CHAPTER-4

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Discrete Wavelet Transform

(4.1) Introduction:

This chapter introduces a type of wavelet representation that has assumed considerable practical significance because of its link to digital filtering. Our aim is to develop the ideas entirely through an example involving the Haar wavelet.

Definition:

If f(t) is any square integral function then continuous time wavelet transform of f(t) with respect to a wavelet $\varphi(t)$ is defined as

$$W(a,b) = \int_{-\infty}^{\infty} f(t) \; (1/\sqrt{|a|})\varphi^*\left[(t-b)/a\right] dt$$

Continuous Wavelet Transform maps a one-dimensional function f(t) to a function W(a,b) of two continuous real variable a and b which are wavelet dilation and translation respectively. The region of support of W(a,b) is defined as the set of ordered pairs (a,b) for which $W(a,b) \neq 0$.

Let us introduce one type of non-redundant wavelet representation of form

$$f(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} d(k,l)\, 2^{-k/2} \varphi(2^{-k} t - l) \qquad \text{-------(4.1.1)}$$

where k is integer and at any dilation 2^k translates takes the values $2^k l$, where l is again integer. The two dimensional sequence $d(k, l)$ is commonly referred as the discrete wavelet transform of f(t). The representation in (4.1.1) is useful in Multiresolution Analysis (MRA) in constructing the approximations to functions in various subspaces of a linear vector space.

(4.2) Approximating Vectors in Nested Subspaces of a Finite Dimensional Linear Vector Space:-

Definition(Inner product)

Given two finite energy signals x(t) and y(t), their inner product, denoted by < x(t), y(t) > and is given by

$$< x(t), y(t) > = \int_{-\infty}^{\infty} x(t)\, \overline{y}(t)\, dt$$

Consider the set of all N-dimensional, real-valued vectors of the form $X = [x_1, x_2, x_3, \ldots, x_N]$. This set forms an N-dimensional linear vector space and there exists N linearly independent basis vectors $a_1, a_2, a_3, \ldots, a_N$ Such that

$X = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_N a_N$ where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are the scalars. Now, we are interested in approximating vectors in V_N by vectors in V_{N-1}. The best possible approximation in the minimum mean squared error or least squares sense is made by choosing the vector say χ_{N-1} in V_{N-1} for which the length of the error vector i.e

$e_{N-1} = X - \chi_{N-1}$ is minimized. This vector is obtained by solving for e_{N-1} in

$$< e_{N-1}, a_k > = 0 \qquad \text{-------}(4.2.2)$$

where $k = 1, 2, 3, . . ., N-1$.

The vector χ_{N-1} is called the orthogonal projection of X on V_{N-1}. The error vector e_{N-1} is orthogonal to every vector in V_{N-1}.

Suppose we now continue this process of projecting throughout the entire sequence of subspaces by projecting χ_{N-1} on V_{N-2} to yield χ_{N-2} and so on. At the end of the process we have a sequence of orthogonal projections of X in the subspaces $V_{N-1}, V_{N-2}, . . ., V_1$. These are $\chi_{N-1}, \chi_{N-2}, . . ., \chi_1$ respectively. Thus approximating vector χ_{k+1} as well as the projection of the vectors $\chi_{k+2}, . . ., \chi_{N-1}$ and X on V_k.
With $X_N = X$, let

$$e_k = \chi_{k+1} - \chi_k \qquad k = 1, 2, . . ., N-1.$$

denote the error between the projections on successive subspaces. The subspace V_{N-1} contains the finest approximation to vectors in V_N whereas V_1 contains the coarsest approximation. Since $X - e_{N-1} = \chi_{N-1}$, the error vector e_{N-1} can be considered as the amount of "detail" that is lost in X in going to its approximation χ_{N-1}. These detail vectors belongs to the W_{N-1}, the orthogonal subspace of V_N. Thus vector in V_N can be expressed as the sum of a vectors V_{N-1} and a vector in W_{N-1}. Also the original vector is reconstructed from the coarsest approximation χ_1 and details at various levels, that is,

$$X = e_{N-1} + e_{N-2} + . . . + e_1 + \chi_1 \qquad \text{-------}(4.2.3)$$

This shows that every vector in V_N can be expressed as in (4.2.3), that is, e_{N-1}, e_{N-2}, . . ., e_1 belongs to W_{N-1}, W_{N-2}, . . ., W_1 respectively and $\chi_1 \in V_1$. Since all these subspaces are mutually orthogonal we can write

$$V_N = W_{N-1} \oplus W_{N-2} \oplus . . . \oplus W_1 \oplus V_1 \qquad \text{-------}(4.2.4)$$

(4.3) Approximating Vectors in Nested Subspaces of an Infinite Dimensional Linear Vector Space

Consider the familiar Fourier Series expansion of periodic signals. A real periodic signal of fundamental periodic T second with finite energy over a period can be expressed as a linear combination of sines and cosines of frequencies that are integer multiples of the fundamental frequency 1/T Hz. The set of all periodic signals of period T with finite energy over a period forms a linear vector space. Let V_n denotes the subspace generated by the DC term, the fundamental and all harmonics upto the n^{th} harmonic. The (2n+1) vectors namely

$\cos(2\pi kt/T)$, $k = 0, 1, \ldots, n$ and

$\sin(2\pi kt/T)$, $k = 1, 2, \ldots, n$

Let the above linear vector space be V_∞ whose bases vectors are mutually orthogonal under

$$<x(t), y(t)> = \int_{-T/2}^{T/2} x(t)\, \bar{y}(t)\, dt \qquad \text{-------(4.3.1)}$$

Here also we have a nested sequence of subspaces and every periodic signal in V_∞ is the sum of the coarsest approximation (DC component) and the detail function (various harmonics).

Example of an Multiresolution Analysis (MRA)

Consider the approximation in nested linear vector subspaces involving a wavelet and is an MRA. Let f(t) be a continuous, real-valued, finite energy signal. We will use piecewise constant functions to build approximations to this function at different levels of resolution. Let us begin with an approximation $(\ell, \ell+1)$, for integer ℓ. let us call this approximation $f_0(t)$. The approximation is best if $\|f(t)-f_0(t)\|^2$ is minimized, that is, the best constant to approximate a function over an interval is average value of

$$f(t)=\int_{l}^{l+1} f(\tau)\,d\tau \qquad l\le t\le l+1 \qquad \text{-------(4.3.2)}$$

the function over that interval.

Let V_0 be the vector space formed by the set of functions that are piecewise constant over unit intervals. Thus function $f(t)\in L^2(R)$ is approximated by function $f_0(t)$ in V_0.

Now if we consider approximating f(t) in the vector space of functions that are piecewise constant over

$$f_1(t)=\frac{1}{2}\int_{2l}^{2l+2} f(\tau)\,d\tau \qquad 2l\le t\le 2l+2 \qquad \text{-------(4.3.3)}$$

intervals of length 2, that is, intervals of the form $2\ell \le t \le 2\ell+2$ for integer ℓ. Let us call this space V_1, proceeding as above and if $f_1(t)$ is that approximation then for integer value of ℓ.

Compared to $f_0(t)$, the functions $f_1(t)$ is a coarser approximation to f(t) and $f_{-1}(t)$ is finer approximation than $f_0(t)$ in the space V_{-1} of functions that are piecewise constant over half unit interval

$$f_{-1}(t)=\frac{1}{2^{1/2}}\int_{l/2}^{l/2+1/2} f(\tau)\,d\tau \qquad l/2\le t\le l/2+1/2 \qquad \text{-------(4.3.4)}$$

for integer value of ℓ.

Proceeding in this way let V_k, for a given integer k be the space of functions, that is, piecewise constant over intervals of length 2^k. Then the least squares approximation to f(t) is given by

$$f_k(t) = \frac{1}{2^k}\int_{2^k l}^{2^k(l+1)} f(\tau)d\tau \qquad 2^k l \le t \le 2^k(l+1) \qquad \text{-------(4.3.5)}$$

for integer value of ℓ.

For two integers i and j, i < j, the approximation $f_i(t)$ is finer than $f_j(t)$. This would suggest that the limiting approximation

$$f_\infty(t) = 0 \qquad \text{for all } t \qquad \text{-------(4.3.6)}$$

Thus $f_\infty(t)$ is the zero vector. We define the detail function at level k as

$$g_k(t) = f_{k-1}(t) - f_k(t) \qquad \text{-------(4.3.7)}$$

From the equation (4.3.6) and repeated application of equation (4.3.7) to successive values of k, we have

$$f_{k-1}(t) = \sum_{j=-\infty}^{\infty} g_j(t) \qquad \text{-------(4.3.8)}$$

Taking the limit k → ∞ yields

$$f(t) = \sum_{k=-\infty}^{\infty} g_k(t) \qquad \text{-------(4.3.9)}$$

(4.4) Bases for the Approximation Subspaces and Haar Scaling Function

The common vector in all the approximation subspaces is the zero vector. It is given by

$$\bigcap_{k=-\infty}^{\infty} V_k = \{0\} \qquad \text{-------(4.4.1)}$$

Another property is that if a function f(t) $\in V_k$ then f(2t) $\in V_{k-1}$ and viceversa. The bases for all the subspaces in this structure can be generated using translation of an appropriately dilation of a single function called the scaling function.

The basis for V_k for any integer k is obtained by translations of an appropriately dilated version of the single function given by

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{-------(4.4.2)}$$

This function, which is a rectangular pulse, is known as the Haar scaling function.

Let

$$C(k,l) = \frac{1}{2^k} \int_{2^k \ell}^{2^k(\ell+1)} f(t)dt \qquad \text{-------(4.4.3)}$$

For some integer k, using equation (4.3.5), $C(k,\ell)$ is simply the average value of f(t) in the interval $2^k\ell \le t < 2^k(\ell+1)$. Based on the fact that,

$$\phi(2^{-k}t)=\begin{cases}1 & 0\le t<2^k\\ 0 & \text{otherwise}\end{cases} \qquad \text{-------}(4.4.4)$$

We have,

$$f_k(t)=\sum_{\ell=-\infty}^{\infty}C(k,\ell)\,\phi(2^{-k}t-\ell) \qquad \text{-------}(4.4.5)$$

Which indicates that any function in V_k can be obtained as a linear combination of a dilation of $\phi(t)$ by a factor of 2^k and its translations by integer multiples of 2^k. Since the set $\{\ \phi(2^{-k}t - \ell)\ :\ k,\ell \text{ integer}\ \}$ is orthogonal it is a basis for V_k.

(4.5) Bases for the Detail Subspace and Haar Wavelet

The basis functions for the subspaces in which the detail functions $g_k(t)$ reside can be obtained using translations and dilations of a single function, which in this case is a wavelet. To generate the basis functions for the details first take,

$$g_0(t) = f_{-1}(t) - f_0(t) \qquad \text{-------}(4.5.1)$$

Since the subspaces are all nested, the basis at one level should be expressible in terms of the basis at the next finer level.

Therefore, $\phi(t) = \phi(2t) + \phi(2t-1)$ -------(4.5.2)

For $0 \le t < 1$, using (4.5.1)

$$f_{-1}(t) = C(-1,0)\phi(2t) + C(-1,1)\phi(2t-1) \quad \text{-------(4.5.3)}$$

and

$$f_0(t) = C(0,0)\phi(t) \quad \text{-------(4.5.4)}$$

We have

$$C(0,\ell) = \tfrac{1}{2}\,[\,C(-1,2\ell) + C(-1, 2\ell+1)\,] \quad \text{-------(4.5.5)}$$

In-particular,

$$C(0,0) = \tfrac{1}{2}\,[\,C(-1,0) + C(-1, 1)\,] \quad \text{-------(4.5.6)}$$

Using (4.5.1) to (4.5.4), we get,

$$g_0(t) = d(0,0)\varphi(t) \quad \text{-------(4.5.7)}$$

Where, $\varphi(t) = \phi(2t) - \phi(2t-1)$ -------(4.5.8)

And $d(0,0) = \tfrac{1}{2}\,[\,C(-1,0) - C(-1,1)\,]$ -------(4.5.9)

The function $\varphi(t)$ is the Haar wavelet. It is given by

$$\varphi(t) = \begin{cases} 1 & 0 \le t < 1/2 \\ -1 & 1/2 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{-------(4.5.10)}$$

As is required $\varphi(t)$ integrates to zero, with

$d(0,\ell) = \tfrac{1}{2}\,[\,C(-1,2\ell) - C(-1,2\ell+1)\,]$ ------(4.5.11)

In-general,

$$g_k(t) = \sum^{\infty} d(k,\ell)\,\varphi(2^{-k}t - \ell) \quad \text{-------(4.5.12)}$$

Where,

$$d(k, \ell) = \tfrac{1}{2}[C(k-1,2\ell)-C(k-1,2\ell+1)] \qquad \text{------(4.5.13)}$$

from equations (4.4.3), (4.5.8), (4.5.10) and (4.5.13) we have,

$$d(k,\ell)=\frac{1}{2^k}\int_{2^k\ell}^{2^k(\ell+1)} f(t)\varphi(2^k t-\ell)\,dt \qquad \text{-------(4.5.14)}$$

Also,

$$<\varphi(2^{-k}t - \ell),\ \varphi(2^{-k}t - m)> = 0 \qquad \text{------(4.5.15)}$$

For $\ell \neq m$, equation (4.5.12) and equation (4.5.15) imply that for a specified integer k the set { $\varphi(2^{-k}t - \ell)$: ℓ is integer }, provides a basis for the space containing the detail function $g_k(t)$. Let us call this subspace W_k.

Symbolically, $V_{k-1} = V_k \oplus W_k$ ------(4.5.16)

The subspace W_k of details of the form $g_k(t)$ is orthogonal to V_k and V_{k+1} and so on, we have,

$$V_K = \bigoplus_{j=k}^{\infty} W_j \qquad \text{-------(4.5.17)}$$

Finally, using equation (4.3.9) and (4.5.15), we have,

$$f(t)=\sum_{k=-\infty}^{\infty}\ \sum_{\ell=-\infty}^{\infty} d(k,\ell)\ \varphi(2^{-k}t-\ell) \qquad \text{-------(4.5.18)}$$

Thus, equation (4.5.18) express the function f(t) in terms of discrete wavelet representation. The orthogonality of the set of dyadically dilated and translated wavelet further makes this by definition an orthogonal wavelet decomposition or orthogonal MRA.

Example: Consider the function

$$F(t) = e^{(-t/4)}u(t)$$

Where u(t) is the step function,

$$u(t) = \begin{cases} 1 \text{ if } t \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

To compute the approximations to the function f(t):

The average value of the given function in the interval $a \leq t \leq b$ for any $a \geq 0$ and $b > a$ is given by

$$\frac{1}{b-a}\int_a^b e^{-t/4}dt = \frac{4}{b-a}\left[e^{-a/4} - e^{-b/4}\right]$$

and the coefficients $c(k,\ell)$ are given by

$$\begin{aligned} C(k,\ell) &= \frac{1}{2^k}\int_{2^k\ell}^{2^k(\ell+1)} f(t)\,dt \\ &= \frac{1}{2^k}\int_{2^k\ell}^{2^k(\ell+1)} e^{-t/4}\,dt \\ &= \frac{4}{2^k}e^{-2^k\ell/4}(1-e^{-2^k/4}) \\ &= f_k(t) \quad , \quad 2^k\ell \leq t \leq 2^k(\ell+1) \end{aligned}$$

This is the approximation to f(t) at resolution level k.

Let us consider the different values of k as k = -2, -1, 0, 1, 2. The corresponding coefficients are given by

1) if k = 2

$$f_2(t) = e^{-\ell}(1 - 1/e) = c(2,\ell) \quad , \; 4\ell \le t \le 4(\ell+1)$$

2) if k = 1

$$f_1(t) = 2e^{-\ell/2}(1 - e^{-1/2}) = c(1,\ell) \quad , \; 2\ell \le t \le 2(\ell+1)$$

3) if k = 0

$$f_0(t) = 4e^{-\ell/4}(1 - e^{-1/4}) = c(0,\ell) \quad , \; \ell \le t \le (\ell+1)$$

4) if k = -1

$$f_{-1}(t) = 8e^{-\ell/8}(1 - e^{-1/8}) = c(-1,\ell) \; , \; \ell/2 \le t \le (\ell+1)/2$$

5) if k = -2

$$f_{-2}(t) = 16e^{-\ell/16}(1 - e^{-1/16}) = c(-2,\ell), \; \ell/4 \le t \le (\ell+1)/4$$

Now the coefficients for the different values of ℓ are given as

k \ ℓ	0	1	2	3
-2	C(-2,0)	C(-2,1)	C(-2,2)	C(-2,3)
-1	C(-1,0)	C(-1,1)	C(-1,2)	C(-1,3)
0	C(0,0)	C(0,1)	C(0,2)	C(0,3)
1	C(1,0)	C(1,1)	C(1,2)	C(1,3)
2	C(2,0)	C(2,1)	C(2,2)	C(2,3)

and using the relation (4.5.12) $g_k(t) = \sum_{\ell=-\infty}^{\infty} d(k,\ell)\varphi(2^{-k}t-\ell)$

where, the coefficients

$$d(k,\ell) = (1/2)\left[C(k-1,2\ell) - C(k-1,2\ell+1)\right]$$

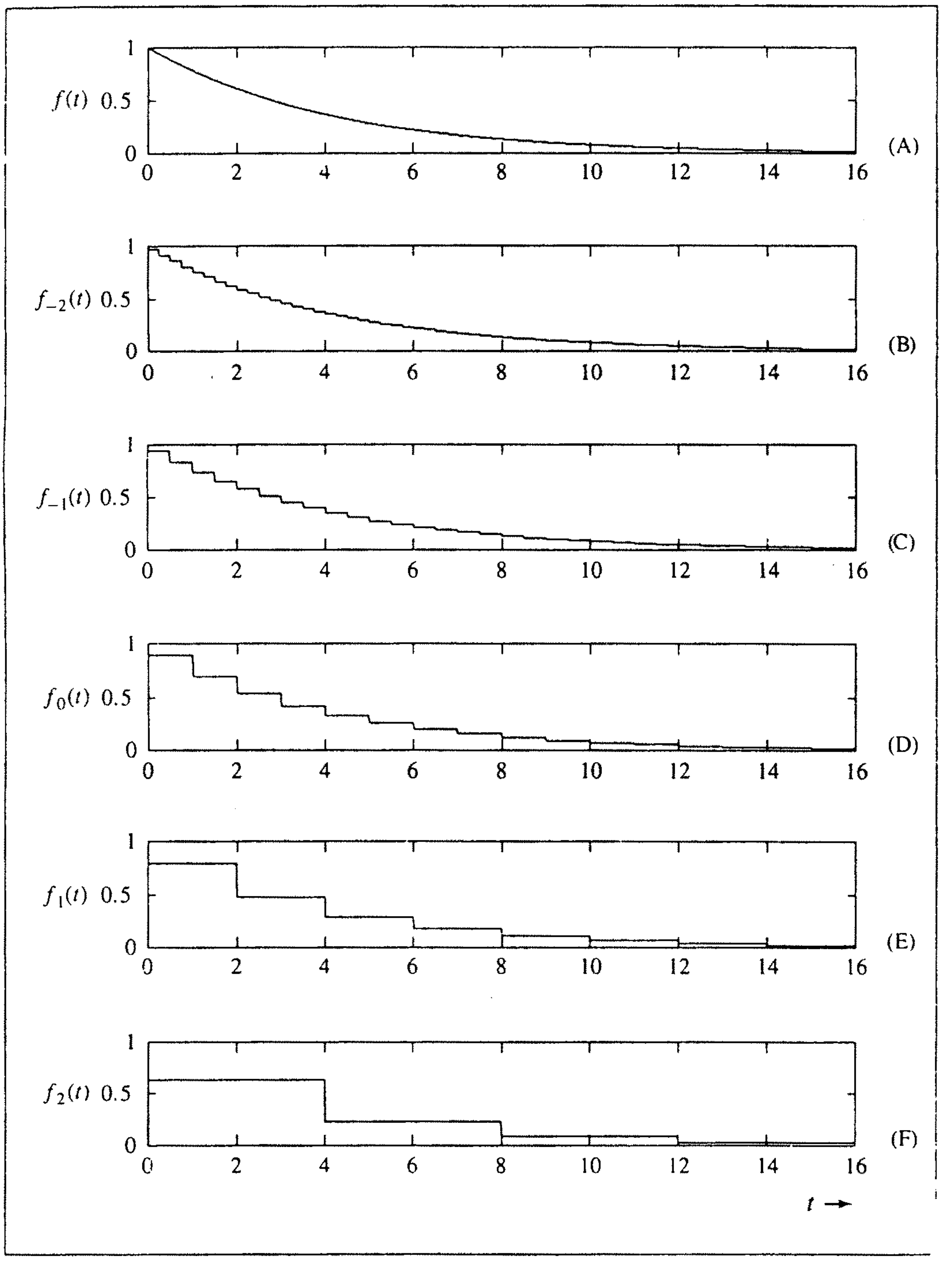
and $\varphi(t)$ the wavelet as in equation (4.5.10)

$$\varphi(t)=\begin{cases}1 & 0\le t<1/2\\ -1 & 1/2\le t<1\\ 0 & \text{otherwise}\end{cases}$$

the corresponding details are computed. Finally using the relations (4.5.18)

$$f(t)=\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} d(k,\ell)\,\varphi(2^{-k}t-\ell)$$

f(t) can be computed. The approximations over the range $0 \le t \le 16$ for values k = -2, -1, 0, 1, 2 are as shown in figure.



1
f(t) 0.5
0
0 2 4 6 8 10 12 14 16
(A)
1
$f_{-2}(t)$ 0.5
0
0 2 4 6 8 10 12 14 16
(B)
1
$f_{-1}(t)$ 0.5
0
0 2 4 6 8 10 12 14 16
(C)
1
$f_0(t)$ 0.5
0
0 2 4 6 8 10 12 14 16
(D)
1
$f_1(t)$ 0.5
0
0 2 4 6 8 10 12 14 16
(E)
1
$f_2(t)$ 0.5
0
0 2 4 6 8 10 12 14 16
(F)
t →