\, Q = R\}$ is called the solution set of $P\, {}^{o}\omega_S\, Q = R$ for P.

Following theorem gives the minimum element of S(Q, R).

**Theorem 3.2.9** [F]: Let $Q \in F(B \times C)$, $R \in F(A \times C)$ be such that the following conditions hold:

1) $\sup_{z \in X} B(z) \leq \min(A(x), C(y))$

2) $\sup_{z \in X} C(z) \leq \min(A(x), B(y))$, for all $x, y \in X$

If $S(Q, R) \neq \phi$, then the fuzzy relation $R\, {}^{o}\omega_S\, Q^{-1} \in F(A \times B)$ is the minimum solution of the fuzzy relation equation $P\, {}^{o}\omega_S\, Q = R$ for P.

Proof: Let $P \in S(Q, R)$ be such that $P\, {}^{o}\omega_S\, R = R$. Then $P \supseteq (P\, {}^{o}\omega_S\, Q)\, {}^{o}\omega_S\, Q^{-1}$.

Thus, $P \supseteq R\, {}^{o}\omega_S\, Q$

Therefore, $P\, {}^{o}\omega_S\, Q \subseteq (R\, {}^{o}\omega_S\, Q^{-1})\, {}^{o}\omega_S\, Q$

i. e. $R \subseteq (R \circ\omega_S Q^{-1}) \circ\omega_S Q$

But by Theorem 3.2.22(ii), $(R \circ\omega_S Q^{-1}) \circ\omega_S Q \subseteq R$

Hence, $(R \circ\omega_S Q^{-1}) \circ\omega_S Q = R$