CHAPTER 2

CHAPTER 2 FUZZY RELATION EQUATIONS ON CRISP SETS

In this chapter we discuss fuzzy relations and study two types of equations relating to fuzzy relations. We discuss some standard methods to solve these equations.

2.1 FUZZY RELATIONS

Definition 2.1.1 [K_2]: A relation on crisp sets $X_1, X_2, ..., X_n$ is a subset of the cartesian product $X_1 \times X_2 \times \ldots \times X_n$.

It is denoted by $R(X_1, X_2, \ldots, X_n)$ or $R(X_i \mid i = 1, 2, \ldots, n)$.

Definition 2.1.2 [F, K_2]: Let X and Y be two crisp sets. A function $R: X \times Y \rightarrow I$ is called a fuzzy binary relation or fuzzy relation from X to Y.

We shall denote the fuzzy relation R from X to Y by R(X, Y).

Definition 2.1.3 [K_2]: Let R(X, Y) be a fuzzy relation. The inverse of R, is a fuzzy relation $R^{-1}(Y, X)$ defined by $R^{-1}(y, x) = R(x, y)$, $\forall\, y \in Y, x \in X$.

A fuzzy relation on finite sets can be represented by a matrix.

Definition 2.1.4 [K_2] :Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the crisp sets and R(X, Y) be a fuzzy relation. Then the matrix $M(R) = [R(x_i, y_j)]_{m \times n}$ is called the matrix of R.

Obviously $M(R^{-1}) = (M(R))^T$, where A^T denotes the transpose of A.

2.2 COMPOSITIONS OF FUZZY RELATIONS

Definition 2.2.1 [K_2]: Let P(X, Y) and Q(Y, Z) be two fuzzy relations. Then the composition of P and Q is a fuzzy relation P o Q(X, Z), which is defined as follows:

$P \circ Q(x, z) = \text{Sup}\ \{\min\ (P(x,y), Q(y, z)) \mid y \in Y\}, \forall\ (x, z) \in X \times Z.$

When Y is finite set the supremum is replaced by maximum and the composition is known as max-min composition.

This composition can be generalized by replacing min by any t-norm.

Definition 2.2.2 [K_2]: Let T be a t-norm and P(X, Y) and Q(Y, Z) be two fuzzy relations. Then the sup-T composition of P and Q is a fuzzy relation $P \circ_T Q(X, Z)$, which is defined as follows:

$P \circ_T Q\ (x, z) = \sup\{T(P(x, y), Q(y, z) \mid y \in Y\},\ \forall\ (x, z) \in X \times Z.$

Theorem 2.2.3 [K_2]: Let P(X, Y), P_j (X, Y), Q(Y, Z), Q_j (Y, Z) and R(Z, V) be fuzzy relations, where j takes the values in the index set J. Then

(i) $(P \overset{o}{T} Q) \overset{o}{T} R = P \overset{o}{T} (Q \overset{o}{T} R)$

(ii) $P \overset{o}{T} (\bigcup_j Q_j) = \bigcup_j (P \overset{o}{T} Q_j)$

(iii) $P \overset{o}{T} (\bigcap_j Q_j) \subseteq \bigcap_j (P \overset{o}{T} Q_j)$

(iv) $(\bigcup_j P_j) \overset{o}{T} Q = \bigcup_j (P_j \overset{o}{T} Q)$

(v) $(\bigcap_j P_j) \overset{o}{T} Q \subseteq \bigcap_j (P_j \overset{o}{T} Q)$

(vi) $(P \overset{o}{T} Q)^{-1} = Q^{-1} \overset{o}{T} P^{-1}$

(vii) $Q_1 \subseteq Q_2 \Rightarrow P \overset{o}{T} Q_1 \subseteq P \overset{o}{T} Q_2$ and $Q_1 \overset{o}{T} R \subseteq Q_2 \overset{o}{T} R$.

Proof: (i) $[(P \overset{o}{T} Q) \overset{o}{T} R] (x, v) = \sup_{z \in Z} T(P \overset{o}{T} Q) (x, z), R(z, v)]$

$$= \sup_{z \in Z} \{T(\sup_{y \in Y} \{T(P(x, y), Q(y, z))\}, R(z, v))\}$$

$$= \sup_{y \in Y,\ z \in Z} \{ T (T(P(x, y), Q(y, z)), R(z, v))\}$$

$$= \sup_{y \in Y} \{ T(P(x, y), \sup_{z \in Z} \{ T(Q(y, z), R(z, v))\})\}$$

$$= \sup_{y \in Y} \{T(P(x, y), (Q \overset{o}{T} R) (y, v)\}$$

$$= P \overset{o}{T} (Q \overset{o}{T} R) (x, v)$$

Thus, $(P \overset{o}{T} Q) \overset{o}{T} R = P \overset{o}{T} (Q \overset{o}{T} R)$

(ii) $P \overset{o}{T} (\bigcup_j Q_j) (x, z)$

$$= \sup_{y \in Y} \{ T(P(x, y), (\bigcup_j Q_j) (y, z))\}$$

$$= \sup_{y \in Y} \{ T(P(x, y), \sup_j Q_j (y, z))\}$$

$$= \sup_{y \in Y} \sup_j \{T(P(x, y), Q_j (y, z))\}$$

$$= \sup_j \sup_{y \in Y} \{ T(P(x, y), Q_j (y, z))\}$$

$$= \sup_j (P \overset{o}{T} Q_j) (x, z)$$

$$= \bigcup_j (P \overset{o}{T} Q_j) (x, z)$$

Thus, $P \underset{T}{\overset{o}{}} (\bigcup_j Q_j) = \bigcup_j (P \underset{T}{\overset{o}{}} Q_j)$

(iii) $P \underset{T}{\overset{o}{}} (\bigcap_j Q_j)\ (x, z)$

$$= \sup_{y \in Y} \{ T(P(x, y), (\bigcap_j Q_j)\ (y, z))\}$$

$$= \sup_{y \in Y} \{ T(P(x, y), \inf_j Q_j (y, z))\}$$

$$\leq \sup_{y \in Y} \inf_j \{ T (P(x, y), Q_j (y, z))\}$$

$$= \inf_j \sup_{y \in Y} \{ T(P(x, y), Q_j (y, z))\}$$

$$= \inf_j (P \underset{T}{\overset{o}{}} Q_j) (x, z)$$

$$= \bigcap_j (P \underset{T}{\overset{o}{}} Q_j) (x, z)$$

Thus, $P (\bigcap_j Q_j) \subseteq \bigcap_j (P \underset{T}{\overset{o}{}} Q_j)$

(iv) $(\bigcup P_j) \underset{T}{\overset{o}{}} Q\ (x, z)$

$$= \sup_{y \in Y} \{ T ((\bigcup_j P_j) (x, y), Q(y, z))\}$$

$$= \sup_{y \in Y} \{ T (\sup_j P_j (x, y), Q(y, z))\}$$

$$= \sup_j \sup_{y \in Y} \{ T(P_j (x, y), Q(y, z))\}$$

$$= \sup_j (P_j \underset{T}{\overset{o}{}} Q)\ (x, z)$$

$$= \bigcup_j (P_j \underset{T}{\overset{o}{}} Q)\ (x, z)$$

Thus, $\bigcup_j (P_j \underset{T}{\overset{o}{}} Q) = \bigcup_j (P_j \underset{T}{\overset{o}{}} Q)$

(v) $(\cap P_j) \underset{T}{\circ} Q)(x, z)$

$$= \sup_{y \in Y} \{T((\cap P_j)(x, y), Q(y, z))\}$$

$$= \sup_{y \in Y} \{T(\inf_j P_j(x, y), Q(y, z))\}$$

$$\leq \sup_{y \in Y} \inf_j \{T(P_j(x, y), Q(y, z))\}$$

$$\leq \inf_j \sup_{y \in Y} \{T(P_j(x, y), Q(y, z))\}$$

$$= \inf_j (P_j \underset{T}{\circ} Q)(x, z)$$

$$= \bigcap_j (P_j \underset{T}{\circ} Q)(x, z)$$

Thus, $(\bigcap_j P_j) \underset{T}{\circ} Q \subseteq \bigcap_j (P_j \underset{T}{\circ} Q)$

(vi) $(P \underset{T}{\circ} Q)^{-1}(z, x) = (P \underset{T}{\circ} Q)(x, z)$

$$= \sup_{y \in Y} \{T(P(x, y), Q(y, z))\}$$

$$= \sup_{y \in Y} \{T(P^{-1}(y, x), Q^{-1}(z, y))\}$$

$$= \sup_{y \in Y} \{T(Q^{-1}(z, y), P^{-1}(y, x))\}$$

$$= (Q^{-1} \underset{T}{\circ} P^{-1})(z, x)$$

Thus, $(P \underset{T}{\circ} Q)^{-1} = Q^{-1} \underset{T}{\circ} P^{-1}$

(vii) Let $Q_1 \subseteq Q_2$

Then $Q_1(y, z) \leq Q_2(y, z)$

$$\Rightarrow T(P(x, y), Q_1(y, z)) \leq T(P(x, y), Q_2(y, z))$$

$$\Rightarrow \sup_{y \in Y} T(P(x, y), Q_1(y, z)) \leq \sup_{y \in Y} T(P(x, y), Q_2(y, z))$$

$$\Rightarrow (P \underset{T}{\circ} Q_1)(x, z) \leq (P \underset{T}{\circ} Q_2)(x, z)$$

$\Rightarrow\ P \overset{o}{T} Q_1 \subseteq P \overset{o}{T} Q_2,$

similarly $Q_1 \overset{o}{T} R \subseteq Q_2 \overset{o}{T} R$

Definition 2.2.4 [K_2]: Let T be a continuous t-norm and P(X, Y), Q(Y, Z) be fuzzy relations. Then inf- w_T composition of P and Q is a fuzzy relation $P \circ_{W_T} Q(X, Z)$ defined as follows:

$(P \circ_{W_T} Q)(x, z) = \inf\{w_T(P(x, y),\ Q(y, z)) \mid y \in Y\}, \forall (x, z) \in X \times Z.$

Theorem 2.2.5 [K_2]: Let P(X, Y), Q(Y, Z) and R(X, Z) be fuzzy relations. Then following are equivalent.

(i) $P \overset{o}{T} Q \subseteq R$

(ii) $Q \subseteq P^{-1} \circ_{W_T} R$

(iii) $P \subseteq (Q \circ_{W_T} R^{-1})^{-1}$

Proof (i) $\Rightarrow$ (ii)

Since $P \overset{o}{T} Q \subseteq R$, $(P \overset{o}{T} Q)(x, z) \leq R(x, z)$

Then, $T[P(x, y), Q(y, z)] \leq R(x, z)$

Therefore, $w_T [P(x, y), R(x, z) \geq Q(y, z)$

Thus, $\inf\limits_{x \in X} w_T [P^{-1}(y, x), R(x, z)] \geq Q(y, z)$

Hence, $(P^{-1} \circ_{W_T} R)\ (y, z) \geq Q(y, z)$

i. e. $Q \subseteq P^{-1} \circ_{W_T} R$

(ii) $\Rightarrow$ (i)

Let $Q \subseteq P^{-1} \circ_{W_T} R$.

Then $Q(y, z) \leq \inf\limits_{x \in X} w_T [P^{-1}(y, x), R(x, z)]$

Thus, $Q(y, z) \leq w_T [P(x, y)\ R(x, z)]$

Therefore, $T[P(x, y),\ Q(y, z)] \leq R(x, z)$

Thus, $\sup\limits_{y \in Y} T[P(x, y), Q(y, z)] \leq R(x, z)$

Hence, $(P \overset{o}{T} Q)(x, z) \leq R(x, z)$

i. e. $P \overset{o}{T} Q \subseteq R$.

(i) $\Rightarrow$ (iii)

$\sup\limits_{y \in Y} T[P(x, y), Q(y, z)] \leq R(x, z)$

Thus, $T[P(x, y),\ Q(y, z)] \leq R(x, z)$

i. e. $T[Q(y, z),\ P(x, y)] \leq R(x, z)$

Therefore, $w_T[Q(y, z),\ R(x, z)] \geq P(x, y)$

Hence, $\inf\limits_{z \in Z} w_T[Q(y, z),\ R^{-1}(z, x)] \geq P(x, y)$

Thus, $Q \circ_{W_T} R^{-1}(y, x) \geq P(x, y)$

i. e. $(Q \circ_{W_T} R^{-1})^{-1}(x, y) \geq P(x, y)$

Hence, $P \subseteq (Q \circ_{W_T} R^{-1})^{-1}$

(iii) $\Rightarrow$ (i)

$P(x, y) \leq (Q \circ_{W_T} R^{-1})^{-1}(x, y)$

i .e. $P(x, y) \leq (Q \circ_{W_T} R^{-1})(y, x)$

Therefore, $P(x, y) \leq \inf\limits_{z \in Z} w_T(Q(y, z), R^{-1}(z, x)]$

Thus, $P(x, y) \leq w_T(Q(y, z), R^{-1}(z, x)]$

Therefore, $T[Q(y, z),\ P(x, y)] \leq R^{-1}(z, x)$

Hence, $\sup\limits_{y \in Y} T[Q(y, z),\ P(x, y)] \leq R^{-1}(z, x)$

i. e. $\sup_{y\in Y} T[P(x, y), Q(y, z)] \leq R(x, z)$

Thus, $P \overset{o}{T} Q(x, z) \leq R(x, z)$

Hence, $P \overset{o}{T} Q \subseteq R$.

Theorem 2.2.6 [K_2]: Let P(X, Y), Pj(X,Y), Q(Y, Z) and Qj(Y, Z) be fuzzy relations where j takes values in an index set J. Then

(i) $(\bigcup_j P_j) \underset{w_T}{\circ} Q = \bigcap_j (P_j \underset{w_T}{\circ} Q)$

(ii) $(\bigcap_j P_j) \underset{w_T}{\circ} Q \supseteq \bigcup_j (P_j \underset{w_T}{\circ} Q)$

(iii) $P \underset{w_T}{\circ} (\bigcap_j Q_j) = \bigcap_j (P \underset{w_T}{\circ} Q_j)$

(iv) $P \underset{w_T}{\circ} (\bigcup_j Q_j) \supseteq \bigcup_j (P \underset{w_T}{\circ} Q_j)$

Proof: (i) $[(\bigcup_j P_j) \underset{w_T}{\circ} Q] (x, z)$

$$= \inf_{y\in Y} w_T[(\bigcup_j P_j)(x, y), Q(y, z)]$$

$$= \inf_{y\in Y} w_T[\sup_j P_j(x, y), Q(y, z)]$$

$$= \inf_{y\in Y} \inf_j w_T[P_j(x, y), Q(y, z)]$$

$$= \inf_j \inf_{y\in Y} w_T[P_j(x, y), Q(y, z)]$$

$$= \inf_j (P_j \circ_{w_T} Q)] (x, z)$$

$$= [\bigcap_j (P_j \underset{w_T}{\circ} Q)] (x, z)$$

Therefore, $(\bigcup_j P_j) \circ_{w_T} Q = \bigcap_j (P_j \underset{w_T}{\circ} Q)$

(ii) $[(\bigcap_j P_j) \underset{w_T}{\circ} Q] (x, z)$

$$= \inf_{y\in Y} w_T[(\bigcap_j P_j)(x, y), Q(y, z)]$$

$$= \inf_{y\in Y} w_T[\inf_j P_j(x, y), Q(y, z)]$$

$$\geq \inf_{y\in Y} \sup_j w_T[P_j(x, y), Q(y, z)]$$

$$\geq \sup_j \inf_{y\in Y} w_T[P_j(x, y), Q(y, z)]$$

$$= \sup_j (P_j \underset{w_T}{\circ} Q)(x, z)$$

$$= \left[\bigcup_j (P_j \circ_{W_T} Q)\right](x, z)$$

Therefore, $(\bigcup_j P_j) \circ_{W_T} Q] = \bigcup_j (P_j \circ_{W_T} Q)$

(iii) $[P \circ_{W_T} (\bigcap_j Q_j)](x, z)$

$$= \inf_{y \in Y} w_T [P(x, y), (\bigcap_j Q_j)(y, z)].$$

$$= \inf_{y \in Y} w_T [P(x, y), \inf_j Q_j(y, z)]$$

$$= \inf_{y \in Y} \inf_j w_T [P(x, y), Q_j(y, z)]$$

$$= \inf_j \inf_{y \in Y} w_T [P(x, y), Q_j(y, z)]$$

$$= \inf_j (P_j \circ_{W_T} Q_j)(x, z)$$

$$= \left[\bigcap_j (P_j \circ_{W_T} Q_j)\right](x, z)$$

Therefore, $[P \circ_{W_T} (\bigcap_j Q_j)] = \bigcap_j (P \circ_{W_T} Q_j)$

(iv) $[P \circ_{W_T} (\bigcup_j Q_j)](x, z)$

$$= \inf_{y \in Y} w_T[P(x, y), \sup_j Q_j(y, z)]$$

$$\geq \inf_{y \in Y} \sup_j w_T[P(x, y), Q_j(y, z)]$$

$$= \bigcup_j (P_j \circ_{W_T} Q)(x, z)$$

Therefore, $P \circ_{W_T} (\bigcup_j Q_j)] \supseteq \bigcup_j (P \circ_{W_T} Q_j)$

Theorem 2.2.7 [K_2]: Let $P(X, Y)$, $Q_1(Y, Z)$, $Q_2(Y, Z)$ and $R(Z, V)$ be fuzzy relations.

If $Q_1 \subseteq Q_2$, then $P \circ_{W_T} Q_1 \subseteq P \circ_{W_T} Q_2$ and $Q_1 \circ_{W_T} R \supseteq Q_2 \circ_{W_T} R$

Proof: Since $Q_1 \subseteq Q_2$, $Q_1 \cap Q_2 = Q_1$ and $Q_1 \cup Q_2 = Q_2$

Therefore, $(P \circ_{W_T} Q_1) \cap (P \circ_{W_T} Q_2) = P \circ_{W_T} (Q_1 \cap Q_2) = P \circ_{W_T} Q_1$

Hence, $P \circ_{W_T} Q_1 \subseteq P \circ_{W_T} Q_2$

Next $(Q_1 \circ_{W_T} R) \cap (Q_2 \circ_{W_T} R) = (Q_1 \cup Q_2) \circ_{W_T} R = Q_2 \circ_{W_T} P$

Hence, $Q_2 \circ_{W_T} R \subseteq Q_1 \circ_{W_T} R$.

Theorem 2.2.8 [K_2]: Let P(X,Y), Q(Y, Z) and R(X, Z) be fuzzy relations. Then

i) $P^{-1} o_T (P \circ_{W_T} Q) \subseteq Q$

ii) $R \subseteq P \circ_{W_T} (P^{-1} o_T R)$

iii) $P \subseteq (P \circ_{W_T} Q) \circ_{W_T} Q^{-1}$

iv) $R \subseteq (R \circ_{W_T} Q^{-1}) \circ_{W_T} Q$

Proof: i) $P \circ_{W_T} Q \subseteq (P^{-1})^{-1} \circ_{W_T} Q$

Set $P \circ_{W_T} Q = Q'$, $P^{-1} = P'$, $Q = R'$,

Then $Q' = (P')^{-1} \circ_{W_T} R'$

Therefore, $P' \overset{o}{T} Q' \subseteq R'$

i. e. $P^{-1} \overset{o}{T} (P \circ_{W_T} Q) \subseteq Q$

ii) $P^{-1} \overset{o}{T} R = P^{-1} \overset{o}{T} R$

Set $P^{-1} = P'$, $R = Q'$, $P^{-1} \overset{o}{T} R = R'$

Then $P' \overset{o}{T} Q' = R'$

Therefore, $Q' \subseteq (P')^{-1} \circ_{W_T} R'$

i. e. $R \subseteq (P^{-1})^{-1} \circ_{W_T} (P^{-1} \overset{o}{T} R)$

Thus, $R \subseteq P \circ_{W_T} (P^{-1} \overset{o}{T} R)$

iii) Since $P^{-1} \overset{o}{T} (P \circ_{W_T} Q) \subseteq Q$, $[P^{-1} \overset{o}{T} (P \circ_{W_T} Q)]^{-1} \subseteq Q^{-1}$

Therefore, $(P \circ_{W_T} Q)^{-1} \overset{o}{T} P \subseteq Q^{-1}$

Set $(P \circ_{W_T} Q)^{-1} = P'$, $P = Q'$, $Q^{-1} = R'$

Then $P' \overset{\circ}{T} Q' \subseteq R'$

Therefore, $Q' \subseteq (P')^{-1} \circ_{W_T} R'$

i. e. $P \subseteq [(P \circ_{W_T} Q)^{-1}]^{-1} \circ_{W_T} Q^{-1}$

Thus, $P \subseteq (P \circ_{W_T} Q) \circ_{W_T} Q^{-1}$

(iv) Replacing P by R and Q by Q^{-1} in (iii), we get

$$R \subseteq (R \circ_{W_T} Q^{-1}) \circ_{W_T} Q$$

2.3 FUZZY RELATION EQUATIONS

The concept of fuzzy relation equation is related with the concept of composition of fuzzy relations. Composition of two fuzzy relations is a fuzzy relation. If any two components in each of these equations are known and one is unknown then these equations are known as fuzzy relation equations and fuzzy relations which satisfy the equation are called solutions of that equation.

Let us consider the fuzzy binary relations P(X, Y), Q(Y,Z) and R(X,Z) which are defined on finte sets $X = \{x_i, | i \in I\}$, $Y = \{y_j | j \in J\}$, and $Z = \{z_k | k \in K\}$, If I, J, K are sets of indices, then $P = [p_{ij}]$, $Q = [q_{jk}]$ and $R = [r_{ik}]$ be the membership matrices of P, Q and R respectively, where $p_{ij} = P(x_i, y_j)$, $q_{jk} = Q(y_j, z_k)$ and $r_{ik} = R(x_i, z_k)$.

2.3.1 Fuzzy relation equations of the type $P \circ Q = R$

The fuzzy relation equation $P \circ Q = R$ is equivalent to the *nk* simultaneous equations $\max_j \min (p_{ij}, q_{jk}) = r_{ik}$

I) If P and Q are given, then R can be determined by performing max-min composition. Obviously R is unique.

II) Suppose Q and R are given and P is to be determine.

The set of all fuzzy relations P satisfying the equation $P \circ Q = R$ is denoted by S(Q, R).

Definition 2.3.1.1 [K_2]: An element $\hat{P} \in S(Q, R)$ is called a maximal solution of the equation $P \circ Q = R$ for P, if $P \in S(Q, R)$ and $P \geq \hat{P} \Rightarrow P = \hat{P}$.

Definition 2.3.1.2 [K_2]: An element $\hat{P} \in S(Q, R)$ is called the maximum solution of the equation $P \circ Q = R$ for P, if $P \in S(Q, R)$ implies $P \leq \hat{P}$.

Definition 2.3.1.3 [K_2]: An element $\check{P} \in S(Q, R)$ is called a minimal solution of the equation $P \circ Q = R$ for P, if $P \in S(Q, R)$ and $P \leq \check{P} \Rightarrow P = \check{P}$.

Definition 2.3.1.4 [K_2]: An element $\check{P} \in S(Q, R)$ is called the minimum solution of the equation $P \circ Q = R$ for P, if $P \in S(Q, R)$ implies $P \geq \check{P}$.

The problem $P \circ Q = R$ can be divided into set of simpler problems by the matrix equation $p_i \circ Q = r_i$, for all i.

Theorem 2.3.1.5[K₂] If $\max_j q_{jk} < \max_i r_{ik}$, for some $k \in K$, then $S(Q, R) = \phi$.

Proof: Let $\max_j q_{jk} < \max_i r_{ik}$, for some $k \in K$.

Then $\max_j q_{jk0} < \max_i r_{ik0}$, for some $k = k_0$.

Therefore, there exists $i_o \in I$ such that

$$r_{i0kj} = \max_j \min (p_{ij}, q_{jk})$$

$$\leq \max_j q_{jko}$$

$$< r_{ioko}$$

Which is a contradiction.

Therefore $S(Q,R) = \phi$.

The following Theorem is given by Klir and Yuan in [K_2]

Theorem 2.3.1.6[K_2] :If $S(Q, r) \neq \phi$ then there is a unique maximum solution

$p^{\wedge} = (p_1, p_2, \ldots, p_m)$, where $p_j^{\wedge} = \min_k \sigma(q_{jk}, r_k)$ and

$$\sigma(q_{jk}, r_k) = \begin{cases} r_k, \text{ if } & \text{if } q_{jk} > r_k \\ 1, \text{ otherwise.} \end{cases}$$

Method for obtaining the minimal solution of $p \circ Q = r$, for p [K_2]

Step 1: Determine the sets

$J_k(\hat{p}) = \{j \in J \mid \min (\hat{p}_j, q_{jk}) = r_k\}$, for all $k \in k$.

Construct their cartesion product

$J(\hat{p}) = \underset{k \in K}{\times} \; J_k(\hat{p})$.

Denote the elements of $J(\hat{p})$ by

$\beta = \{\beta_k \mid k \in K\}$

Step 2: For each $\beta \in J(\hat{p})$ and each $j \in J$ determine the set

$$K(\beta, j) = \{k \in K \mid \beta_k = j\}$$

Step 3: For each $\beta \in J(\hat{p})$, generate the m-tuple by taking $g(\beta) = \{g_j(\beta) \mid j \in J\}$, where

$$g_j(\beta) = \begin{cases} \max\limits_{k \in K(\beta, j)} r_k, & \text{if } K(\beta, j) \neq \phi \\ 0, & \text{Otherwise} \end{cases}$$

Step 4: From all the m-tuples $g(\beta)$ generated in step 3, select only the minimal ones by pairwise comparison. The resulting set of m-tuples is the set $\check{S}(Q, r)$ of all minimal solutions of $p \circ Q = r$.

III) If P and R are given, then the set $S(P, R) = \{Q \mid P \text{ o } Q = R\}$ is called the solution set of $P \text{ o } Q = R$ for Q. The solution of this type of equation can be obtained by using above type and by using inverse relation. For example we want to find a solution of $P \text{ o } Q = R$ for Q. Consider the equation $Q^{-1} \text{ o } P^{-1} = R^{-1}$, the solution of this equation, Q^{-1}, can be obtained by using above type (II). The fuzzy relation $Q = (Q^{-1})^{-1}$ will be required solution of $P \text{ o } Q = R$ for Q.

2.3.2 Fuzzy relation equations of the type $P \circ_T Q = R$

In this section T denotes a continuous t-norm.

I) If P and Q are given, than R can be obtained by using the Definition 2.2.2. Obviously R is unique.

II) Suppose Q and R are given and P is to be determine

From the equation $P \circ_T Q = R$, we write

$$\max_j T(p_{ij}, q_{jk}) = r_{ik}$$

Let $S(Q, R) = \{P \mid P \circ_T Q = R\}$.

Theorem 2.3.2.1 [K_2]: If $S(Q, R) \neq \phi$, then $\hat{P} = (Q \circ_{W_T} R^{-1})^{-1}$ is the maximum solution of $P \circ_T Q = R$, for P.

Proof: Let $P_1 \in S(Q, R)$

Then $P_1 \circ_T Q = R$.

Therefore, $P_1 \subset (Q \circ_{wT} R^{-1})^{-1} = \hat{P}$

Let $G = (Q \circ_{wT} R^{-1})^{-1} \circ_T Q$

Then $G = \hat{P} \circ_T Q$ and $G^{-1} = Q^{-1} \circ_T (Q \circ_{\omega T} R^{-1}) \subseteq R^{-1}$

Thus, $G \subseteq R$

Also $G = \hat{P} \circ_T Q \supseteq P \circ_T Q = R$

Hence, $\hat{P} \circ_T Q = R$

i. e. $\hat{P} \in S(Q, R)$.

Theorem 2.3.2.2 [K_2]: Let $P_1, P_2 \in S(Q, R)$, Then

i) $P_1 \subseteq P \subseteq P_2 \Rightarrow P \in S(Q, R)$

ii) $P_1 \cup P_2 \in S(Q, R)$.

Proof: Let $P_1, P_2 \in S(Q, R)$. Then $P_1 \overset{o}{T} Q = R$ and $P_2 \overset{o}{T} Q = R$.

i) Let $P_1 \subseteq P \subseteq P_2$. Then $R = P_1 \overset{o}{T} Q \subseteq P \overset{o}{T} Q \subseteq P_2 \overset{o}{T} Q = R$

Therefore, $P \overset{o}{T} Q = R$

Hence, $P \in S(Q, R)$.

ii) $(P_1 \cup P_2) \overset{o}{T} Q = (P_1 \overset{o}{T} Q) \cup (P_2 \overset{o}{T} Q)$

$= R \cup R = R$

Hence, $(P_1 \cup P_2) \overset{o}{T} Q = R$

Therefore, $P_1 \cup P_2 \in S(Q, R)$.

III) If P and R are given, then the set $S(P, R) = \{Q \mid P \overset{o}{T} Q = R\}$ is called the solution set of $P \overset{o}{T} Q = R$ for Q. The solution of this type of equation can be obtained by using above type and by using inverse relation. For example we want to find a solution of $P \overset{o}{T} Q = R$ for Q. Consider the equation $Q^{-1} \overset{o}{T} P^{-1} = R^{-1}$, the solution of this equation, Q^{-1}, can be obtained by using above type (II). The fuzzy relation $Q = (Q^{-1})^{-1}$ will be required solution of $P \overset{o}{T} Q = R$ for Q.

2.3.3 Fuzzy relation equations of the type $P \circ w_T Q = R$

In this section T denotes a continuous t-norm.

I) If P and Q are given R can be obtained by using Definition of inf - w_T. Obviously R is unique.

II) If Q and R are given, then P is to determine

Let $S(Q, R) = \{ P \mid P \circ w_T Q = R \}$.

Theorem 2.3.3.1 [K_2]: If $S(Q, R) \neq \phi$, then $\hat{P} = R \circ w_T Q^{-1}$ is the maximum solution of fuzzy the relation equation $P \circ w_T Q = R$.

Proof: Let $P \in S(Q, R)$. Then $P \circ w_T Q = R$

Therefore, $R \circ w_T Q^{-1} = (P \circ w_T Q) \circ w_T Q^{-1} \supseteq P$

By Theorem 2.2.8 (iv), $R \subseteq (R \circ w_T Q^{-1}) \circ w_T Q \subseteq \hat{P} \circ w_T Q = R$

Hence, $(R \circ w_T Q^{-1}) \circ w_T Q = R$.

III) If P and R are given, then Q is to determine

Let $S(P, R) = \{ Q \mid P \circ w_T Q = R \}$.

Theorem 2.3.3.2 [K_2]: If $S(P, R) \neq \phi$, then $\check{Q} = P^{-1} \overset{o}{T} R$ is the minimum solution of fuzzy relation equation $P \circ w_T Q = R$.

Proof: Let $Q \in S(P, R)$. Then $P \circ w_T Q = R$

Now $P^{-1} \overset{o}{T} R = P^{-1} \overset{o}{T} (P \circ w_T Q) \subseteq Q$

Since, $R \subseteq P\ {}^{\circ}w_T\ (P^{-1}\ \overset{o}{T}\ R)$, $R \subseteq P\ {}^{\circ}w_T\ (P^{-1}\ \overset{o}{T}\ R) \subseteq P\ {}^{\circ}w_T\ Q = R$

Therefore, $P\ {}^{\circ}w_T\ (P^{-1}\ \overset{o}{T}\ R) = R$

Hence, $P^{-1}\ \overset{o}{T}\ R \in S(P, R)$

Theorem 2.3.3.3 [K_2]: For the fuzzy relation equation $P\ {}^{\circ}w_T\ Q = R$, the following hold.

(i) If $P_1, P_2 \in S(Q, R)$, then $P_1 \cup P_2 \in S(Q, R)$

(ii) If $P_1, P_2 \in S(Q, R)$ and $P_1 \subseteq P \subseteq P_2$, then $P \in S(Q, R)$

(iii) If $Q_1, Q_2 \in S(P, R)$, then $Q_1 \cap Q_2 \in S(P, R)$

(iv) If $Q_1, Q_2 \in S(P, R)$ and $Q_1 \subseteq Q \subseteq Q_2$, then $Q \in S(P, R)$

Proof: (i) Let $P_1, P_2 \in S(Q, R)$

Then $P_1\ {}^{\circ}w_T\ Q = R$ and $P_2\ {}^{\circ}w_T\ Q = R$

Now $(P_1 \cup P_2)\ {}^{\circ}w_T\ Q = (P_1\ {}^{\circ}w_T\ Q) \cap (P_2\ {}^{\circ}w_T\ Q)$

$= R \cap R = R$

Therefore, $(P_1 \cup P_2)\ {}^{\circ}w_T\ Q = R$

Hence, $P_1 \cup P_2 \in S(Q, R)$.

(ii) Let$P_1, P_2 \in S(Q, R)$ and $P_1 \subseteq P \subseteq P_2$.

Then $P_1\ {}^{\circ}w_T\ Q = R$ and $P_2\ {}^{\circ}w_T\ Q = R$

Now $R = P_1\ {}^{\circ}w_T\ Q \supseteq P\ {}^{\circ}w_T\ Q \supseteq P_2\ {}^{\circ}w_T\ Q = R$

Hence, $P\ {}^{\circ}w_T\ Q = R$

Therefore, $P \in S(Q, R)$

(iii) Let $Q_1, Q_2 \in S(P, R)$

Then $P\ {}^{c}w_T\ Q_1 = R$ and $P\ {}^{\circ}w_T\ Q_2 = R$

Now $P\ {}^{\circ}w_T\ (Q_1 \cap Q_2) = (P\ {}^{\circ}w_T\ Q_1) \cap (P\ {}^{\circ}w_T\ Q_2)$

$= R \cap R = R$

Hence, $P \circ w_T (Q_1 \cap Q_2) = R$

Therefore, $Q_1 \cap Q_2 \in S(P, R)$

(iv) Let $Q_1, Q_2 \in S(P, R)$ and $Q_1 \subseteq Q \subseteq Q_2$

Then $P \circ w_T Q_1 = R$ and $P \circ w_T Q_2 = R$

Now $R = P_1 \circ w_T Q_1 \supseteq P \overset{\circ}{wt} Q \supseteq P_2 \circ w_T Q_2 = R$

Therefore, $P \circ w_T Q = R$

Hence, $Q \in S(P, R)$.