CHAPTER 5

CHAPTER 5 EQUATIONS OF FUZZY NUMBERS

In this chapter we study real and complex fuzzy numbers and discuss fuzzy equations of these numbers. Necessary and sufficient conditions for the existence of the solutions are given.

5.1 FUZZY NUMBERS

In this Section we define fuzzy numbers and discuss the existence of solutions of linear equations based on fuzzy numbers.

Definition 5.1.1 [K_1, K_2]: Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be any two closed bounded intervals. Then

1) $A + B = [a_1 + b_1, a_2 + b_2]$

2) $A - B = [a_1 - b_2, a_2 - b_1]$

3) $AB = [a, b]$, where $a = \min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$ and $b = \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$

4) $A/B = [a, b]$, where $a = \min\left\{\frac{a_1}{b_2}, \frac{a_1}{b_1}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right\}$ and $b = \max\left\{\frac{a_1}{b_2}, \frac{a_1}{b_1}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right\}$, if $0 \notin B$

5) $-A = [-a_2, -a_1]$

6) $1/A = [1/a_2, 1/a_1]$, if $0 \notin A$.

Definition 5.1.2 [K_2]: A fuzzy set A: 3 $\rightarrow$ I is a fuzzy number, if

1) A is a normal fuzzy set

2) ${}^{\alpha}A$ is a closed interval, $\forall \alpha \in (0, 1]$

3) Support of A is bounded set.

Remark 5.1.3: 1) It is obvious that fuzzy numbers may or may not have continuous membership functions.

2) Some authors [K_1, K_2] call fuzzy numbers with continuous membership functions as continuous fuzzy numbers.

3) Throughout this chapter we consider fuzzy numbers with continuous membership functions.

Theorem 5.1.4 [K_2]: Let A, B be two fuzzy numbers and $* \in \{+, -, \cdot, /\}$. Define $A * B : 3 \rightarrow [0, 1]$ as follows:

$$A * B(z) = \sup_{z = x * y} \{\min \{A(x), B(y)\}\}, \text{ for all } z \in 3$$

Then ${}^{\alpha}(A * B) = {}^{\alpha}A * {}^{\alpha}B$, $\alpha \in I$

Proof: Proof of this theorem can be found in [K_2].

Theorem 5.1.5 [K_2]: Let A, B be two fuzzy numbers and $* \in \{+, -, \cdot, /\}$. Then $A * B$ is a fuzzy number.

Proof: Proof of this theorem can be found in [K_2].

Remark 5.1.6: Real numbers and real intervals are special cases of fuzzy numbers.

5.2 SOLVING A + X = B

Now we discuss solution of the equation $A + X = B$, when fuzzy numbers A and B are known. Following example shows that $X = B - A$ need not be a solution.

Example 5.2.1 [K_2]: Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two fuzzy numbers. Then $X = B - A$ need not be a solution of $A + X = B$.

Solution: Since, $A + (B - A) = [a_1 + b_1 - a_2, a_2 + b_2 - a_1] \neq B$.

Theorem 5.2.2 [K_2]: Let A and B be fuzzy numbers. Let α - cut of A be ${}^{\alpha}A = [a_1^{\alpha}, a_2^{\alpha}]$ and that of B be ${}^{\alpha}B = [b_1^{\alpha}, b_2^{\alpha}]$. Then $A + X = B$ has a solution if and only if

i) $b_1^{\alpha} - a_1^{\alpha} \leq b_2^{\alpha} - a_2^{\alpha}$, for every $\alpha \in (0, 1]$

ii) $\alpha \leq \beta \Rightarrow b_1^{\alpha} - a_1^{\alpha} \leq b_1^{\beta} - a_1^{\beta} \leq b_2^{\beta} - a_2^{\beta} \leq b_2^{\alpha} - a_2^{\alpha}$.

Proof: Suppose that $A + X = B$ has a solution X.

Let ${}^{\alpha}X = [x_1^{\alpha}, x_2^{\alpha}]$ be α - cut of X.

Since, $A + X = B$, ${}^{\alpha}(A + X) = {}^{\alpha}B$

Thus, ${}^{\alpha}A + {}^{\alpha}X = {}^{\alpha}B$

Therefore, $[a_1^{\alpha}, a_2^{\alpha}] + [x_1^{\alpha}, x_2^{\alpha}] = [b_1^{\alpha}, b_2^{\alpha}]$

i. e. $[a_1^{\alpha} + x_1^{\alpha}, a_2^{\alpha} + x_2^{\alpha}] = [b_1^{\alpha}, b_2^{\alpha}]$

Therefore, $a_1^{\alpha} + x_1^{\alpha} = b_1^{\alpha}$ and $a_2^{\alpha} + x_2^{\alpha} = b_2^{\alpha}$

Thus, $x_1^{\alpha} = b_1^{\alpha} - a_1^{\alpha}$ and $x_2^{\alpha} = b_2^{\alpha} - a_2^{\alpha}$

But $x_1^{\alpha} \leq x_2^{\alpha}$

Hence, $b_1^{\alpha} - a_1^{\alpha} \leq b_2^{\alpha} - a_2^{\alpha}$.

Let $\alpha, \beta \in [0, 1]$ be such that $\alpha \leq \beta$.

Then ${}^{\beta}X \subseteq {}^{\alpha}X$,

Thus, $[x_1^{\beta}, x_2^{\beta}] \subseteq [x_1^{\alpha}, x_2^{\alpha}]$

Therefore, $x_1^{\alpha} \leq x_1^{\beta} \leq x_2^{\beta} \leq x_2^{\alpha}$

Hence, $b_1^{\alpha} - a_1^{\alpha} \leq b_1^{\beta} - a_1^{\beta} \leq b_2^{\beta} - a_2^{\beta} \leq b_2^{\alpha} - a_2^{\alpha}$

Conversely, suppose that conditions hold.

Take $x_1^\alpha = b_1^\alpha - a_1^\alpha$ and $x_2^\alpha = b_2^\alpha - a_2^\alpha$

Then $[x_1^\alpha, x_2^\alpha]$ is an interval

Choose ${}^\alpha X = [x_1^\alpha, x_2^\alpha]$

Since ${}^\beta X \subseteq {}^\alpha X$, for all $\alpha \leq \beta$, $\{{}^\alpha X\}$ is a nested sequence of intervals.

Now $a_1^\alpha + x_1^\alpha = b_1^\alpha$ and $a_2^\alpha + x_2^\alpha = b_2^\alpha$

Therefore, ${}^\alpha(A + X) = {}^\alpha B$, for all $\alpha \in (0, 1]$

Thus, $A + X = B$

Hence, X is a solution of $A + X = B$.

5.3 SOLVING A · X = B

Now we discuss solution of the equation $A \cdot X = B$, when fuzzy numbers A and B are known.

Theorem 5.3.1 [K₂]: Let A and B be fuzzy numbers in $\Im^+$ and ${}^\alpha A = [a_1^\alpha, a_2^\alpha]$, ${}^\alpha B = [b_1^\alpha, b_2^\alpha]$ be α - cuts of A and B respectively. Then $AX = B$ has a solution if and only if

i) $b_1^\alpha/a_1^\alpha \leq b_2^\alpha/a_2^\alpha$, $\forall\ \alpha \in (0, 1]$

ii) $\alpha \leq \beta \Rightarrow b_1^\alpha/a_1^\alpha \leq b_1^\beta/a_1^\beta \leq b_2^\beta/a_2^\beta \leq b_2^\alpha/a_2^\alpha$

Proof: Suppose that X is a solution of $AX = B$

Let ${}^\alpha X = [x_1^\alpha, x_2^\alpha]$ be α -cut of X.

Since $AX = B$, ${}^\alpha(AX) = {}^\alpha B$

i. e. ${}^\alpha A\, {}^\alpha X = {}^\alpha B$

Thus, $[a_1^\alpha, a_2^\alpha]\, [x_1^\alpha, x_2^\alpha] = [b_1^\alpha, b_2^\alpha]$

Since $a_1^\alpha > 0$ and $b_1^\alpha > 0$, $[a_1^\alpha x_1^\alpha, a_2^\alpha x_2^\alpha] = [b_1^\alpha, b_2^\alpha]$

Thus, $a_1^\alpha x_1^\alpha = b_1^\alpha$ and $a_2^\alpha x_2^\alpha = b_2^\alpha$

i. e. $x_1^\alpha = b_1^\alpha / a_1^\alpha$ and $x_2^\alpha = b_2^\alpha / a_2^\alpha$

But $x_1^\alpha \leq x_2^\alpha$

Hence, $b_1^\alpha/a_1^\alpha \leq b_2^\alpha/a_2^\alpha$

Let $\alpha \leq \beta$. Then ${}^\beta X \subseteq {}^\alpha X$.

Therefore, $[x_1^\beta, x_2^\beta] \subseteq [x_1^\alpha, x_2^\alpha]$

i. e. $x_1^\alpha \leq x_1^\beta \leq x_2^\beta \leq x_2^\alpha$

Therefore, $b_1^\alpha/a_1^\alpha \leq b_1^\beta/a_1^\beta \leq b_2^\beta/a_2^\beta \leq b_2^\alpha/a_2^\alpha$

Conversely, suppose that the conditions hold

Take $x_1^\alpha = b_1^\alpha/a_1^\alpha$ and $x_2^\alpha = b_2^\alpha/a_2^\alpha$

Thus, ${}^\alpha X = [x_1^\alpha, x_2^\alpha]$ is an interval

Since ${}^\beta X \subseteq {}^\alpha X$, for all $\alpha \leq \beta$, $\{{}^\alpha X\}$ is a nested sequence of intervals

Now $a_1^\alpha x_1^\alpha = b_1^\alpha$ and $a_2^\alpha x_2^\alpha = b_2^\alpha$

Therefore, ${}^\alpha A {}^\alpha X = {}^\alpha B$

i. e. ${}^\alpha(AX) = {}^\alpha B$

Hence, AX = B.

5.4 TRIANGULAR FUZZY NUMBERS

Definition 5.4.1 [B1]: A fuzzy number N: 3 $\rightarrow$ I, is called triangular if, there exist real numbers $n_1 < n_2 < n_3$ such that

1) $N(x) = 0$, if $x \notin (n_1, n_3)$

2) $N(x) = 1$, if $x = n_2$

3) N(x) is continuous and monotonically increasing from 0 to 1 on $[n_1, n_2]$

4) N(x) is continuous and monotonically decreasing from 1 to 0 on $[n_2, n_3]$

This triangular fuzzy number is denoted as $N = (n_1| n_2| n_3)$.

Definition 5.4.2 [B_1]: Let $N = (n_1| n_2| n_3)$ be a triangular fuzzy number. Then

1) $N \geq 0$, if $n_1 \geq 0$

2) $N > 0$, if $n_1 > 0$

3) $N < 0$, if $n_3 < 0$

4) $N \leq 0$, if $n_3 \leq 0$

Proposition 5.4.3 [B_1]: Let $N = (n_1| n_2| n_3)$ be a triangular fuzzy number. Then for $\alpha \in I$ an α-cut of N is given by ${}^{\alpha}N = [n_1^{\alpha}, n_2^{\alpha}]$ and $\dot{n}_1^{\alpha} = (n_2 - n_1)$, $\dot{n}_2^{\alpha} = (n_2 - n_3)$, where $\dot{n}_1^{\alpha} = \frac{d}{d\alpha}(n_i^{\alpha})$.

Proof: Consider the points A, B, C, D, E as $A \equiv (n_1, 0)$, $B \equiv (n_2, 1)$, $C \equiv (n_3, 0)$, $D \equiv (n_1^{\alpha}, \alpha)$, $E \equiv (n_2^{\alpha}, \alpha)$

Equation of line AB is: $x - (n_2 - n_1)y - n_1 = 0$

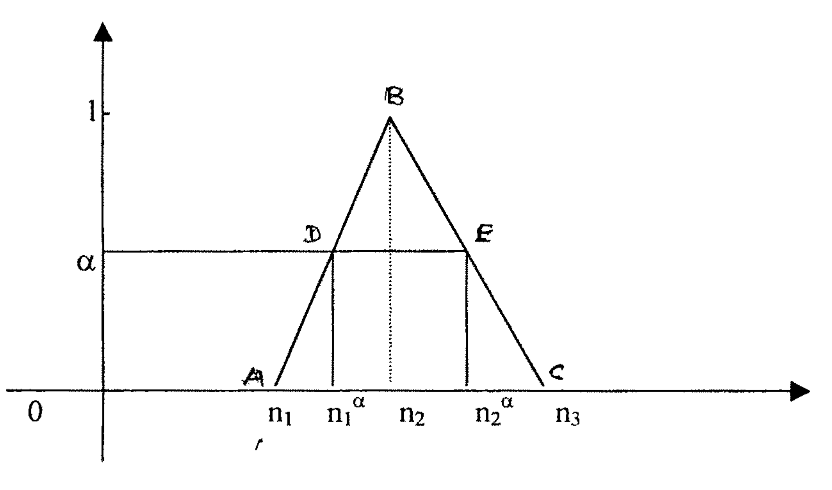
Since $D(n_1^{\alpha}, \alpha)$ lies on line AB, $n_1^{\alpha} - (n_2 - n_1)\alpha - n_1 = 0$

Therefore, $n_1^{\alpha} = n_1 + (n_2 - n_1)\alpha$

Similarly $E(n_2^{\alpha}, \alpha)$ lies on BC gives, $n_2^{\alpha} = n_3 + (n_2 - n_3)\alpha$

Thus, α - cut of N is $[n_1 + (n_2 - n_1)\alpha, n_3 + (n_2 - n_3)\alpha]$

Differentiating n_1^{α} and n_2^{α} with respect to α, we get $\dot{n}_1^{\alpha} = (n_2 - n_1)$, $\dot{n}_2^{\alpha} = (n_2 - n_3)$.



Theorem 5.4.4 [K_1]: If M and N are triangular fuzzy numbers, then so is M + N.

Proof: Let $M = (m_1| m_2| m_3)$ and $N = (n_1| n_2| n_3)$.

Let ${}^{\alpha}M = [m_1{}^{\alpha}, m_2{}^{\alpha}]$ and ${}^{\alpha}N = [n_1{}^{\alpha}, n_2{}^{\alpha}]$ be α-cuts of M and N respectively.

Since, ${}^{\alpha}(M + N) = {}^{\alpha}M + {}^{\alpha}N$,

${}^{\alpha}(M + N) = [m_1 + \alpha(m_2 - m_1), m_3 + \alpha(m_2 - m_3)] + [n_1 + \alpha(n_2 - n_1), n_3 + \alpha(n_2 - n_3)]$

$= [(m_1 + n_1) + \alpha(m_2 + n_2 - m_1 - n_1), (m_3 + m_3) + \alpha(m_2 + n_2 - m_3 - n_3)]$

For $\alpha = 0$, ${}^{0}(M + N) = [m_1 + n_1, m_3 + n_3]$.

Let $x \in {}^{\alpha}(M + N)$. Then $(m_1 + n_1) + \alpha(m_2 + n_2 - m_1 - n_1) \le x$ and

$x \le (m_3 + n_3) + \alpha(m_2 + n_2 - m_3 - n_3)$

$\alpha \le [x - (m_1 + n_1)]/[m_2 + n_2 - m_1 - n_1]$ and $\alpha \le (m_3 + n_3) - x]/[m_3 + n_3 - m_2 - n_2]$

But $0 \le \alpha \le 1$

Therefore, $0 \le [x - (m_1 + n_1)]/[m_2 + n_2 - m_1 - n_1] \le 1$ and

$0 \le [(m_3 + n_3) - x]/[m_3 + n_3 - m_2 - n_2] \le 1$

Therefore, $m_1 + n_1 \le x \le m_2 + n_2$ and $m_2 + n_2 \le x \le m_3 + n_3$

Thus, $\alpha \le [x - (m_1 + n_1)]/[m_2 + n_2 - m_1 - n_1]$, for $m_1 + n_1 \le x \le m_2 + n_2$ and

$\alpha \le [(m_3 + n_3) - x]/[m_3 + n_3 - m_2 - n_2]$, for $m_2 + n_2 \le x \le m_3 + n_3$.

Also $M + N(x) = 0$, for all $x \notin (m_1 + n_1, m_3 + n_3)$

Hence, M + N is

$(M + N)(x) = 0,$ if $x \notin (m_1 + n_1, m_3 + n_3)$

$= [x - (m_1 + n_1)]/[(m_2 + n_2) - (m_1 + n_1)],$ if $m_1 + n_1 \le x \le m_2 + n_2$

$= 1,$ if $x = m_2 + n_2$

$= [m_3 + n_3 - x]/[(m_3 + n_3) - (m_2 + n_2)],$ if $m_2 + n_2 \le x \le m_3 + n_3$.

Now we discuss the solution of linear and quadratic fuzzy equations based on triangular fuzzy numbers.

Theorem 5.4.5 [B_1]: Let $A = (a_1 \mid a_2 \mid a_3)$ and $B = (b_1 \mid b_2 \mid b_3)$ be triangular fuzzy numbers. If $A + X = C$, then X is triangular fuzzy number.

Proof: Let ${}^{\alpha}A = [a_1^{\alpha}, a_2^{\alpha}]$, ${}^{\alpha}C = [c_1^{\alpha}, c_2^{\alpha}]$ and ${}^{\alpha}X = [x_1^{\alpha}, x_2^{\alpha}]$ be α - cuts of A, C and X respectively.

Now ${}^{\alpha}(A + X) = {}^{\alpha}C$

i. e. ${}^{\alpha}A + {}^{\alpha}X = {}^{\alpha}C$

Therefore, $[a_1^{\alpha}, a_2^{\alpha}] + [x_1^{\alpha}, x_2^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$

i. e. $[a_1^{\alpha} + x_1^{\alpha}, a_2^{\alpha} + x_2^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$

Thus, $a_1^{\alpha} + x_1^{\alpha} = c_1^{\alpha}$ and $a_2^{\alpha} + x_2^{\alpha} = c_2^{\alpha}$.

Hence, $x_1^{\alpha} = c_1^{\alpha} - a_1^{\alpha}$ and $x_2^{\alpha} = c_2^{\alpha} - a_2^{\alpha}$.

Therefore, $x_1^{\alpha} = [c_1 + \alpha (c_2 - c_1)] - [a_1 + \alpha (a_2 - a_1)]$

and $x_2^{\alpha} = [c_3 + \alpha (c_2 - c_3)] - [a_3 + \alpha (a_2 - a_3)]$

i. e. $x_1^{\alpha} = (c_1 - a_1) + \alpha [(c_2 - a_2) - (c_1 - a_1)]$ and $x_2^{\alpha} = (c_3 - a_3) + \alpha [(c_2 - a_2) - (c_3 - a_3)]$

But ${}^{\alpha}X = [x_1^{\alpha}, x_2^{\alpha}]$

For $\alpha = 0$, ${}^{0}X = [c_1 - a_1, c_3 - a_3]$

Therefore, $X(x) = 0$, for all $x \notin (c_1 - a_1, c_3 - a_3)$

Let $x \in {}^{\alpha}X$. Then $x \in [x_1^{\alpha}, x_2^{\alpha}]$

Therefore, $(c_1 - a_1) + \alpha [(c_2 - a_2) - (c_1 - a_1)] \leq x$

and $x \leq (c_3 - a_3) + \alpha [(c_2 - a_2) - (c_3 - a_3)]$

$\alpha \leq [x - (c_1 - a_1)/[(c_2 - a_2) - (c_1 - a_1)]$

$\alpha \leq [(c_3 - a_3) - x]/[(c_3 - a_3) - (c_2 - a_2)]$

But $0 \leq \alpha \leq 1$. Therefore

$0 \leq [x - (c_1 - a_1)]/[(c_2 - a_2) - (c_1 - a_1)] \leq 1$ and

$0 \leq [(c_3 - a_3) - x]/[(c_3 - a_3) - (c_2 - a_2)] \leq 1$

Therefore, $c_1 - a_1 \leq x \leq c_2 - a_2$ and $c_2 - a_2 \leq x \leq c_3 - a_3$

Thus, $\alpha \le [x - (c_1 - a_1)]/ [(c_2 - a_2) - (c_1 - a_1)]$, for $c_1 - a_1 \le x \le c_2 - a_2$

and $\alpha \le [(c_3 - a_3) - x]/ [(c_3 - a_3) - (c_2 - a_2)]$, for $c_2 - a_2 \le x \le c_3 - a_3$

Hence, by Theorem 1.1.16,

$X(x) = 0$; if $x \notin (c_1 - a_1,\ c_3 - a_3)$

$= \sup\{\alpha \mid \alpha \le [x - (c_1 - a_1)]/ [(c_2 - a_2) - (c_1 - a_1)],\ c_1 - a_1 \le x \le c_2 - a_2.$

$= \sup \{ \alpha / \alpha \le 1 \}, x = c_2 - a_2$

$= \sup\{\alpha \mid \alpha \le [(c_3 - a_3) - x]/ [(c_3 - a_3) - (c_2 - a_2)]$, for $c_2 - a_2 \le x \le c_3 - a_3$

$X(x) = 0,$	if $x \notin (c_1 - a_1,\ c_3 - a_3)$
$= [x - (c_1 - a_1)]/ [(c_2 - a_2) - (c_1 - a_1)],$	if $c_1 - a_1 \le x \le c_2 - a_2$
$= 1,$	if $x = c_2 - a_2$
$= [(c_3 - a_3) - x]/ [(c_3 - a_3) - (c_2 - a_2)],$	if $c_2 - a_2 \le x \le c_3 - a_3$

Hence, X is a triangular fuzzy number.

Theorem 5.4.6 [B_1]: Let $A = (a_1 \mid a_2 \mid a_3)$ and $B = (b_1 \mid b_2 \mid b_3)$ be triangular fuzzy numbers. An equation $A + X = C$ has a solution X if and only if $c_1 - a_1 < c_2 - a_2 < c_3 - a_3$.

Proof: Since $A + X = C$, ${}^{\alpha}(A + X) = {}^{\alpha}C$

i.e. ${}^{\alpha}A + {}^{\alpha}X = {}^{\alpha}C$, by Theorem 5.1.4

Thus, $[a_1^{\alpha}, a_2^{\alpha}] + [x_1^{\alpha}, x_2^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$

Hence, $[a_1^{\alpha} + x_1^{\alpha}, a_2^{\alpha} + x_2^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$

Thus, $a_1^{\alpha} + x_1^{\alpha} = c_1^{\alpha}$ and $a_2^{\alpha} + x_2^{\alpha} = c_2^{\alpha}$

Hence, $x_1^{\alpha} = c_1^{\alpha} - a_1^{\alpha}$ and $x_2^{\alpha} = c_2^{\alpha} - a_2^{\alpha}$.

Since X is a triangular fuzzy number, $x_1 < x_2 < x_3$ and $\dot{x}_1^{\alpha} > 0$, $\dot{x}_2^{\alpha} < 0$

iff $\frac{d}{d\alpha}(c_1^{\alpha} - a_1^{\alpha}) > 0$ and $\frac{d}{d\alpha}(c_2^{\alpha} - a_2^{\alpha}) < 0$

iff $\dot{c}_1^{\alpha} - \dot{a}_1^{\alpha} > 0$ and $\dot{c}_2^{\alpha} - \dot{a}_2^{\alpha} < 0$

iff $c_2 - c_1 > a_2 - a_1$, and $c_2 - c_3 < a_2 - a_3$

iff $c_1 - a_1 < c_2 - a_2$, and $c_2 - a_2 < c_3 - a_3$

iff $c_1 - a_1 < c_2 - a_2 < c_3 - a_3$

Hence, the equation $A + X = C$ has a solution iff $c_1 - a_1 < c_2 - a_2 < c_3 - a_3$.

We now discuss solution of the fuzzy equation $AX = C$, when A, C are triangular fuzzy numbers and $0 \notin supp(A)$. i. e. either $A > 0$ or $A < 0$.

Theorem 5.4.7 [B_1]: (a) Suppose zero does not belong to the support of C. Then there exists a solution X to the equation $AX = C$ if and only if

i) $a_1c_2 > c_1a_2$ and $a_3c_2 > c_3a_2$, when $A > 0,\ C \geq 0$.

ii) $a_1c_2 < c_1a_2$ and $a_3c_2 > c_3a_2$, when $A < 0,\ C \leq 0$.

iii) $a_3c_2 > c_1a_2$ and $a_1c_2 < c_3a_2$, when $A > 0,\ C \leq 0$.

iv) $a_3c_2 < c_1a_2$ and $a_1c_2 > c_3a_2$, when $A < 0,\ C \geq 0$.

(b) Suppose zero belongs to the support of C ($c_2 = 0$). Then there is a solution X if and only if zero belongs to the support of X ($x_2 = 0$).

Proof: (a) Since $AX = C$, ${}^{\alpha}(AX) = {}^{\alpha}C$.

i. e. ${}^{\alpha}A\,{}^{\alpha}X = {}^{\alpha}C$

Thus, $[a_1^{\alpha}, a_2^{\alpha}]\,[x_1^{\alpha}, x_2^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$, where ${}^{\alpha}A = [a_1^{\alpha}, a_2^{\alpha}]$, ${}^{\alpha}X = [x_1^{\alpha}, x_2^{\alpha}]$, ${}^{\alpha}C = [c_1^{\alpha}, c_2^{\alpha}]$ are α-cuts of A,X and C respectively.

i. e. $[\min S, \max S] = [c_1^{\alpha}, c_2^{\alpha}]$, where $S = \{a_1^{\alpha} x_1^{\alpha}, a_1^{\alpha} x_2^{\alpha}, a_2^{\alpha} x_1^{\alpha}, a_2^{\alpha} x_2^{\alpha}\}$

(i) If $A > 0$, $C \geq 0$, then $X \geq 0$

Now $[a_1^{\alpha} x_1^{\alpha}, a_2^{\alpha} x_2^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$

Therefore, $a_1^{\alpha} x_1^{\alpha} = c_1^{\alpha}$, $a_2^{\alpha} x_2^{\alpha} = c_2^{\alpha}$

Hence, $x_1^{\alpha} = c_1^{\alpha} / a_1^{\alpha}$, $x_2^{\alpha} = c_2^{\alpha} / a_2^{\alpha}$.

Then $\dot{x}_1^{\alpha} > 0$ iff $a_1^{\alpha} \dot{c}_1^{\alpha} > c_1^{\alpha} \dot{a}_1^{\alpha}$

iff $a_1^\alpha\ (c_2 - c_1) > c_1^\alpha\ (a_2 - a_1)$

iff $[a_1 + \alpha\,(a_2-a_1)]\,(c_2-c_1) > [c_1 + \alpha\,(c_2-c_1)](a_2-a_1)$

iff $a_1\,(c_2 - c_1) > c_1\,(a_2 - a_1)$

iff $a_1 c_2 > c_1 a_2$

Similarly $\dot{x}_2^\alpha < 0$ iff $a_3 c_2 > c_3 a_2$.

(ii) Suppose $A < 0$, $C \le 0$ $(X \ge 0)$

Then $[a_1^\alpha\ x_2^\alpha,\ a_2^\alpha\ x_1^\alpha] = [c_1^\alpha, c_2^\alpha]$

Thus, $x_1^\alpha = c_2^\alpha / a_2^\alpha$, $x_2^\alpha = c_1^\alpha / a_1^\alpha$

But $\dot{x}_1^\alpha > 0$ iff $a_2^\alpha\ \dot{c}_2^\alpha > c_2^\alpha\ \dot{a}_2^\alpha$.

iff $a_2^\alpha\ (c_2 - c_3) > c_2^\alpha\ (a_2 - a_3)$

iff $[a_3 + \alpha\,(a_2-a_3)]\,(c_2-c_3) > [c_3 + \alpha\,(c_2-c_3)](a_2-a_3)$

iff $a_3\,(c_2 - c_3) > c_3\,(a_2 - a_3)$

iff $a_3 c_2 > c_3 a_2$.

Similarly $\dot{x}_2^\alpha < 0$ iff $a_1 c_2 < c_1 a_2$.

(iii) Suppose $A > 0$, $C \le 0$ $(X \le 0)$

Then $[a_2^\alpha\ x_1^\alpha,\ a_1^\alpha\ x_2^\alpha] = [c_1^\alpha, c_2^\alpha]$

Thus, $x_1^\alpha = c_1^\alpha / a_2^\alpha$, $x_2^\alpha = c_2^\alpha / a_1^\alpha$

$\dot{x}_1^\alpha > 0$ iff $a_2^\alpha\ \dot{c}_1^\alpha > c_1^\alpha\ \dot{a}_2^\alpha$.

iff $a_2^\alpha\ (c_2 - c_1) > c_1^\alpha\ (a_2 - a_3)$

iff $[a_3 + \alpha\,(a_2-a_3)]\,(c_2-c_1) > [c_1 + \alpha\,(c_2-c_1)](a_2-a_3)$

iff $a_3\,(c_2 - c_1) > c_1\,(a_2 - a_3)$

iff $a_3 c_2 > c_1 a_2$.

Similarly $\dot{x}_2^\alpha < 0$ iff $a_1 c_2 < c_3 a_2$.

(iv) Let $A < 0$, $C \ge 0$ $(X \le 0)$

Then $[a_2^\alpha\ x_2^\alpha , a_1^\alpha\ x_1^\alpha] = [c_1^\alpha, c_2^\alpha]$

Thus, $x_1^\alpha = c_2^\alpha / a_1^\alpha$, $x_2^\alpha = c_1^\alpha / a_2^\alpha$

$\dot{x}_1^\alpha > 0$ iff $a_1^\alpha\ \dot{c}_2^\alpha > c_2^\alpha\ \dot{a}_1^\alpha$.

iff $a_1^\alpha\ (c_2 - c_3) > c_2^\alpha\ (a_2 - a_1)$

iff $[a_1 + \alpha\ (a_2-a_1)]\ (c_2-c_3) > [c_3 + \alpha\ (c_2-c_3)](a_2-a_1)$

iff $a_1\ (c_2 - c_3) > c_3\ (a_2 - a_1)$

iff $a_1 c_2 > c_3 a_2$.

Similarly $\dot{x}_2^\alpha < 0$ iff $a_3 c_2 < c_1 a_2$.

(b) Let $0 \in$ supp (C). Then $C(0) > 0$.

So take $c_2 = 0$

Since for $X \leq 0$ or $X \geq 0$ and $A > 0$ or $A < 0$, $AX \geq 0$ or $AX \leq 0$.

But $c_2 = 0$

Therefore let us assume that $x_2 = 0$

Now ${}^\alpha A {}^\alpha X = {}^\alpha C$

i. e. $[a_1^\alpha, a_2^\alpha]\ [x_1^\alpha, x_2^\alpha] = [c_1^\alpha, c_2^\alpha]$

As $c_2 = 0$, we have only two cases given below:

i) Suppose $A > 0$

$[a_2^\alpha\ x_1^\alpha,\ a_2^\alpha\ x_2^\alpha] = [c_1^\alpha, c_2^\alpha]$

Therefore, $x_1^\alpha = c_1^\alpha / a_2^\alpha$, $x_2^\alpha = c_2^\alpha / a_2^\alpha$

Then $\dot{x}_1^\alpha = \frac{1}{(a_2^\alpha)^2}\ [a_2^\alpha\ \dot{c}_1^\alpha - c_1^\alpha\ \dot{a}_2^\alpha]$

Now $a_2^\alpha\ \dot{c}_1^\alpha - c_1^\alpha\ \dot{a}_2^\alpha = a_2^\alpha\ (c_2 - c_1) - c_1^\alpha\ (a_2 - a_3)$

$= [a_3+\alpha\ (a_2 - a_3)]\ (c_2 - c_1) - [c_1+\alpha(c_2-c_1)]\ (a_2 - a_3)$

$= a_3\ (c_2 - c_1) - c_1\ (a_2 - a_3)$

$= a_3\ c_2 - c_1\ a_2$

$= a_3\ (0) - c_1\ (a_2)$

$= - c_1 a_2 > 0$ as $c_1 < 0$

Therefore, $\dot{x}_1^{\alpha} > 0$

Now $\dot{x}_2^{\alpha} = (1/a_2)^2(a_2^{\alpha}\dot{c}_2^{\alpha} - c_1^{\alpha}\dot{a}_2^{\alpha})$

But $a_2^{\alpha}\dot{c}_1^{\alpha} - \dot{c}_1^{\alpha} a_2^{\alpha} = a_2\alpha (c_2 - c_3) - c_2^{\alpha} (a_2 - a_3)$

$= [a_3 + \alpha (a_2 - a_3)] (c_2 - c_3) - [c_3 + \alpha (c_2 - c_3)] (a_2 - a_3)$

$= a_3 (c_2 - c_3) - c_3 - (a_2 - a_3)$

$= a_3c_2 - c_3 a_2$

$= a_3 (0) - c_3a_2$

$= - c_3a_2 < 0.$

Therefore $\dot{x}_2^{\alpha} < 0$.

ii) Suppose $A < 0$

$[a_1^{\alpha}x_2^{\alpha}, a_1^{\alpha}x_1^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$

Then $x_1^{\alpha} = c_2^{\alpha}/a_1^{\alpha}$, $x_2^{\alpha} = c_1^{\alpha}/a_1^{\alpha}$

Thus, $\dot{x}_1^{\alpha} = (1/ a_1^{\alpha})^2 [a_1^{\alpha} \dot{c}_2^{\alpha} - c_2^{\alpha} \dot{a}_1^{\alpha}]$

But $a_1^{\alpha}\dot{c}_2^{\alpha} - c_2^{\alpha}\dot{a}_1^{\alpha} = a_1^{\alpha} (c_2 - c_3) - c_2^{\alpha} (a_2 - a_1)$

$= [a_1 + \alpha (a_2 -a_1)] (c_2 - c_3) - [c_3 + \alpha (c_2 -c_3)] (a_2 - a_1)$

$= a_1 (c_2 - c_3) - c_3 (a_2 - a_1)$

$= a_1c_2 - c_3a_2$

$= a_1 (0) - c_3a_2$

$= - c_3a_2 > 0$

Therefore, $\dot{x}_1^{\alpha} > 0$

Similarly $\dot{x}_2^{\alpha} < 0$.

Now we will discuss fuzzy quadratic equations. Here onwards we assume that $0 \notin$ support of A. i. e. either $A > 0$ or $A < 0$.

Theorem 5.4.8 [B1]: Fuzzy numbers $X_1 \geq 0$ and $X_2 = - X_1$ are solutions of $AX^2 = C$ if and only if

i) $a_1c_2 > c_1 a_2$ and $a_3 c_2 < c_3 a_2$, when $A > 0, C \geq 0$

ii) $a_1c_2 < c_1a_2$ and $a_3 c_2 > c_3a_2$, when $A < 0, C \leq 0$.

Proof: Set $U = X^2$. Then $U \geq 0$ and $AU = C$.

Let $X_1 = \sqrt{U} > 0$ and $X_2 = - \sqrt{U} < 0$

Now ${}^{\alpha}(AU) = {}^{\alpha}C$

i. e. ${}^{\alpha}A{}^{\alpha}U = {}^{\alpha}C$

$[a_1^{\alpha}, a_2^{\alpha}]\ [u_1^{\alpha}, u_2^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$, where ${}^{\alpha}A = [a_1^{\alpha}, a_2^{\alpha}]$, ${}^{\alpha}U = [u_1^{\alpha}, u_2^{\alpha}]$, ${}^{\alpha}C = [c_1^{\alpha}, c_2^{\alpha}]$

are α - cuts of A, U and C respectively.

i) Suppose $A > 0$

Now $[a_1^{\alpha}u_1^{\alpha}, a_2^{\alpha}u_2^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$

Thus, $u_1^{\alpha} = c_1^{\alpha}/a_1^{\alpha}$, $u_2^{\alpha} = c_2^{\alpha}/a_2^{\alpha}$

Then there exist solutions iff $\dot{u}_1^{\alpha} > 0$ and $\dot{u}_2^{\alpha} < 0$

iff $a_1^{\alpha}\ \dot{c}_1^{\alpha} > c_1^{\alpha}\dot{a}_1^{\alpha}$ and $a_2^{\alpha}\ \dot{c}_2^{\alpha} < c_2^{\alpha}\ \dot{a}_2^{\alpha}$

iff $a_1^{\alpha}\ (c_2 - c_1) > c_1^{\alpha}\ (a_2 - a_1)$ and $a_2^{\alpha}\ (c_2 - c_3) < c_2^{\alpha}\ (a_2 - a_3)$

iff $[a_1 + \alpha\ (a_2 - a_1)]\ (c_2 - c_1) > [c_1 + \alpha\ (c_2 - c_1)]\ (a_2 - a_1)$ and

$[a_3 + \alpha\ (a_2 - a_3)]\ (c_2 - c_3) < [c_3 + \alpha\ (c_2 - c_3)]\ (a_2 - a_3)$

iff $a_1c_2 > c_1a_2$ and $a_3c_2 < c_3 a_2$. and $a_3\ (c_2 - c_3) < c_3(a_2 - a_3)$

iff $a_1c_2 > c_1a_2$ and $a_3c_2 < c_3a_2$

ii) Suppose $A < 0, C \leq 0$

Now $[a_1^{\alpha}\ u_2^{\alpha}, a_2^{\alpha}\ u_1^{\alpha}] = [c_1^{\alpha}, c_2^{\alpha}]$

Then $u_1^{\alpha} = c_2^{\alpha}/a_2^{\alpha}$, $u_2^{\alpha} = c_1^{\alpha}/a_1^{\alpha}$

Therefore, there exists a solution iff $\dot{u}_1^{\alpha} > 0$ and $\dot{u}_2^{\alpha} < 0$.

iff $a_2^{\alpha}\dot{c}_2^{\alpha} > c_2^{\alpha}\ \dot{a}_2^{\alpha}$ and $a_1^{\alpha}\dot{c}_1^{\alpha} < c_1^{\alpha}\ \dot{a}_1^{\alpha}$

iff $a_2^\alpha (c_2 - c_3) > c_2^\alpha (a_2 - a_3)$ and $a_1^\alpha (c_2 - c_1) < c_1^\alpha (a_2 - a_1)$

iff $[a_3 + \alpha (a_2 - a_3)] (c_2 - c_3) > [c_3 + \alpha (c_2 - c_3)] (a_2 - a_3)$ and

$[a_1 + \alpha (a_2 - a_1)] (c_2 - c_1) < [c_1 + \alpha (c_2 - c_1)] (a_2 - a_1)$

iff $a_3 (c_2 - c_3) > c_3 (a_2 - a_3)$ and $a_1 (c_2 - c_1) < c_1 (a_2 - a_1)$

iff $a_3c_2 > c_3a_2$ and $a_1 c_2 < c_1a_2$

5.5 EQUATIONS OF FUZZY COMPLEX NUMBERS

Throughout this section $\forall$ stands for a set complex numbers.

Definition 5.5.1 [B_1, B_2, B_3, B_4]**:** A fuzzy set $X: \forall \to I$ is called a complex fuzzy number, if

i) X is continuous

ii) ${}^\alpha X = \{z \in \forall \mid X(z) > \alpha\}$, for $0 \leq \alpha < 1$ is open, bounded, connected and simply connected, for all $0 \leq \alpha < 1$

iii) ${}^1X = \{z \in \forall \mid X(z) = 1\}$ is non-empty, compact, arc wise connected and simply connected

Definition 5.5.2 [B_1, B_2]: Let X_1 and X_2 be real fuzzy numbers. Define a fuzzy set $X: \forall \to I$ by $X(z) = \min\{ X_1(x), X_2(y) \}$, where $z = x + iy$. Then X is a complex fuzzy number which we shall denote as $X = X_1 + iX_2$ and call it a rectangular complex fuzzy number.

If A and B are crisp sets, then $A + iB = \{ x + iy \mid x \in A, y \in B\}$.

Theorem 5.5.3 [B_1, B_2]: If X and Y are rectangular complex fuzzy numbers, then

i) $^{\alpha}X = {}^{\alpha}X_1 + i\,{}^{\alpha}X_2$

ii) $X + Y = (X_1 + Y_1) + i(X_2 + Y_2)$

iii) $^{\alpha}(X + Y) = ({}^{\alpha}X_1 + {}^{\alpha}Y_1) + i({}^{\alpha}X_2 + {}^{\alpha}Y_2)$

Proof: Let $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$ be rectangular complex fuzzy number. Let $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers.

i) $z \in {}^{\alpha}X \Leftrightarrow X(z) > \alpha$

$\Leftrightarrow \min\{X_1(x), X_2(y)\} > \alpha$

$\Leftrightarrow X_1(x) > \alpha$ and $X_2(y)\} > \alpha$

$\Leftrightarrow x \in {}^{\alpha}X_1$ and $y \in {}^{\alpha}X_2$

$\Leftrightarrow z = x + iy \in {}^{\alpha}X_1 + i^{\alpha}(X_2)$

Hence, $^{\alpha}X = {}^{c}X_1 + i^{\alpha}(X_2)$

ii) $X + Y(z) = \sup_{z = z_1 + z_2} \{\min\{X(z_1), Y(z_2)\}\}$

$= \sup_{z = z_1 + z_2} \{\min\{\min\{X_1(x_1), X_2(y_1)\}, \min\{Y_1(x_2), Y_2(y_2)\}\}$

$= \min\{ \sup_{z = z_1 + z_2} \{ \min\{X_1(x_1), X_2(y_1)\}, \min\{Y_1(x_2), Y_2(y_2)\}\}$

$= \min\{ \sup_{z = z_1 + z_2} \{\min\{X_1(x_1), Y_1(x_2)\}, \min\{X_2(y_1), Y_2(y_2)\}\}$

$= \min\{ \sup_{x = x_1 + x_2} \{\min\{X_1(x_1), Y_1(x_2)\}\}, \sup_{y = y_1 + y_2} \{\min\{X_2(y_1), Y_2(y_2)\}\}\}$

$= \min\{X_1 + Y_1(x_1 + x_2), X_2 + Y_2(y_1 + y_2)\}$

$= \min\{X_1 + Y_1(x), X_2 + Y_2(y)\}$

$= (X_1 + Y_1) + i(X_2 + Y_2)(z)$

iii) $z \in {}^{\alpha}(X + Y) \Leftrightarrow X + Y(z) > \alpha$

$\Leftrightarrow \min\{X_1 + Y_1(x), X_2 + Y_2(y)\} > \alpha$

$\Leftrightarrow X_1 + Y_1(x) > \alpha$ and $X_2 + Y_2(y) > \alpha$

$\Leftrightarrow x \in {}^{\alpha}(X_1 + Y_1)$ and $y \in {}^{\alpha}(X_2 + Y_2)$

$\Leftrightarrow x \in ({}^{\alpha}X_1 + {}^{\alpha}Y_1)$ and $y \in ({}^{\alpha}X_2 + {}^{\alpha}Y_2)$

$\Leftrightarrow z = x + iy \in ({}^{\alpha}X_1 + {}^{\alpha}Y_1) + i({}^{\alpha}X_2 + {}^{\alpha}Y_2)$

Definition 5.5.4 [B_1]: Let R and θ be real fuzzy numbers. Then a fuzzy set X: $\forall \rightarrow I$ defined by $X(re^{i\theta}) = \min\{R(r), \theta(\theta)\}$ is a complex fuzzy number. We shall denote $X = X = Re^{i\theta}$ and call it a polar complex fuzzy number.

Theorem 5.5.5 [B_1]: If X and Y are polar complex fuzzy numbers. Then

i) ${}^{\alpha}X = {}^{\alpha}R_x\, e^{i({}^{\alpha}\theta x)}$

ii) $XY = (R_x R_y)\, e^{i(\theta x + \theta y)}$

iii) ${}^{\alpha}(XY) = ({}^{\alpha}R_x {}^{\alpha}R_y)\, e^{i({}^{\alpha}\theta x + {}^{\alpha}\theta y)}$

Proof: Let $X = R_x e^{i\theta x}$, $Y = R_y e^{i\theta y}$, $z = re^{i\theta}$, $z_1 = r_1e^{i\theta 1}$ and $z_2 = r_2e^{i\theta 2}$.

i) $z \in {}^{\alpha}X \Leftrightarrow X(z) > \alpha$

$\Leftrightarrow \min\{R_x(r), \theta_x(\theta)\} > \alpha$

$\Leftrightarrow R_x(r) > \alpha$ and $\theta_x(\theta) > \alpha$

$\Leftrightarrow r \in {}^{\alpha}R_x$ and $\theta \in {}^{\alpha}\theta_x$

$\Leftrightarrow z = re^{i\theta} \in {}^{\alpha}R_x\, e^{i({}^{\alpha}\theta x)}$

ii) $XY(z) = \sup_{z = z_1 z_2} \{\min\{X(z_1), Y(z_2)\}\}$

$= \sup_{z = z_1 z_2} \{\min\{\min\{R_x(r_1), \theta_x(\theta_1)\}, \{\min\{R_y(r_2), \theta_y(\theta_2)\}\}$

$$= \min\{ \sup_{z = z_1 z_2} \{\min\{R_x(r_1), \theta_x(\theta_1)\}, \min\{R_y(r_2), \theta_y(\theta_2)\}\}\}$$

$$= \min\{ \sup_{r = r_1 r_2} \{\min\{R_x(r_1), R_y(r_2) \}, \sup_{\theta = \theta_1 + \theta_2} \{\min\{ \theta_x(\theta_1), \theta_y(\theta_2)\}\}\}$$

$$= \min\{ R_x R_y(r_1 r_2), \theta_x + \theta_y(\theta)\}$$

$$= (R_x R_y)\, e^{i(\theta x + \theta y)}(re^{i\theta})$$

iii) $z \in {}^{\alpha}(XY) \Leftrightarrow \in XY(z) > \alpha$

$\Leftrightarrow (R_x R_y)\, e^{i(\theta x + \theta y)} > \alpha$

$\Leftrightarrow \min \{R_x R_y(r), \theta_x + \theta_y(\theta)\} > \alpha$

$\Leftrightarrow R_x R_y(r) > \alpha$ and $\theta_x + \theta_y(\theta) > \alpha$

$\Leftrightarrow r \in {}^{\alpha}(R_x R_y)$ and $\theta \in {}^{\alpha}(\theta_x + \theta_y)$

$\Leftrightarrow z = re^{i\theta} \in {}^{\alpha}(\check{R}_x \check{R}_y)\, e^{i\,(\check{\theta} x + \check{\theta} y)}$

A fuzzy complex number X is neither rectangular complex fuzzy number nor polar complex fuzzy number, if the equation contains both multiplication and addition.

If the equation contains only additions and/or subtractions and all parameters are rectangular complex fuzzy numbers, then X is also a rectangular complex fuzzy number.

If the equation only multiplication and all the parameters are polar complex fuzzy numbers, then X is also a complex fuzzy numbers.

Theorem 5.5.6 [B_1]: If A + X = C is fuzzy equation where A, C are rectangular complex fuzzy numbers and X is a complex fuzzy numbers, then $A_i + X_i = C_i$, i = 1, 2.

Proof: Let $A = A_1 + iA_2$, $C = C_1 + iC_2$ and $X = X_1 + i X_2$.

Then A + X = C gives $(A_1 + iA_2) + (X_1 + i X_2) = C_1 + iC_2$

Therefore, $(A_1 + X_1) + i(A_2 + X_2) = C_1 + iC_2$

Thus, $A_1 + X_1 = C_1$ and $A_2 + X_2 = C_2$

Theorem 5.5.7 [B_1]: Let A and C be rectangular complex fuzzy numbers. Then the equation A + X = C has a solution X, a rectangular complex fuzzy numbers iff the equation $A_i + X_i = C_i$ have solution X_i, i = 1, 2.

Proof: Let $A = A_1 + iA_2$, $C = C_1 + iC_2$ and $X = X_1 + i X_2$.

Suppose A + X = C has a solution X, which is a rectangular complex fuzzy number.

Since A + X = C, $(A_1 + iA_2) + (X_1 + i X_2) = C_1 + iC_2$

Therefore, $(A_1 + X_1) + i(A_2 + X_2) = C_1 + iC_2$

Thus, $A_1 + X_1 = C_1$ and $A_2 + X_2 = C_2$

Conversely suppose Xi is a solution of $A_i + X_i = C_i$, i = 1, 2

Therefore, $A_1 + X_1 = C_1$ and $A_2 + X_2 = C_2$

Thus $A + X = (A_1 + iA_2) + (X_1 + i X_2) = (A_1 + X_1) + i(A_2 + X_2) = C_1 + iC_2 = C$

Hence, X is solution of A + X = C.

Theorem 5.5.8 [B_1]: If AX = C is a fuzzy equation, where A and C are polar complex fuzzy numbers and X is a complex fuzzy number, then $R_a R_x = R_c$ and $\theta_a + \theta_x = \theta_c$.

Proof: Let $A = R_a e^{i\theta a}$, $C = R_c e^{i\theta c}$ and $X = R_x e^{i\theta x}$.

Then AX = C gives $R_a e^{i\theta a} R_x e^{i\theta x} = R_c e^{i\theta c}$

Therefore, $R_a R_x e^{i(\theta a + \theta x)} = R_c e^{i\theta c}$

Hence, $R_a R_x = R_c$ and $\theta_a + \theta_x = \theta_c$.

Theorem 5.5.9 [B_1]: The equation AX = C has a solution X, a polar complex fuzzy number if and only if

i) $R_a R_x = R_c$ has a solution for R_x

ii) $\theta_a + \theta_x = \theta_c$ has a solution for θ_x.

Proof: Let $A = R_a e^{i\theta a}$, $C = R_c e^{i\theta c}$ and $X = R_x e^{i\theta x}$.

Suppose the equation AX = C has a solution X, a polar complex fuzzy number.

Taking α-cuts of both sides ${}^{\alpha}(AX) = {}^{\alpha}C$

i. e. ${}^{\alpha}A^{\alpha}X = {}^{\alpha}C$

Thus ${}^{\alpha}(R_a e^{i\theta a})\, {}^{\alpha}(R_x e^{i\theta x}) = {}^{\alpha}(R_c e^{i\theta c})$

Therefore, ${}^{\alpha}(R_a R_x)\, e^{i\,{}^{\alpha}(\theta a + \theta x)} = {}^{\alpha}R_c e^{i\,{}^{\alpha}\theta c}$

Thus, ${}^{\alpha}(R_a R_x) = {}^{\alpha}R_c$ and ${}^{\alpha}(\theta_a + \theta_x) = {}^{\alpha}(\theta_c)$

Hence, $R_a R_x = R_c$ and $\theta_a + \theta_x = \theta_c$.

Conversely $R_a R_x = R_c$ and $\theta_a + \theta_x = \theta_c$.

Then $AX = R_a e^{i\theta a} R_x e^{i\theta x} = R_a R_x e^{i(\theta a + \theta x)} = R_c e^{i\theta c} = C$

Hence X is solution of AX = C.