CHAPTER 1

CHAPTER 1 PRELIMINARIES

In this chapter we discuss some basic definitions and results in fuzzy set theory, which will be used in the remaining part of the dissertation.

1.1 FUZZY SETS:

Throughout this work X stands for the universal set and I for the unit interval of reals [0, 1].

Definition 1.1.1 [D_1, D_2, K_2, R]: Let X be the universal set. A fuzzy set A in X is a function A: X $\rightarrow$ I.

The set of all fuzzy sets of X is denoted by F(X)

Remark 1.1.2: Clearly every set can be considered as a fuzzy set by identifying it with its characteristic function, when it is necessary to distinguish a set from a "proper" fuzzy set it will be called a crisp set.

Definition 1.1.3 [D_1, D_2, K_2]: Let A and B be two fuzzy sets in X

(i) A fuzzy set A is a subset of fuzzy set B if, $A(x) \leq B(x)$, $\forall x \in X$ and is denoted by $A \subseteq B$.

(ii) A fuzzy set A is said to be a proper subset of fuzzy set B, if $A(x) < B(x)$, $\forall x \in X$, and is denoted by $A \subset B$.

(iii) A fuzzy set A is equal to a fuzzy set B, if $A(x) = B(x)$, $\forall x \in X$.

Remark 1.1.4: The inclusion relation '$\subseteq$' defined in the Definition 1.1.3 is a partial order relation on F(X).

Definition 1.1.5 [D_1, D_2, K_2]: Let A and B be two fuzzy sets in X.

(i) The union of fuzzy sets A and B is a fuzzy set, $A \cup B$, in X defined as

$A \cup B(x) = \max\{A(x), B(x)\}$, $\forall$ x $\in$ X.

(ii) The intersection of fuzzy sets A and B is a fuzzy set, $A \cap B$, in X defined as

$A \cap B(x) = \min\{A(x), B(x)\}$, $\forall$ x $\in$ X.

Union and intersection of fuzzy sets can be easily extended for infinite fuzzy sets by replacing max and min by supremum and infimum respectively.

Definition 1.1.6 [D_1, D_2, K_2]: Let A be a fuzzy set in X. The complement of A is a fuzzy set, A′, in X defined as $A'(x) = 1 - A(x)$, $\forall$ x $\in$ X.

Definition 1.1.7 [K_2]: Let A be a fuzzy set in X and $\alpha \in I$. Then the α - cut of A is defined as a crisp set $\{x \in X \mid A(x) \geq \alpha\}$.

We shall denote it by ${}^{\alpha}A$.

Thus, ${}^{\alpha}A = \{x \in X \mid A(x) \geq \alpha\}$

Definition 1.1.8 [K_2]: Let A be a fuzzy set in X and $\alpha \in I$. The strong (strict) α-cut of A is defined as a crisp set $\{x \in X \mid A(x) > \alpha\}$.

We shall denote it by ${}^{\alpha+}A$.

Thus, ${}^{\alpha+}A = \{x \in X \mid A(x) > \alpha\}$

Definition 1.1.9 [K_2]: Let A be a fuzzy set in X. The support of A is a crisp set,

$\{x \in X \mid A(x) > 0\}$.

We shall denote it by Supp(A).

Thus, $\text{Supp}(A) = \{x \in X \mid A(x) > 0\}$.

Definition 1.1.10 [K_2]: The $\sup\{A(x) \mid x \in X\}$ is called the height, h(A), of a fuzzy set A.

Thus, $h(A) = \sup_{x \in X} \{A(x)\}$

When $\{A(x) \mid x \in X\}$ is finite, the height of A, h(A), is the maximum value attended by A.

Definition 1.1.11 [K_2]: A fuzzy set A is said to be normal if, $h(A) = 1$. Otherwise it is called subnormal.

Theorem 1.1.12 [K_2]: Let $A, B \in F(X)$. Then, for all $\alpha, \beta \in I$

i) ${}^{\alpha+}A \subseteq {}^{\alpha}A$

ii) $\alpha \le \beta \Rightarrow {}^{\beta}A \subseteq {}^{\alpha}A$ and ${}^{\beta+}A \subseteq {}^{\alpha+}A$

iii) ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$

iv) ${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$

v) ${}^{\alpha+}(A \cap B) = {}^{\alpha+}A \cap {}^{\alpha+}B$

vi) ${}^{\alpha+}(A \cup B) = {}^{\alpha+}A \cup {}^{\alpha+}B$

vii) ${}^{\alpha}(A') = ({}^{(1-\alpha)+}A)'$.

Theorem 1.1.13 [K_2]: Let A, B $\in$ F(X). Then, for all $\alpha \in$ I

i) $A \subseteq B$ if and only if ${}^{\alpha}A \subseteq {}^{\alpha}B$

ii) $A \subseteq B$ if and only if ${}^{\alpha+}A \subseteq {}^{\alpha+}B$

iii) $A = B$ if and only if ${}^{\alpha}A = {}^{\alpha}B$

iv) $A = B$ if and only if ${}^{\alpha+}A = {}^{\alpha+}B$.

Definition 1.1.14 [K_2]: Let X and Y be two non-empty sets and let f: X $\rightarrow$ Y be a function. Then f: F(X) $\rightarrow$ F(Y) and f^{-1} : F(Y) $\rightarrow$ F(X) are defined as follows:

i) $f(A)(y) = \sup_{x \mid y = f(x)} \{A(x)\}, \quad \forall A \in F(X)$

ii) $f^{-1}(B)(x) = B(f(x)), \forall B \in F(Y)$.

Definition 1.1.15 [K_2]: Let A be a fuzzy set in X and $\alpha \in$ I. Define a fuzzy set ${}_{\alpha}A$ in X as:

$$
\begin{aligned}
{}_{\alpha}A(x) &= \alpha, \quad \text{if } A(x) \geq \alpha \\
&= 0, \quad \text{otherwise}
\end{aligned}
$$

Theorem 1.1.16 [K_2]: (First Decomposition Theorem) If A $\in$ F(X), then $A = \bigcup\{{}_{\alpha}A \mid \alpha \in I\}$, where $\cup$ denotes union of fuzzy sets.

1.2 TRIANGULAR NORMS:

Union and Intersection of fuzzy sets defined in the Definition 1.1.5 was introduced by Zadeh in 1965. There are several ways to generalize these concepts with the help of triangular norms.

Definition 1.2.1 [F, K_2, M]: A binary operation T on I is said to be a triangular norm or t-norm, if for all $x, y, z \in I$

i) $T(T(x, y), z) = T(x, T(y, z))$ (Associative Law)

ii) $T(x, y) = T(y, x)$ (Commutative Law)

iii) $y \leq z \Rightarrow T(x, y) \leq T(x, z)$ (Monotonicity)

iv) $T(x, 1) = x$ (Boundary condition)

Definition 1.2.2 [F, K_2, M]: A binary operation S on I is said to be a triangular conorm or t-conorm, if for all $x, y, z \in I$

i) $S(S(x, y), z) = S(x, S(y, z))$ (Associative Law)

ii) $S(x, y) = S(y, x)$ (Commutative Law)

iii) $y \leq z \Rightarrow S(x, y) \leq S(x, z)$ (Monotonicity)

iv) $S(x, 0) = x$ (Boundary condition)

Theorem 1.2.3 [K_2]: Minimum is the largest t-norm and maximum is the smallest t-conorm.

Theorem 1.2.4[F]: Let $T: I \times I \rightarrow I$. Define $T': I \times I \rightarrow I$ by $T'(x, y) = 1-T(1-x, 1-y)$, $\forall\, x, y \in I$. Then T is a t-norm if and only if T' is a t-conorm.

Proof: Let $x, y, z \in I$.

(i) $T'(T'(x, y), z) = T'(1 - T(1 - x, 1 - y), z)$

$= 1 - T(1-(1 - T(1 - x, 1 - y)), 1 - z)$

$= 1 - T(T(1 - x, 1 - y), 1 - z)$

Now, $T'(x, T'(y, z)) = T'(x, 1 - T(1 - y, 1 - z))$

$= 1 - T(1 - x, T(1 - y, 1 - z))$

$= 1 - T(T(1 - x, 1 - y), 1 - z)$ By associativity of T

Thus, $T'(T'(x, y), z) = T'(x, T'(y, z))$

(ii) $T'(x, y) = 1 - T(1 - x, 1 - y)$

$= 1 - T(1 - y, 1 - x)$ By commutativity of T

$= T'(y, x)$

(iii) Since $y \leq z$, $1 - y \geq 1 - z$

$\Rightarrow T(1 - x, 1 - y) \geq T(1 - x, 1 - z)$

$\Rightarrow 1 - T(1 - x, 1 - y) \leq 1 - T(1 - x, 1 - z)$

$\Rightarrow T'(x, y) \leq T'(x, z)$

(iv) $T'(x, 0) = 1 - T(1 - x, 1 - 0)$

$= 1 - T(1 - x, 1)$

$= 1 - (1 - x)$

$= x$

Hence, T' is a t-conorm on I

Similarly we prove the converse.

Definition 1.2.5[F]: T and T' defined in the Theorem 1.2.4 are called dual of each other.

Following are some triangular norms with their duals:

Example 1.2.6: (i) $T_1(x, y) = \min(x, y)$, $T_1'(x, y) = \max(x, y)$

(ii) $T_2(x, y) = x \cdot y$, $T_2'(x, y) = x + y - x \cdot y$.

(iii) $T_3(x, y) = \max(0, x + y - 1)$, $T_3'(x, y) = \min(1, x + y)$.

Definition 1.2.7 [K_2]: Let T be a continuous t-norm. Then define $w_T: I \times I \to I$ as follows:

$$w_T(a, b) = \sup\{x \in I \mid T(a, x) \leq b\}, \forall\, a, b \in I$$

Theorem 1.2.8 [K_2]: Let $a, a_j, b, c \in I$. Then

i) $T(a, b) \leq c$ if and only if $w_T(a, c) \geq b$

ii) $w_T(w_T(a, b)\, b) \geq a$

iii) $w_T(T(a, b), c) = w_T(a, w_T(b, c))$

iv) $a \leq b \Rightarrow w_T(a, c) \geq w_T(b, c)$ and $w_T(c, a) \leq w_T(c, b)$

v) $T(w_T(a, b), w_T(b, c)) \leq w_T(a, c)$

vi) $w_T(\inf_j a_j, b) \geq \sup_j w_T(a_j, b)$

vii) $w_T(\sup_j a_j, b) = \inf_j w_T(a_j, b)$

viii) $w_T(b, \sup_j a_j) \geq \sup_j w_T(b, a_j)$

ix) $w_T(b, \inf_j a_j) = \inf_j w_T(b, a_j)$

x) $T(a, w_T(a, b)) \leq b$

xi) $w_T(a, T(a, b)) \geq b$

Definition 1.2.9[F]: Let S be a continuous t-conorm. Then define ω_S: I x I → I an operator as follows: $\omega_S(a, b) = \inf\{x \in I \mid S(a, x) \geq b\}, \forall\, a, b \in I$

Theorem 1.2.10 [K_2]: Let a, a_j, b, c ∈ I. Then

i) $S(a, b) \geq c$ if and only if $\omega_S(a, c) \leq b$

ii) $\omega_S(\omega_S(a, b)\ b) \leq a$

iii) $\omega_S(S(a, b), c) = \omega_S(a, \omega_S(b, c))$

iv) $a \leq b \Rightarrow \omega_S(a, c) \geq \omega_S(b, c)$ and $\omega_S(c, a) \leq \omega_S(c, b)$

v) $\omega_S(\inf_j a_j, b) = \sup_j \omega_S(a_j, b)$

vi) $\omega_S(\sup_j a_j, b) \leq \inf_j \omega_S(a_j, b)$

vii) $\omega_S(b, \sup_j a_j) \leq \sup_j \omega_S(b, a_j)$

viii) $\omega_S(b, \inf_j a_j) \leq \inf_j \omega_S(b, a_j)$

ix) $S(a, \omega_S(a, b)) \geq b$

x) $\omega_S(a, S(a, b)) \leq b$

xi) $\omega_S(a, b) \leq \max(a, b)$.

Proof: i) Let $S(a, b) \geq c$. Then $b \in \{x \in I \mid S(a, x) \geq c\}$

Therefore, $b \geq \inf\{x \in I \mid S(a, x) \geq b\}$

Hence, $b \geq \omega_S(a, c)$

Conversely let $b \geq \omega_S(a, c)$. Then $S(a, \omega_S(a, c)) \leq S(a, b)$.

$S(a, \omega_S(a, c)) = S(a, \inf\{x \in I \mid S(a, x) \geq c\}) = \inf\{S(a, x) \mid S(a, x) \geq c\} \geq c$

Hence, $S(a, b) \geq c$.

ii) $S(\omega_S(a, b)\ a) = S(\inf\{x \in I \mid S(a, x) \geq b\}, a) = \inf\{S(a, x) \mid S(a, x) \geq b\} \geq b$.

Therefore, by (i), $\omega_S(\omega_S(a, b)\, b) \le a$

iii) $S(a, x) \ge \omega_S(b, c) \Leftrightarrow S(b, S(a, x)) \ge c \Leftrightarrow S(S(a, b), x) \ge c \Leftrightarrow \omega_S(S(a, b), c) \le x$

Thus, $\omega_S(a, \omega_S(b, c)) = \inf\{x \in I \mid S(a, x) \ge \omega_S(b, c)\}$

$= \inf\{x \in I \mid x \ge \omega_S(S(a, b), c)\} = \omega_S(S(a, b), c)$.

iv) Let $a \le b$. Then $S(a, x) \le S(b, x)$.

Therefore, $\{x \in I \mid S(a, x) \ge c\} \subseteq \{x \in I \mid S(b, x) \ge c\}$

Thus, $\inf\{x \in I \mid S(a, x) \ge c\} \ge \inf\{x \in I \mid S(b, x) \ge c\}$

Hence, $\omega_S(a, c) \ge \omega_S(b, c)$.

Now $\{x \in I \mid S(c, x) \ge b\} \subseteq \{x \in I \mid S(c, x) \ge a\}$

Thus, $\inf\{x \in I \mid S(c, x) \ge b\} \ge \inf\{x \in I \mid S(c, x) \ge a\}$

Hence, $\omega_S(c, b) \ge \omega_S(c, a)$.

v) $\inf\limits_j a_j \le a_j$, for all j

Thus, $\omega_S(\inf\limits_j a_j, b) \ge \omega_S(a_j, b)$, for all j

Therefore, $\omega_S(\inf\limits_j a_j, b) \ge \sup\limits_j \omega_S(a_j, b)$

Conversely $\sup\limits_j \omega_S(a_j, b) \ge \omega_S(a_j, b)$, for all j

Therefore, $S(a_j, \sup\limits_j \omega_S(a_j, b)) \ge b$, for all j

Thus, $\inf\limits_j S(a_j, \sup\limits_j \omega_S(a_j, b)) \ge b$

Then, $S(\inf\limits_j a_j, \sup\limits_j \omega_S(a_j, b)) \ge b$

Therefore, $\omega_S(\inf\limits_j a_j, b) \le \sup\limits_j \omega_S(a_j, b)$

vi) $\sup\limits_j a_j \ge a_j$, for all j

Thus, $\omega_S(\sup\limits_j a_j, b) \le \omega_S(a_j, b)$, for all j

Therefore, $\omega_S(\sup_j a_j, b) \leq \inf_j \omega_S(a_j, b)$

vii) $\sup_j a_j \geq a_j$, for all j

Thus, $\omega_S(b, \sup_j a_j) \geq \omega_S(b, a_j)$, for all j

Hence, $\omega_S(b, \sup_j a_j) \geq \sup_j \omega_S(b, a_j)$

Next $\sup_j \omega_S(b, a_j) \geq \omega_S(b, a_j)$, for all j

Therefore, $S(b, \sup_j \omega_S(b, a_j)) \geq a_j$, for all j

Thus, $S(b, \sup_j \omega_S(b, a_j)) \geq \sup_j a_j$,

Therefore, $\omega_S(b, \sup_j a_j) \leq \sup_j \omega_S(b, a_j)$

viii) $\inf_j a_j \leq a_j$, for all j

Thus, $\omega_S(b, \inf_j a_j) \leq \omega_S(b, a_j)$, for all j

Therefore, $\omega_S(b, \inf_j a_j) \leq \inf_j \omega_S(b, a_j)$

ix) Since $\omega_S(a, b) \leq \omega_S(a, b)$, $S(a, \omega_S(a, b)) \geq b$.

x) Since, $S(a, b) \geq S(a, b)$, $\omega_S(a, S(a, b)) \leq b$

xi) Since $b \leq \max\{a, b\}$, $\omega_S(a, b) \leq \omega_S(a, \max(a, b))$ and

$\max\{a, b\} \leq \max\{a, \max\{a, b\}\}$

Now $\omega_S(a, \max(a, b)) \leq \omega_S(a, \max\{a, \max\{a, b\}\}) \leq \max(a, b)$ (By property x)

Thus, $\omega_S(a, b) \leq \max(a, b)$.

Theorem 1.2.11 [F]: Let S be the dual of a t-norm T. Then $\omega_S(a, b) = 1 - w_T(1-a, 1-b)$, for all $a, b \in I$.

Example 1.2.12: Let T_1, T_2, T_3, T_1', T_2', T_3' be the triangular norms given in the Example 1.2.6. Then

$$w_{T1}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$$

$$w_{T2}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b \end{cases}$$

$$w_{T3}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b - a + 1, & \text{if } a > b \end{cases}$$

$$\omega_{S1}(a, b) = \begin{cases} 0, & \text{if } a \geq b \\ b, & \text{if } a < b \end{cases}$$

$$\omega_{S2}(a, b) = \begin{cases} 0, & \text{if } a \geq b \\ b - a / (1 - a), & \text{if } a < b \end{cases}$$

$$\omega_{S3}(a, b) = \begin{cases} 0, & \text{if } a \geq b \\ b - a, & \text{if } a < b \end{cases}$$