CHAPTER 2

GENERALISED WEAKLY PSEUDO-IDEALS IN NEAR-RINGS

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GENERALISED WEAKLY PSEUDO-IDEALS IN NEAR-RINGS

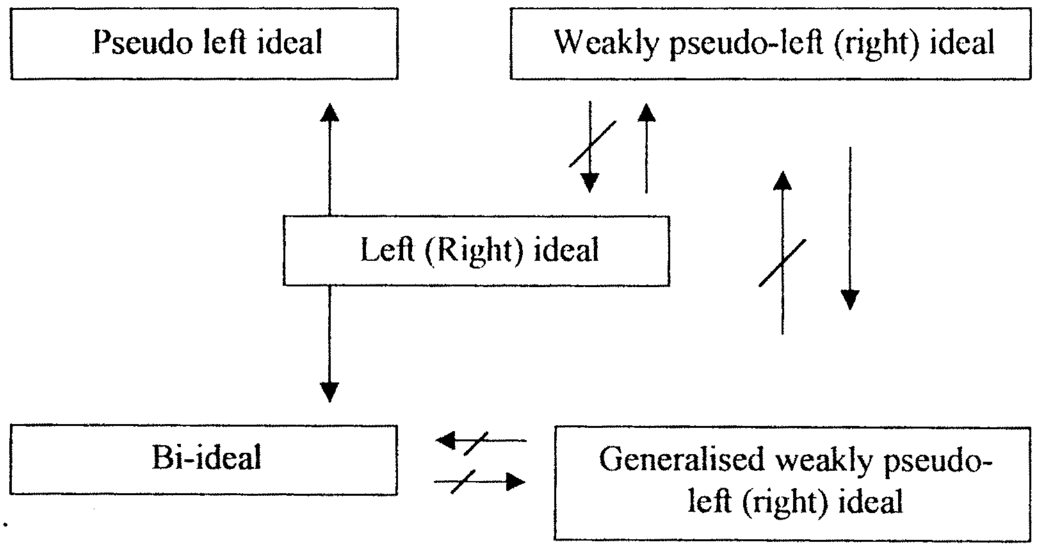
§ 2.0 Introduction:

Throughout this chapter N denotes a right near-ring. Generalization of the concept of weakly pseudo-left (right) ideal in N is done in this chapter, we name it generalised weakly pseudo-left (right) ideal in N, a help is taken of the paper 'On Generalised semi-ideals of rings' by T.K. Dutta [5].

In this chapter we have studied some interesting properties of generalised weakly pseudo-ideal in near-ring and near-fields. Efforts are also made to give necessary and sufficient conditions for a commutative near-ring without any divisors of zero to be a near-field.

It has been observed in near ring N, generalised weakly pseudo-left ideal, generalised weakly pseudo-right ideal and subnear-ring are independent concepts. But in a near-field with $N_c = \{0\}$ all these three concepts coincide.

The diagramatic representation of the relationship between ideals, pseudo-left ideals, weakly pseudo-ideals, bi-ideals and generalised weakly pseudo-ideals in N is as follows.


Pseudo left ideal
Weakly pseudo-left (right) ideal
Left (Right) ideal
Bi-ideal
Generalised weakly pseudo-left (right) ideal

§ Generalised weakly pseudo-left Ideal in a near-ring

§ 2.1 Definition and examples :

In this section we define generalised weakly pseudo-left ideal and give some examples of generalised weakly pseudo-left ideals in a near-ring N

Definition 2.1.1 :

Let < N, +, . > be a near-ring . A non-empty subset A of N with zero is called a generalised weakly pseudo-left ideal of N if it satisfies the follwing conditions.

1) $a + b \in A$, $\forall\, a,b \in A$ and
2) $n^2.a - n^2.0 \in A$, $\forall\, a \in A$ and $\forall\, n \in N$

Some examples of generalised weakly pseudo-left ideals in near-rings are given below.

Example 2.1.2 : (Pilz , page - 408)

Consider the near-ring N = {0, a, b, c } with addition and multiplication as given by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

The subsets {0,a}, { 0,b}, {0,c} are generalised weakly pseudo-left ideals in N.

□

The following example shows that not every subset I of a near-ring N containing zero is generalised weakly pseudo-left ideal of N.

<u>Example 2.1.3:</u> (Clay, 2.2, 2)

Consider the near-ring N = { 0, a, b, c} with addition and multiplication as given by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

Let I = { 0, b }. For $b \in I$ and $c \in N$, $c^2.b - c^2.0 = c.b - 0 = c \notin I$. Hence I is not a generalised weakly pseudo-left ideal in N .

□

§ <u>2.2 Properties of generalised weakly pseudo-left ideals</u>:

In this section we collect some properties of generalised weakly pseudo-left ideals of a near-ring N.

From the definitions of weakly pseudo-left ideal (by 1.1.1) and generalised weakly pseudo-left ideal, it is clear that every weakly pseudo-left ideal in a near-ring N is a generalised weakly pseudo -left ideal. But converse need not be true. This we establish by the following example.

Example 2.2.1 :-(Clay, 2.5, 29)

Consider the near-ring N = { 0, a, b, c, x, y } with addition and multiplication is given by the follwoing tables.

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	a	a	0	0
c	0	a	a	a	0	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0

Let I={ 0, a}. Here I is a generalised weakly pseudo -left ideal in N .

But as, $x + a - x = b + y = c \notin I$, for $a \in I$ and $x \in N$ we get $< I, + >$ is not normal subgroup of $< N,+ >$ Therefore I is not a weakly pseudo-left ideal .

□

From the definition of left ideal of a near-ring (see 0.1.8) , it is clear that every left ideal in a near-ring is a generalised weakly pseudo-left ideal. But converse need not be true. This is established by the following example.

Example 2.2.2 :

In example 2.2.1. consider I={ 0, a } , I is a generalised weakly pseudo-left ideal in N. But as, $x + a - x = b + y = c \notin I$, for

$a \in I$ and $x \in N$, we get $< I, + >$ is not a normal subgroup of $< N, + >$. Therefore I is not a left ideal in N.

□

Result 2.2.3 : Intersection of any collection of generalised weakly pseudo-left ideals in a near-ring N is a generalised weakly pseudo - left ideal.

Proof : - Take $I = \cap \{ I_i \ / \ I_i$, is generalised weakly pseudo-left ideal in N $\}$. To prove that I is a generalised weakly pseudo-left ideal.

since $0 \in I_i$, where I_i is generalised weakly pseudo -left ideal in N,

Therefore $0 \in \cap_i I_i \Rightarrow \cap_i I_i \neq \emptyset$ ------------------ (1)

Let $a,b \in I = \cap_i I_i$

Therefore $a,b \in I_i$, $\forall I_i$

$\Rightarrow a+b \in I_i$, $\forall$ generalised weakly pseudo left ideal I_i

$\Rightarrow a+b \in \cap_i I_i = I$

Hence $a+b \in I$, $\forall a,b \in I$ ------------------ (2)

Now let $a \in I$, $n \in N$

Therefore $a \in \cap_i I_i$, $n \in N$

$\Rightarrow a \in I_i$, $n \in N$, $\forall I_i$

$\Rightarrow n^2 . a - n^2 .0 \in I_i$, $\forall I_i$

[Since I_i is a generalised weakly pseudo -left ideal in N]

Therefore $n^2.a - n^2.0 \in \cap_i I_i = I$

Hence $n^2.a - n^2.0 \in I$, $\forall a \in I$ and $\forall n \in N$ ------------------ (3)

Therefore, from (1), (2) and (3). I is a generalised weakly pseudo-left ideal of N.

□

As N itself is a generalised weakly pseudo-left ideal, by definition of a Moore family of subsets of a given set (see 0.1.15) and by the Result 2.2.3 , we get the following corollary.

Corollary 2.2.4 : Set of all generalised weakly pseudo-left ideals in a near-ring N forms a Moore family of subsets of N .

□

Union of any two generalised weakly pseudo-left ideals need not be a generalised weakly pseudo-left ideal. For this consider the following example.

Example 2.2.5 (Clay, 2.2, 13)

Consider the near-ring N = { 0, a, b, c} under the addition and multiplication defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Here A={ 0, a } and B= { 0,b } are generalised weakly pseudo-left ideals in N. Hence AUB = { 0, a, b}

As , a + b = c $\notin$ AUB , for a,b $\in$ AUB

Therefore AUB is not a generalised weakly pseudo-left ideal in N .

□

§ Generalised weakly pseudo-right Ideal in a near-ring

§ 2.3 Definition and examples:

In this section we define generalised weakly pseudo-right ideal and give some examples of generalised weakly pseudo-right ideals in a near-ring N.

Definition 2.3.1 :

Let < N, +, . > be a near-ring . A non-empty subset A of N with 0 is called a generalised weakly pseudo-right ideal in N if it satisfies the following conditions .

(1) $a+b \in A$, $\forall$ $a,b \in A$ and

(2) $a.n^2 \in A$, $\forall$ $a \in A$ and $\forall$ $n \in N$.

□

Some examples of generalised weakly pseudo-right ideals in near-rings are given below.

Example 2.3.2 : (Pilz, page -408)

Consider the near-ring N= { 0, a, b, c } with addition and multiplication as given by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Let I ={ 0, a} . Here I is a generalised weakly pseudo-right ideals in N. Also { 0, b} and { 0, c} are generalised weakly pseudo-right ideals in N .

□

Every generalised weakly pseudo-left ideal in a near-ring N need not be a generalised weakly pseudo-right ideal. This we establish by the following example .

Example 2.3.3 (Clay, 2.2,13)

Consider the near-ring N = { 0, a, b, c} with addition and multiplication defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Let I ={ 0, a} . Here I is generalised weakly pseudo-left ideal of N but not a generalised weakly pseudo-right ideal of N. Because $a.c^2 = a.c = c \notin I$, for $a \in I$ and $c \in N$.

□

Every generalised weakly pseudo-right ideal in a near-ring N need not be a generalised weakly pseudo-left ideal. This we establish by the following example.

Example 2.3.4 : (Clay, 2.2.2)

Consider the near-ring N = { 0, a, b, c } with addition and multiplicaton as given by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

Let I = { 0, b} . Here I is a generalised weakly pseudo-right ideal of N but not a generalised weakly pseudo-left ideal of N. Because $c^2.b - c^2.0 = c.b - 0 = c \notin I$, for $b \in I$ and $c \in N$.

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§ 2.4 Properties of generalised weakly pseudo-right ideals :

In this section we collect some properties of generalised weakly pseudo-right ideals of a near-ring N.

From the definition of weakly pseudo-right ideal (by 1.3.1) it is clear that every weakly pseudo-right ideal in a near-ring is a generalised weakly pseudo-right ideal. But converse need not be true. This we establish in the following example.

Example 2.4.1 :(Clay, 2.5, 29)

Consider the near-ring N= { 0,a,b,c,x,y } with addition and multiplication as given by the following tables.

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	a	a	0	0
c	0	a	a	a	0	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0

Let I = {0,a}. Here I is a generalised weakly pseudo-right ideal in N. But as, x+a-x= b+y =c$\notin$I, for a$\in$I. and x $\in$ N. Hence I is not a weakly pseudo-right ideal of N.

□

From the definition of generalised weakly pseudo-right ideal in a near-ring . It is clear that every right ideal in a near-ring is a generalised weakly pseudo-right ideal . But converse need not be true. This is established in the following example.

<u>Example 2.4.2</u> : (Pilz, page - 408)

Consider the near-ring N = { 0, a, b, c } with addition and multiplication as given by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Let $I = \{0,a\}$. I is a generalised weakly pseudo-right ideal of N. But as , $a.a = b \notin I$ for $a \in I$ and $a \in N$. Hence I is not a right ideal of N.

As usual we get,

Result 2.4.3 : Intersection of any collection of generalised weakly pseudo-right ideals in a near-ring N is a generalised weakly pseudo-right ideal.

Proof: Take $I = \cap \{ I_i / I_i$ is generalised weakly pseudo-right ideal in N$\}$. To prove that I is generalised weakly pseudo-right ideal in N. Since, $0 \in I_i$, Where I_i is a generalised weakly pseudo-right ideal.

Therefore , $0 \in \cap_i I_i = I$

$\Rightarrow \cap_i I_i \neq \varnothing$ ----------- (1)

Let $a,b \in \cap_i I_i$

Therefore, $a, b \in I_i$, $\forall I_i$

$\Rightarrow a + b \in I_i$, $\forall$ Generalised weakly pseudo-right ideal I_i .

Hence , $a + b \in \cap_i I_i = I$

Therefore, $a + b \in I$, $\forall a , b \in I$ ----------- (2)

Let $a \in I$, $n \in N$

Therefore , $a \in \cap_i I_i$, $n \in N$

$\Rightarrow a \in I_i$. , $n \in N$, $\forall I_i$

$\Rightarrow a.n^2 \in I_i$, $\forall I_i$ (Since , I_i is generalised weakly pseudo-right ideal).

$\Rightarrow a.n^2 \in \cap_i I_i = I$

Hence $a.n^2 \in I$, $\forall a \in I$ and $\forall n \in N$ ----------- (3)

Therefore from (1) , (2) and (3) . we get I is a generalised weakly pseudo-right ideal of N.

□

As N itself is a generalised weakly pseudo-right ideal of N, by definition of a Moore family of subsets of a given set (See 0.1.15) and by the Result 2.4.3. we get the following corollary.

Corollary 2.4.4 : Set of all generalised weakly pseudo-right ideals in a near-ring N forms a Moore family of subsets of N.

□

Union of any two generalised weakly pseudo -right ideals need not be a generalised weakly pseudo - right ideal. For this consider the following example.

Example 2.4.5 : (Pilz, page-408)

Consider the near-ring N= { 0, a, b, c} with addition and multiplication is defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Let A = {0, a} and B= {0 , b} be two generalised weakly pseudo-right ideals of N.

Therefore, AUB = { 0 ,a , b}

As, $a+b=c \notin A \cup B$

Hence AUB is not a generalised weakly pseudo - right ideal of N .

□

For any two non-empty subsets A and B of N we define, $A+B = \{x+y \mid x \in A,\ y \in B\}$. When A and B are generalised weakly pseudo-right ideals we get,

Result 2.4.6 : Let $< N, +, . >$ be an abelian near-ring . Let A and B be two generalised weakly pseudo-right ideals of N. Then A+B is the smallest generalised weakly pseudo-right ideal of N containing both A and B.

Proof : Here $A+B = \{a+b / a \in A, b \in B)$. Since , $0 \in A$, $0 \in B$.

Therefore $0=0+0 \in A+B$ Hence $A+B \neq \emptyset$ -------------------- (1)

Let $x,y \in A+B$

Therefore, $x = a_1+b_1$ and $y = a_2+b_2$, for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Therefore, $x+y = (a_1+b_1) + (a_2+b_2)$

$= a_1+(b_1 + a_2) +b_2$

$= a_1+(a_2 + b_1) + b_2$ (N being abelian)

$= (a_1+a_2) + (b_1 + b_2)$

Hence $x+y \in A +B$. [Since , $a_1+a_2 \in A$ and $b_1 + b_2 \in B$]

Therefore, $x+y \in A+B , \forall x, y \in A+ B$ ---------- (2)

Let $n \in N$ and $x \in A+B$.

Therefore $x = a_1 + b_1$ for some $a_1 \in A$ and $b_1 \in B$

As , A and B are generalised weakly pseudo-right ideals of N.we get,

$x. n^2 = (a_1 + b_1).n^2 = a_1.n^2 + b_1. n^2 \in A + B.$

Therefore, $x \cdot n^2 \in A+B$, $\forall$ $x \in A+B$ and $\forall$ $n \in N$ --------- (3)

Hence from (1) , (2) and (3). A+B is a generalised weakly pseudo-right ideal of N. Since, $0 \in B$ for any $a \in A$, $a = a+0 \in A+B$. Therefore $A \subseteq A+B$. Similarly $B \subseteq A+B$. Let C be any generalised weakly pseudo-right ideal of N such that $A \subseteq C$ and $B \subseteq C$.

To prove that $A+B \subseteq C$.

Let $x \in A+B$.

Therefore $x = a + b$ for some $a \in A$ and $b \in B$.

Since $A \subseteq C$, therefore $a \in C$ and $B \subseteq C$ therefore $b \in C$.

Hence, $x = a + b \in C$. [Since C is generalised weakly pseudo-right ideal of N]

Therefore $A+B \subseteq C$. Hence A+B is the smallest generalised weakly pseudo-right ideal containing both A and B .

□

For any two generalised weakly pseudo-right ideals A and B we define

$$AoB = \{ \sum_{i=1}^{n} a_i b_i \ / \ a_i \in A , \ b_i \in B , n \text{ is finite} \}$$

An interesting property of AoB is stated in the following result.

Result 2.4.7 : Let < N, +, . > be a near-ring. Let A and B be two generalised weakly pseudo-right ideals of N. Then AoB is a generalised weakly pseudo-right ideal of N.

Proof : Here

$$AoB = \{ \sum_{i=1}^{n} a_i b_i \,/\, a_i \in A,\; b_i \in B,\; n \text{ is finite} \}$$

Since, $0 \in A$ and $0 \in B$.

Therefore $0 \in AoB$.

Hence $AoB \neq \varnothing$ ------------------------ (1)

Let $x, y \in AoB$

Therefore $x = \sum_{i=1}^{r} a_i b_i$ and $y = \sum_{i=r+1}^{m} a_i b_i$,

where $a_i \in A$ and $b_i \in B$, r and m are finite.

$$\text{Therefore } x+y = \sum_{i=1}^{r} a_i b_i + \sum_{i=r+1}^{m} a_i b_i ,$$

$$= \sum_{i=1}^{m} a_i b_i \in AoB$$

Thus , $x+y \in AoB$, $\forall x, y \in AoB$ ---------- (2)

Let $x \in AoB$ and $n \in N$.

Therefore $x = \sum_{i=1}^{r} a_i b_i$ and $n \in N$, where $a_i \in A$ and $b_i \in B$

$$\text{Hence } x.n^2 = \left(\sum_{i=1}^{r} a_i b_i \right) .n^2$$

$$= \sum_{i=1}^{r} a_i \,(b_i .n^2) \in AoB$$

[Since $b_i \in B$, $n \in N \Rightarrow b_i n^2 \in B$

Therefore $x . n^2 \in AoB$, $\forall$ $x \in AoB$ and $\forall$ $n \in N$.

Thus , from (1). , (2) and (3) we get AoB is a generalised weakly pseudo-right ideal.

□

A sufficient condition for a subnear-ring A of a near-ring N to be a generalised weakly pseudo-right ideal is given in the following result.

Result 2.4.8 : Let A and B be two subnear-ring of a near-ring N such that $A^2 = A$ and let A be a right ideal of B.Let B be a generalised weakly-right ideal of N Then A is a generalised weakly pseudo-right ideal of N.

Proof : Obivously $a + b \in A$ whenever a, $b \in A$ .Now let $a \in A$. Then a $= a_1 . a_2$ where $a_1, a_2 \in A$ .Now for any $x \in N$, $a . x^2 = (a_1 .a_2) x^2 = a_1 (a_2.x^2) \in a_1 .B \subseteq A$ [Since $a_2 \in A \subset B$ and B is generalised weakly pseudo-right ideal of N and A is a right ideal of B]. Hence A is generalised weakly pseudo-right ideal of N.

□

One more sufficient condition for a subnear-ring I of N to be a generalised weakly pseudo-right ideal is furnished in the following result.

Result 2.4.9 : Let $N \neq \{0\}$ be a zero-symmetric regular near-ring with identity. Let J be a generalised weakly pseudo-right ideal and also a subnear-ring of N without nilpotent elements and let I be a right ideal of J,then I is a generalised weakly pseudo-right ideal of N.

Proof : Since, the zero-symmetric regular near-ring N does not contains nilpotent elements , therefore all the idempotents of N are central.[See Result 0.2.4.] First we shall show that J is regular . Let $a \in J \subset N$.Then there exists an element $x \in N$ such that $a = a.x.a$ [Since N is regular (see Def. 0.1.16)] Let $b = x.a.x$. Therefore $b = x.a.x. = x.(a.x) = (a.x).x = a(x.x) = a.x^2 \in J$.[Since a.x being an idempotent element of N is a central idempotent , therefore $x.(a.x) = (a.x).x$ (See Result 0.2.4 and Def. 0.1.20)] Hence $a.b.a = a.(x.a.x).a = (a.x.a).(x.a) = a.(x.a) = a.x.a = a$.

i.e. If $a \in J$ then there exists $b = x.a.x \in J$ such that $a.b.a = a$. Therefore J is regular.Now we shall show that $I^2 = I$.

Let $a \in I \subseteq J$. Since J is regular, there exists an element $b \in J$ such that $a = a.b.a = (a.b).a \in I^2$ [I is a right ideal of J, therefore $a.b \in I$ for $a \in I$ and $b \in J$]

Thus $I \subseteq I^2$ ------ (1)

Obviously $I^2 \subseteq I$ ------ (2)

Hence from (1) and (2) $I^2 = I$

Thus from Result 2.4.8. I is generalised weakly pseudo-right ideal of N.

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In a commutative near-ring (see definition 0.1.3) the two concepts of generalised weakly pseudo-right ideal and of generalised weakly pseudo-left ideal coinside. This we prove in the following result.

Result 2.4.10 : In a commutative near-ring N,a non-empty subset A of N is generalised weakly pseudo-left ideal iff A is generalised weakly pseudo-right ideal.

Proof : Let < N, +, .> be a commutative near-ring . Let A be a generalised weakly pseudo-left ideal of N. To prove that A is generalised weakly pseudo-right ideal of N. Obviously a+b $\in$ A ,$\forall$ a,b $\in$ A. Let x $\in$ N and a $\in$ A,therefore $a.x^2 = x^2.a = x^2.a - 0 = x^2.a - 0.x^2 = x^2.a - x^2.0$ $\in$ A [Since , N is commutative and A is generalised weakly pseudo-left ideal and 0.n = 0 $\forall$ n $\in$ N .(See Result 0.2.1)] Hence A is generalised weakly pseudo-right ideal.

Conversely, let A be a generalised weakly pseudo-right ideal. To prove that A is generalised weakly pseudo-left ideal. Obviously a+b $\in$ A , $\forall$ a,b $\in$ A.let x$\in$N , a $\in$ A, therefore $x^2.a - x^2.0 = a.x^2 - 0.x^2 = a.x^2 - 0 = a.x^2$ $\in$ A [Since N is commutative , A is generalised weakly pseudo-right ideal and 0.n =0, $\forall$ n $\in$ N (See Result 0.2.1)] Hence A is generalised weakly pseudo-left ideal of N.

□

§ Generalised weakly pseudo-ideal in a near-ring

§ 2.5 Definition and examples :

In this section we define generalised weakly pseudo-ideal in a near-ring N and give some examples of a generalised weakly pseudo-ideal in a near-ring.We know every generalised weakly pseudo-right ideal in a near-ring need not be a genralised weakly pseudo-left ideal and every generalised weakly pseudo-left ideal in a near-ring need not be a generalised weakly pseudo-right ideal (see Example 2.3.4 and Example 2.3.3) . This motivates us to define.

Definition 2.5.1 :

Let $< N, +, . >$ be a near- ring . A non-empty subset A of N with zero is called a generalised weakly pseudo-ideal if it satisfies the following conditions.

(1) $a + b \in A$, $\forall\, a , b \in A$.

(2) $n^2. a - n^2.0 \in A$, $\forall\, a \in A$ and $\forall\, n \in N$.

(3) $a.n^2 \in A$, $\forall\, a \in A$ and $\forall\, n \in N$.

□

Some examples of generalised weakly pseudo-ideals in near-ring are given below.

Example 2.5.2 : (Pilz , page - 408)

Consider the near-ring N = { 0, a, b, c} with addition and multiplication defined by the following tables,

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Let I = { 0, a }. Here I is generalised weakly pseudo-left ideal as well as generalised weakly pseudo-right ideal in N. Hence I is generalised weakly pseudo-ideal in N.

□

Example 2.5.3 : (Clay, 2.2, 2)

Consider the near-ring N = { 0, a, b, c } with addition and multiplication defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

Let I = { 0, a } . Here I is generalised weakly pseudo-left ideal as well as generalised weakly pseudo- right ideal in N. Hence I is generalised weakly pseudo-ideal in N.

□

From the definition of generalised weakly pseudo-ideal of a near-ring , it is clear that every weakly pseudo-ideal in a near-ring is a generalised weakly pseudo-ideal . But converse need not be true. This is established by the following example.

Example 2.5.4 : (Clay, 2.5, 29)

Consider the near-ring N = { 0, a, b, c, x, y } with addition and multiplication are defined as.

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	a	a	0	0
c	0	a	a	a	0	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0

Let I = { 0, a}. Here I is a generalised weakly pseudo-ideal in N . But as, $x + a - x = b + y = c \notin I$, for $a \in I$ and $x \in N$ Hence < I, +> is not a normal subgroup of <N + >. Therefore I is not a weakly pseudo-ideal.

□

From the definition of generalised weakly pseudo-ideal of a near-ring, it is clear that every ideal of a near-ring is a generalised weakly pseudo-ideal. But converse need not be true. This we establish by the following example.

Example 2.5.5 :

In Example 2.5.4 Consider I = { 0, a}. < I, + > is not a normal subgroup of < N, + >. Hence I is not an ideal of N. But I is a generalised weakly pseudo- ideal of N.

| |

§ 2.6 Properties of generalised weakly pseudo-ideals:

In this section we collect some properties of generalised weakly pseudo-ideals of a near-ring N.

From Result 2.2.3 and Result 2.4.3 we get the following result.

Result 2.6.1 : Intersection of any collection of generalised weakly pseudo-ideals of N is a generalised weakly pseudo-ideal of N.

| |

From Result 2.6.1 and from definition of Moore family (see Def.0.1.15) we get,

Result 2.6.2 : Set of all generalised weakly pseudo ideals in a near-ring N forms a Moore family of subsets of N.

| |

Union of any two generalised weakly-pseudo-ideals of N need not be a generalised weakly pseudo-ideal .This we establish in the following example.

Example 2.6.3 : (Pilz, page - 408)

Consider the near-ring N = { 0, a, b, c} with addition and multiplication defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Let A= { 0, a } , B = { 0, b } be two generalised weakly pseudo-ideals of N.

AUB = { 0, a, b }

For a, b $\in$ AUB , a + b = c $\notin$ AUB.

Therefore AUB is not a generalised weakly pseudo-ideal of N.

□

Now we establish the fact that the three concepts (1) generalised weakly pseudo-left ideal of N (2) generalised weakly pseudo-right ideal of N and (3) subnear-ring of N are completely independent in a near-ring N.

(1) Generalised weakly pseudo-left ideal in a near-ring N need not be generalised weakly pseudo-right ideal in N (From 2.3.3)

(2) Generalised weakly pseudo-right ideal in a near-ring N need not be generalised weakly pseudo-left ideal in N (From 2.3.4)

(3) Every generalised weakly pseudo-ideal in a near-ring N need not be a subnear-ring of N. This we establish in the following example.

Example 2.6.4 : (Pilz, page – 408)

Consider the near-ring N = { 0, a, b, c } with addition and multiplication defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Let A= { 0, a}. Here A is a generalised weakly pseudo-ideal of N. But as, a.a = b $\notin$ A , for a $\in$ A.Hence A is not a subnear-ring of N.

□

(4) Every subnear-ring of N need not be a generalised weakly pseudo-ideal of N. This we establish by the following example.

Example 2.6.5 : (Clay, 2.2, 2)

Consider the near-ring N = { 0, a, b, c } addition and multiplication defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

Let I = { 0, b }.Here I is a subnear-ring of N. But as , $c^2.b-c^2.0 = c.b - 0 = c \notin I$. Hence I is not a generalised weakly pseudo- ideal of N

□

But these concepts coincide when N is a near-field with $N_c = \{0\}$. This we prove in the following result.

Result 2.6.6 : Let S be a subnear-ring of a near-field N for which $N_c = \{0\}$. Then the following are equivalent.

(a) S is a generalised weakly pseudo-left ideal.

(b) S is a generalised weakly pseudo-right ideal.

(c) S is a generalised weakly pseudo-ideal.

(d) S is a subnear-field and generalised weakly pseudo-left ideal.

Proof :

(a) ⇒ (b)

Let S be a generalised weakly pseudo-left ideal of N and $a \in S$ ($a \neq 0$). Since N is a near-field for which $N_c = \{0\}$ we get $n.0 = 0$ $\forall\, n \in N$ (See Result 0.2.5). Therefore, $a^{-1} = a^{-1} - 0 = (a^{-1})^2.a - (a^{-1})^2 . 0$. Hence $a^{-1} \in S$ as S is a generalised weakly pseudo-left ideal.

As $a \in S$ and $a^{-1} \in S$ we get $a \,.\, a^{-1} \in S$ i.e. $1 \in S$. Now for any $x \in N$, $x^2 = x^2\text{-}0 = x^2.1 - x^2.0$ S being generalised weakly pseudo-left ideal. $x^2 \in S$. Thus $a \in S$ and $x^2 \in S$ imply $a.x^2 \in S$. Consequently S is also a generalised weakly pseudo-right ideal.

(b) $\Rightarrow$ (c)

Let S be a generalised weakly pseudo-right ideal of N. We show that S is a generalised weakly pseudo-left ideal. Let $0 \neq a \in S$. Then $a^{-1} = 1.a^{-1} = (a.\, a^{-1}).a^{-1} = a.(a^{-1}\, .a^{-1}\,) = a.(a^{-1})^2 \in S$. S being generalisd weakly pseudo-right ideal.. Thus $a \in S$ and $a^{-1} \in$. S will imply $a.a^{-1} \in S$. i.e. $1 \in S$. Now for any $x \in N$, $x^2 = 1.\, x^2 \in S$. S being generalised weakly pseudo-right ideal. Since S is a subnear-ring . Therefore $x^2.a\text{-}x^2.0 \in S$. Hence S is generalised weakly pseudo-left ideal. Therefore S is generalised weakly pseudo-ideal.

(c) $\Rightarrow$ (d)

Let S be a generalised weakly pseudo-ideal. Let $a(\neq 0) \in S$ Now $a^{-1} = 1.a^{-1} = (\, a.\, a^{-1}) \,.a^{-1} = a\,(\,a^{-1}\,.a^{-1}) = a.(a^{-1})^2$ .S being generalised weakly pseudo-right ideal. we get $a^{-1} \in S$. Thus $a \in S$ and $a^{-1} \in S$ will imply $1 = a.a^{-1} \in S$. Therefore S is a subnear-field of N and S is generalised weakly pseudo-left ideal.

(d)$\Rightarrow$ (a) is obviously true.

Thus (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and hence all the statements are equivalent.

□

A necessary and sufficient condition for a subnear-ring of a near-field N with $N_c = \{0\}$ to be a generalised weakly pseudo-ideal is in the following result.

Result 2.6.7 : In a near-field N for which $N_c = \{0\}$, a subnear-ring S will be a generalised weakly pseudo-ideal iff $rSr \subseteq S$ and $rSr^{-1} \subseteq S$ for every $r(\neq 0) \in N$.

Proof : Let a subnear-ring S of a near-field N be a generalised weakly pseudo ideal , then from the Result 2.6.6 . it follows that S is a subnear-field . Let $0 \neq a \in S$ then $r.a.r = (r.a.r).1 = (r.a.r)(a.a^{-1}) = (r.a).(r.a).a^{-1} = (r.a)^2.a^{-1} - 0 = (r.a)^2.\ a^{-1} - (r.a)^2.0$ [$0=(r.a)^2.0$ since N is a near-field for which $N_c=\{0\}$ therefore $n.0 = 0$, $\forall\ n \in N$. (See Result 0.2.5)]. Thus $r.a.r \in S$, $\forall\ 0 \neq a \in S$ and if $a = 0 \in S$, then $r.a.r = 0 \in S$.

Thus , $r.a.r \in S$, $\forall\ a \in S$.

Hence $rSr \subseteq S$, $\forall\ 0 \# r \in N$.

Also $r.a.r^{-1} = (r.a).1.r^{-1} = (r.a).(r.r^{-1})(r^{-1}) = (r.a.r).(r^{-1})^2 \in S$ [Since $r.a.r \in S$ and S is generalised weakly pseudo-right ideal].

Thus $r.a.r^{-1} \in S \quad \forall\ a \in S$.

Therefore $rSr^{-1} \subseteq S$, $\forall\ r(\neq 0) \in N$.

Conversly let S be a subnear-ring of N such that $rSr \subseteq S$ and $rSr^{-1} \subseteq S$ for every $r(\neq 0) \in N$. Let $a \in S$ and let $r \in N$.

If $r = 0$ then $r^2.a - r^2.0 = 0 \in S$.

If $r \neq 0$ then $r^2.a - r^2.0 = r^2.a - 0 = r^2.a = r(r.a.).1 = r.(r.a).(r.r^{-1}) = r(r.a.r).r^{-1} \in rSr^{-1} \subseteq S$ [Since $r^2.0=0$(see Result 0.2.5)]

Therefore $r^2.a - r^2.0 \in S$, $\forall\ r \in N$ and $\forall\ a \in S$

Hence S is a generalised weakly pseudo-left ideal of N.

Therefore from Result 2.6.6. S is a generalised weakly pseudo-ideal.

□

In a commutative near-ring with identity element we have,

Result 2.6.8 : Let J be a generalised weakly pseudo-ideal and also a subnear-ring of a commutative near-ring N with identity element then an ideal I of J is a generalised weakly pseudo-ideal of N.

Proof : First we shall show that J is regular. Let $a \in J \subseteq N$ there exists $x \in N$ such that $a = a.x.a$ [since N is regular]. Let $b = x.a.x$. Thus $b = x.a.x = (x.a).x = (a.x).x = a.x^2 \in J$. Now , $a.b.a = a.(x.a.x).a = (a.x.a).(x.a.) = a.x.a=a$. Thus given $a \in J$ there exists $b= x.a.x$ in J such that $a.b.a = a$.This shows that J is regular.

Let $a \in I \subseteq J$. Since J is regular , there exists an element $b \in J$ such that $a = a.b.a$, $a = a.b.a = (a.b).a \in I^2$ [Since I is an ideal of J therefore $a.b \in I$ $\forall a \in I$ and $\forall b \in J$]

Therefore $I \subseteq I^2$

obviously $I^2 \subseteq I$

Hence $I^2 = I$

From result 2.4.8. It follows that I is a generalised weakly pseudo-right ideal of N. Therefore I is a generalised weakly pseudo-right ideal of N [Result 2.4.10].

Hence I is generalised weakly pseudo-ideal of N.

□

For any subset T of N we define $\overline{T} = \{ t^2 / t \in T \}$.In connection with this we have.

Result 2.6.9 : If a commutative near-ring N is regular then $A \cap B = A \circ \overline{B}$, where A is generalised weakly pseudo-ideal and B is a bi-ideal of N .

Proof : Let N be a commutative regular near-ring . We know

$$A \circ \overline{B} = \{ \sum_{i=1}^{n} a_i b_i^2 / a_i \in A , b_i \in B \quad , n \text{ is finite}\}$$

Since A is generalised weakly pseudo ideal. $a_i.b_i^2 \in A, \forall i$ So $A \circ \overline{B} \subseteq A$.

Again since each $ai.bi^2 = (a_i.b_i).b_i = (b_i.a_i).b_i$ [by commutativity of N].

Therefore $a_i.b_i^2 \in BNB , \forall i$ [Since $bi.ai.b_i \in BNB ,\forall i$].

$$\begin{aligned} \text{Now } a_i.b_i^2 &= b_i.a_i.b_i \\ &= (b_i.a_i).(0+b_i) && \text{[See Result 0.2.1]} \\ &= (b_i.a_i).(0 + b_i) - 0.b_i.a_i && \text{[Since } 0.n=0 \ \forall n \in N \text{]} \\ &= (b_i.a_i).(0+b_i) - b_i.a_i.0 && \text{[by commutativity of N]} \\ &= b_i.a_i.(0+b_i) - b_i.a_i .0 \in (BN) * B && \text{[See definition 0.1.13]} \end{aligned}$$

Therefore $a_i.b_i^2 \in BNB \cap (BN) * B \subseteq B$ [See definition 0.1.13]

Hence $\sum_{i=1}^{n} a_i b_i^2 \in B$

Therefore $A \circ \overline{B} \subseteq B$

Therefore $A \circ \overline{B} \subseteq A \cap B$ ------------------ (1)

On the other hand , let $a \in A \cap B \subseteq N$

Since N is a regular near-ring.

Thus for $a \in N$ there exists an element $x \in N$ such that $a=a.x.a = a.(x.a)^4$

[Since x.a is an idempotent see Result 0.2.7]

$= a(x.a.x.a)^2$

$= a(a.x.x.a)^2$ [By commutativily of N]

$=a\ (a.x^2.a)^2 \in A \circ \bar{B}$

[Since $a\in B$, $x^2\in N \Rightarrow a.x^2a \in BNB$ and $a.x^2 \in BN$, $a.x^2a=a.x^2(0+a) =$ $a.x^2(0+a) - 0 = a.x^2(0+a) - 0.(a.x^2) = a.x^2\ (0+a) - (a.x^2).0 \in BN * B$.

Therefore $a.x^2.a \in BNB \cap (BN)^* B \subseteq B$. See Def. 0.1.13]

Therefore $A\cap B \subseteq A \circ \bar{B}$.

From (1) and (2) we get $A\cap B = A\circ \bar{B}$.

□

Every bi-ideal in a near-ring N need not be a generalised weakly pseudo-ideal . This we establish in the following example.

Example 2.6.10 : (Clay , 2.2 13)

Consider the near-ring N ={ 0,a,b,c} wiith addition and multiplication defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Let I ={0,a} Here I is a bi-ideal of N. But I is not a generalised weakly pseudo-ideal of N. Because , for $a\in I$ and $c\in N$, $a.c^2 = a.c = c$ .Therefore $a.c^2 \notin I$.

□

Every generalised weakly pseudo-ideal in a near-ring need not be a bi-ideal . This is a proved in the following example

Example 2.6.11 : Let < N, +, . > be the near-ring of all integers. Let A = {a / a ≥ 0, a is an integer}. Here A is a generalised weakly pseudo-ideal of N. But additive inverse of any element in A does not exists in A. Hence < A,+> is not a subgroup of < N, +>. Therefore A is not a bi-ideal of N.

□

Result 2.6.12 : A Commutative near-ring N without any divisor of zero and $N_d \neq \{0\}$ will be a near-field iff for any generalised weakly pseudo ideal A, $a \in N \backslash A$ (the complement of A in N) and $x(\neq 0) \in N$ implies $x^2a \in N \backslash A$

Proof : First suppose commutative near-ring N without any divisor of zero will be a near-field . For any $a \in N \backslash A$ and $x(\neq 0) \in N$ $x^2.a \in A$ implies $a=1.a= (x^{-1}.x).a = (x^{-1}.x)^2.a=(x^{-1})^2 .x^2.a = x^2.a. (x^{-1})^2 \in A$ (Since A is generalised weakly pseudo- ideal of N) Which is contradiction .Therefore $x^2.a \notin A$ Hence $x^2.a \in N \backslash A$

Conversely, let the given condition hold in N. Let $a(\neq 0) \in N$. We shall show Na=N.

If possible let $Na \neq N$ and $b \in N \backslash Na$. Then $a^2b \in N \backslash Na$ but $a^2.b=b.a^2 = (ba).a \in Na$.

This is contradiction so Na=N. Hence N is a near- field (From Result 0.2.8)

◈ ◈ ◈ ◈ ◈ □