C H A P T E R - II

" THE DEFORMATION TENSOR FIELD"

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1. INTRODUCTION :

The theory of rheology comprises the study of flow and deformation of matter. According to Fredrickson (1964) the development has led to a division of rheological research on three levels

a) Physico-Chemical Research

b) Engineering Research

c) Mathematical Research

In rheology matter is treated as continuum (structurless substance) and its molecular structure is ignored so that kinematical variables associated with a fixed point in space may be regarded as contineous function of the spatial co-ordinates of the points. The dynamical state of a body is determined if we know the forces which act on an arbitrary located and arbitrary oriented element of surface in the body.

The "Relativistic Rheology" is the study of deformation and flow of ponderable matter at high speed comparable with velocity of light (Radhakrishna, 1978). In general relativistic rheology the strong and rapidly changing gravitational field can not be studied in terrestrial laboratory. An attempt is made to overcome these difficulties by Narlikar (1978) using the relations

$$\frac{GM}{C^2R} \approx 1 , \quad \frac{p}{C^2\rho} \approx 1, \quad v^2/C^2 \approx 1,$$

Where G is universal gravitational constant, R is the radius of the gravitating body, and M is the mass of the body.

The subject being of recent origin, we find very few persons working on relativistic rheology. The propagation equation for the expansion parameter is obtained by Ray-Chauduri (1955), Grot and Eringen (1966), Paria (1967) have investigated some aspects in special relativistic rheology. The propagation equation of Shear Tensor is expressed by Greenberg (1970). Carter and Quintana (1977) has obtained the equation for strain tensor in general relativity. Using the Ricci identity the kinematical strain variation equation is workedout by Radhakrishna and Singh (1983).

The kinematical strain variation equation along the time like vector $\bar{u}$ is presented in section 2, while the section 3 deals with the dynamical form of strain tensor for ferrofluid system. The strain variation equation along the space like vector $\bar{h}$ is obtained in section 4. The last section speakes about the dynamical form of strain variation equation for ferrofluid.

2. RELATIVISTIC STRAIN VARIATION EQUATION :

From the 3-space operator h_{ab} the expression for the 3-space projection of the flow gradient $u_{a;b}$ can be expressed as follows (Trautman, 1964).

$$u_{\bar{a};\bar{b}} = h^e_a \, h^f_b \, u_{e;f}, \qquad \text{.....(2.1)}$$

$$= (\delta^e_a - u^e u_a)\,(\delta^f_b - u^f u_b) u_{e;f}.$$

(by definition of h_{ab})

This can be simplified by using the relations $u_{a;b}u^a = 0$, $u_{a;b}u^b = \dot{u}^a$, $\dot{u}_a u^a = 0$ in the form

$$u_{\bar{a};\bar{b}} = u_{a;b} - \dot{u}_a u_b. \qquad \text{.....(2.2)}$$

The second rank tensor which is written in the form of symmetric part of $u_{\bar{a};\bar{b}}$ is the strain tensor θ_{ab}(Greenberg, 1970 a).
This means that

$$\theta_{ab} = 1/2\ (u_{\bar{a};\bar{b}} + u_{\bar{b};\bar{a}}) \qquad \text{.....(2.3)}$$

From the equation (2.2) we write the equation (2.3) in the form

$$\theta_{ab} = 1/2\ (u_{a;b} + u_{b;a} - \dot{u}_a u_b - u_a\dot{u}_b). \qquad \text{.....(2.4)}$$

This equation implies that the Strain Tensor θ_{ab} can be expressed in term of the gradient of flow vector and acceleration vector.

The rotation tensor W_{ab} is defined as the antisymmetric part of the $u_{\bar{a};\bar{b}}$

i.e., $W_{ab} = 1/2\ (u_{\bar{a};\bar{b}} - u_{\bar{b};\bar{a}})$. .....(2.5)

From equations (2.2) and (2.5) we get

$$W_{ab} = 1/2\ [u_{a;b} - u_{b;a} - (\dot{u}_a u_b - u_a\dot{u}_b)]. \qquad \text{.....(2.6)}$$

The Shear tensor σ_{ab} has the expression

$$\sigma_{ab} = 1/2\ [u_{a;b} + u_{b;a} - \dot{u}_a u_b - u_a \dot{u}_b) - \frac{1}{3} h_{ab}\,\theta\ .$$

.....(2.7)

This is trace free and u-orthogonal.

So that from (2.3) and (2.5) we have

$$u_{\bar{a};\bar{b}} = \theta_{ab} + W_{ab}. \qquad .....(2.8)$$

To derive the strain variation equation we prove the following Lemma regarding the derivatives of material tensor. Any u- orthogonal tensor M_{ab} is called material tensor.

Lemma : For the material tensor M_{ab}

$$\underset{u}{L}\ M_{ab} = M_{\bar{a}\bar{b};c}u^c + M_{ac}u^{\bar{c}}{}_{;\bar{b}} + M_{cb}u^{\bar{c}}{}_{;\bar{a}}. \qquad .....(2.9)$$

Proof : By using the definition of Lie derivative (vide 0.5.1) we have

$$\underset{u}{L}\ M_{ab} = M_{ab;c}\ u^c + M_{ac}u^c{}_{;b} + M_{cb}u^c{}_{;a}.$$

Further from equation (2.2) the above equation becomes

$$\underset{u}{L}\ M_{ab} = \delta^e_a\ \delta^f_b\ M_{ef;c}u^c + M_{ac}(u^{\bar{c}}{}_{;\bar{b}} + \dot{u}^c u_b) + M_{cb}(u^{\bar{c}}{}_{;\bar{a}} + \dot{u}^c u_a).$$

.....(2.10)

Also M_{ab} is material tensor, hence we write

$$M_{ab}u^b = 0. \qquad .....(2.11)$$

By substituting the value $\delta^a_b = h^a_b + u^a u_b$ in equation (2.10) we get

$$\underset{u}{L} M_{ab} = (h^e_a + u^e u_a)\,(h^f_b + u^f u_b)\, M_{ef;c} u^c + \\ + M_{ac}(u^{\bar{c}}{}_{;\bar{b}} + \dot{u}^c u_b) + M_{cb}(u^{\bar{c}}{}_{;\bar{a}} + \dot{u}^c u_a),$$

i.e., $$\underset{u}{L} M_{ab} = (h^e_a\, h^f_b + h^e_a\, u^f u_b + h^f_b\, u^e u_a + u^e u^f u_a u_b)\, M_{ef;c} u^c + \\ + M_{ac} u^{\bar{c}}{}_{;\bar{b}} + M_{cb} u^{\bar{c}}{}_{;\bar{a}} + M_{ac}\dot{u}^c u_b + M_{cb}\dot{u}^c u_a. \qquad .....(2.12)$$

On using the results (2.1) and (2.11) the equation (2.12) reduces to

$$\underset{u}{L} M_{ab} = M_{\bar{a}\bar{b};c} u^c + h^e_a\, \dot{M}_{ef} u^f u_b + h^f{}_b\, \dot{M}_{ef}\, u^e u_a + \\ + \dot{M}_{ef} u^e u^f u_a u_b + M_{ac} u^{\bar{c}}{}_{;b} + M_{cb} u^{\bar{c}}{}_{;a} + \\ + M_{ac} u^{\bar{c}}{}_{;\bar{b}} + M_{ac}\dot{u}^c u_b + M_{cb}\dot{u}^c u_a. \qquad .....(2.13)$$

Further we have

$$h^e_a \dot{M}_{ef} u^f u_b = (\delta^e_a - u^e u_a)\, \dot{M}_{ef}\, u^f u_b\ ,$$

i.e., $h^e_a\, \dot{M}_{ef}\, u^f u_b = \dot{M}_{af} u^f u_b$, (vide (2.11)

i.e., $h^e_a\, \dot{M}_{ef} u^f u_b = -M_{af}\dot{u}^f u_b$.

This with equation (2.13) reduces to

$$\underset{u}{L}\ M_{ab} = M_{\bar{a}\bar{b};c}\, u^c + M_{ac} u^{\bar{c}}{}_{;\bar{b}} + M_{bc}\, u^{\bar{c}}{}_{;\bar{a}}\ .$$

This is the required result of Lemma.

Now for obtaining the expression for deformation tensor θ_{ab} we use the above Lemma and write

$$\underset{u}{L} \theta_{ab} = h^e_a h^f_b \theta_{ef;c} u^c + \theta_{ac} u^{\bar{c}}_{;\bar{b}} + \theta_{cb} u^{\bar{c}}_{;\bar{a}} .$$

By the definition of θ_{ab} (vide equation (2.4)) the above equation becomes

$$\underset{u}{L} \theta_{ab} = 1/2\, h^e_a h^f_b [u_{e;f} + u_{f;e} - \dot{u}_e u_f - u_e \dot{u}_f]_{;c} u^c + \\ + \theta_{ac} u^{\bar{c}}_{;\bar{b}} + \theta_{cb} u^{\bar{c}}_{;\bar{a}} ,$$

i.e., $$\underset{u}{L} \theta_{ab} = 1/2\, h^e_a h^f_b [u_{e;f;c} u^c + u_{f;e;c} u^c - \dot{u}_e \dot{u}_f - \\ - \dot{u}_e \dot{u}_f - (\dot{u}_e)_{;c} u^c u_f - (\dot{u}_f)_{;c} u^c u_e] + \\ + \theta_{ac} u^{\bar{c}}_{;\bar{b}} + \theta_{cb} u^{\bar{c}}_{;\bar{a}} . \qquad \text{.....(2.14)}$$

The contracted Ricci identity for the flow vector u_a is given by

$$u_{e;f;c}\, u^c - u_{e;c;f}\, u^c = u^c u^d R_{defc} , \qquad \text{.....(2.15)}$$

On addding and substracting $u_{e;c}\, u^c_{;f}$ and $u_{f;c} u^c_{;e}$ in equation (2.14) and using (2.15) we get

$$\underset{u}{L} \theta_{ab} = 1/2\, h^e_a h^f_b [(u_{e;c;f}\, u^c + u_{e;c}\, u^c_{;f}) + \\ + (u_{f;c;e}\, u^c + u_{f;c}\, u^c_{;e}) - u_{e;c}\, u^c_{;f} - \\ - u_{f;c}\, u^c_{;e} + u^c u^d (R_{defc} + R_{dfec}) - \\ - 2\, \dot{u}_e \dot{u}_f] + \theta_{ac} u^{\bar{c}}_{;\bar{b}} + \theta_{bc} u^{\bar{c}}_{;\bar{a}} ,$$

(Since $\dot{P}_{ab} u^a = 0$)

i.e., $\underset{u}{L} \theta_{ab} = 1/2\ h^e_a h^f_b [\dot{u}_{e;f} + \dot{u}_{f;e}] + 1/2\ h^e h^f [-2\dot{u}_e\dot{u}_f - u_{e;c} u^c_{;f} - u_{f;c}u^c_{;e} + 2u^c u^d R_{defc}] + \theta_{ac} u^{\bar{c}}_{;\bar{b}} + \theta_{cb} u^{\bar{c}}_{;\bar{a}}$ . .....(2.16)

Further by using the equation (2.16) and then making use of the definition of P_{ab} we write

$$\underset{u}{L} \theta_{ab} = h^e_a h^f_b \dot{u}_{(e;f)} + 1/2\ (\delta^e_a \delta^f_b - \delta^e_a u^f u_b - \delta^e_b u^e u_a + u^e u^f u_a u_b)\ (-2\dot{u}_e\dot{u}_f - u_{e;c} u^c_{;f} - u_{f;c} u^c_{;e} + 2u^c u^d R_{defc}] + \theta_{ac} u^{\bar{c}}_{;\bar{b}} + \theta_{cb} u^{\bar{c}}_{;\bar{a}} .$$

After simplification we get

$$\underset{u}{L} \theta_{ab} = h^e_a h^f_b \dot{u}_{(e;f)} + 1/2\ [-2\dot{u}_a\dot{u}_b - u_{a;c} u^c_{;b} - u_{b;c} u^c_{;a} + 2 u^c u^d R_{dabc} + u_{a;c} \dot{u}^c u_b + u_{b;c} \dot{u}^c u_a - 2u^c u^d u^f u_b R_{dafc} - 2u^c u^d u^e u_a R_{debc}] + \theta_{ac} u^{\bar{c}}_{;\bar{b}} + \theta_{cb} u^{\bar{c}}_{;\bar{a}} . \quad .....(2.17)$$

But we note the results

$$u_{\bar{a};\bar{c}} \dot{u}^c u_b = (u_{a;c} - \dot{u}_a u_c)\ \dot{u}^c u_b ,$$

i.e., $u_{\bar{a};\bar{c}}\ \dot{u}^c u_b = u_{a;c} \dot{u}^c u_b$, Similarly $u_{a;c}u^{c}{}_{;b} = u_{\bar{a};\bar{c}}u^{c}{}_{;b}$,

and $u^{c}u^{d}u^{e}\,R_{defc} = 0.$

The expression (1.17) with these results yields

$$\underset{u}{L}\,\theta_{ab} = h^{e}_{a}\,h^{f}_{b}\,\dot{u}_{(e;f)} - \dot{u}_{a}\,\dot{u}_{b} + 1/2\,(-\,u_{\bar{a};\bar{c}}\,u^{\bar{c}}{}_{;\bar{b}} -$$

$$-\,u_{\bar{b};\bar{c}}\,u^{\bar{c}}{}_{;\bar{a}}) + \theta_{ac}\,u^{\bar{c}}{}_{;\bar{b}} + \theta_{cb}\,u^{\bar{c}}{}_{;\bar{a}} +$$

$$+\,u^{c}\,u^{d}\,R_{dabc}.$$

This immediately gives with the definition of $u_{\bar{a};\bar{b}}$ (vide (2.8))

$$\underset{u}{L}\,\theta_{ab} = h^{e}_{a}\,h^{f}_{b}\,\dot{u}_{(e;f)} - \dot{u}_{a}\,\dot{u}_{b} + 1/2\,[-(\theta_{ac} + W_{ac})\times$$

$$\times(\theta^{c}{}_{.b} + W^{c}{}_{.b}) - (\theta_{bc} + W_{bc})\,(\theta^{c}{}_{.a} + W^{c}{}_{.a})] +$$

$$+\,\theta_{ac}(\theta^{c}{}_{.b} + W^{c}{}_{.b}) + \theta_{cb}(\theta^{c}{}_{.a} + W^{c}{}_{.a}) +$$

$$+\,u^{c}u^{d}\,R_{dabc}\ ,$$

i.e., $$\underset{u}{L}\theta_{ab} = h^{e}_{a}\,h^{f}_{b}\,\dot{u}_{(e;f)} - \dot{u}_{a}\,\dot{u}_{b} + W_{ac}\,W^{c}{}_{.b} +$$

$$+\,\theta_{ac}W^{c}{}_{.b} + \theta_{cb}\theta^{c}{}_{.a} + \theta_{cb}\,W^{c}{}_{.a} + u^{c}u^{d}\,R_{dabc}\ ,$$

i.e. $$\underset{u}{L}\theta_{ab} = h^{e}\,h^{f}\,\dot{u}_{(e;f)} - \dot{u}_{a}\,\dot{u}_{b} + (\theta^{c}{}_{.a} + W^{c}{}_{.a})\,(\theta_{cb} + W_{cb}) +$$

$$+\,u^{c}u^{d}\,R_{dabc},$$

i.e., $$\underset{u}{L}\,\theta_{ab} = h^{e}_{a}\,h^{f}_{b}\,\dot{u}_{(e;f)} - \dot{u}_{a}\dot{u}_{b} + u^{\bar{c}}{}_{;\bar{a}}\,u_{\bar{c};\bar{b}} + u^{c}u^{d}\,R_{dabc}.$$

.....(2.18)

This equation is described as the kinematical equation for deformation tensor field. Carter and Quintana (1977) has derived this equation for the signature (+, +, +, -). Where as we have chosen the metric signature (-, -, -, +).

The Weyl's Tensor has the defining expression

$$C_{abcd} = R_{abcd} + 1/2\,(g_{ac} R_{bd} - g_{ad} R_{bc} + g_{bd}R_{ac} - g_{cb}R_{ad}) - R/6\,(g_{ad}g_{cb} - g_{ac}g_{bd}). \quad .....(2.19)$$

So that the eequation (2.18) is transformed as

$$\underset{u}{L}\,\theta_{ab} = h^e_a\, h^f_b\, \dot{u}_{(e;f)} - \dot{u}_a\, \dot{u}_b + u^{\bar{c}}{}_{;\bar{a}}u_{\bar{c};\bar{b}} - u^e u^f C_{aebf} + 1/2\, h_{ab}[R_{ef}u^e u^f - R/3] + 1/2\, h^e_a\, h^f_b\, R_{ef}. \quad .....(2.20)$$

This is the strain variation equation along flow.

3. DYNAMICAL FORM OF STRAIN VARIATION EQUATION IN FERROFLUID SYSTEM :

The stress energy tensor for ferrofluid is characterized by (0.3.1)

$$T_{ab} = (\rho + p + \mu H^2)\, u_a u_b - (P + 1/2\,\mu H^2)\, g_{ab} - \mu H_a H_b. \quad .....(3.1)$$

We recall the value of Rici Tensor for ferrofluid from equation (0.3.3) in the form

$$R_{ab} = - K [(\rho + p + \mu H^2) u_a u_b - 1/2 (\rho - p + \mu H^2) \times$$

$$\times g_{ab} - \mu H_a H_b] \qquad .....(3.2)$$

The twice contracted Bianchi identity yield for ferrofluid

$$(\rho + p + \mu H^2) \dot{u}^a - (p + 1/2 \mu H^2)_{;b} h^{ab} -$$

$$- (\mu H^b)_{;b} H^a = 0, \qquad \text{(Vide (0.4.9))}$$

i.e., $u^a = A^{-1} [(p + 1/2 \quad H^2)_{;b} h^{ab} - (\mu H^b)_{;b} H^a]. \qquad .....(3.3)$

Here the value of A is given by

$$A = (\rho + p + \mu H^2).$$

Now recall the kinematical form of Strain variation equation

$$\underset{\bar{u}}{L} \theta_{ab} = h^e_a h^f_b \dot{u}_{(e;f)} - \dot{u}_a \dot{u}_b + u^{\bar{c}}_{;\bar{a}} u_{\bar{c};\bar{b}} -$$

$$- u^e u^f C_{aebf} + 1/2 h_{ab} [R_{ef} u^e u^f - R/3]$$

$$+ 1/2 h^e_a h^f_b R_{ef}. \qquad .....(3.4)$$

Let us write this in the form

$$\underset{\bar{u}}{L} \theta_{ab} = L_1 + L_2 + L_3 + L_4 \qquad .....(3.5)$$

Where $L_1 = h^e_a h^f_b \dot{u}_{(e;f)}$,

$L_2 = - \dot{u}_a \dot{u}_b + u^c_{;\bar{a}} u_{\bar{c};\bar{b}}$,

$L_3 = u^e u^f C_{aebf}$,

$L_4 = 1/2 h_{ab} [R_{ef} u^e u^f - R/3] + 1/2 h^e_a h^f_b R_{ef}$.

<u>To simplify</u> L_1 : We note the following trivial result

$$h^e_a h^f_b \dot{u}_{(e;f)} = h_{e(a}h_{b)f}\, u^e{}_{;f} ,$$

This result under the expression (3.3) provides the value of L_1 as

$$L_1 = h_{e(a}h_{b)f} \{ A^{-1}[(p + 1/2\,\mu H^2)_{;c}\, h^{ce} +$$

$$+ (\mu H^c)_{;c} H^e] \}_{;f},$$

i.e., $L_1 = h_{e(a}h_{b)f}\, A^{-2} \{ A[(p + 1/2\,\mu H^2)_{;c} h^{ce} +$

$$(\mu H^c)_{;c} H^e]_{;f} - A_{;f}[(p + 1/2\,\mu H^2)_{;c} h^{ce} + (\mu H^c)_{;c} H^e]\} ,$$

i.e., $L_1 = h_{e(a}h_{b)f}\, A^{-2} \{ (p + 1/2\,\mu H^2)_{;c}[A h^{ce}{}_{;f} - A_{;f} h^{ce}] +$

$$+ (\mu H^c)_{;c}[A H^e{}_{;f} - A_{;f} H^e] +$$

$$+ A\, [(p + 1/2\,\mu H^2)_{;cf} h^{ce} + (\mu H^c)_{;cf} H^e]\} . \qquad .....(3.6)$$

<u>To simplify</u> L_2 : We have

$$L_2 = -\dot{u}_a \dot{u}_b + u^{\bar{c}}{}_{;\bar{a}}\, u_{\bar{c};\bar{b}} ,$$

i.e., $L_2 = - \dot{u}_a \dot{u}_b + (\theta^c_a + W^c_a) (\theta_{cb} + W_{cb})$. .....(3.7)

<u>To simplify</u> L_3 : We write L_3 in the form

$$L_3 = u^e u^f C_{aebf},$$

If we denote the expression $C_{aebf} u^e u^f$ by C_{ab}

Hence $L_3 = C_{ab}$. .....(3.8)

To simplify L_4 : We have

$$L_4 = 1/2\ h_{ab}\ [R_{ef}u^e u^f - R/3\ ] + 1/2\ h^e_a\ h^f_b\ R_{ef}.$$

This with (3.2) takes the form

$$L_4 = 1/2\ h_{ab}\{-K\ [\rho + p + \mu H^2) - 1/2(\rho - p + \mu H^2)\,] -$$
$$- K/3\ (\rho - 3p)\} + 1/2\ (\delta^e_a - u^e u_a)\ (\delta^f_b - u^f u_b)$$
$$\{-K\ [\ (\rho - p + \mu H^2)u_e u_f - 1/2\ (\rho - p + \mu H^2)g_{ef} -$$
$$- H_e H_f]\}\ ,\quad \text{(by definition of } h_{ab})$$

After simplification we get

$$L_4 = -K/2\ [\ (\rho/3 + p)\ h_{ab} - \mu H_a H_b]. \qquad .....(3.9)$$

By substituting the values (3.6), (3.7), (3.8) and (3.9) in equation (3.5) we get

$$\underset{u}{L}\ \theta_{ab} = A^{-2} h_{e(a} h_{b)}{}^f \{\ (\ p + 1/2\ \mu H^2)_{;c}(Ah^{ce}{}_{;f} - A_{;f}h^{ce}) +$$
$$+ (\mu H^c)_{;c}\ (AH^e{}_{;f} - A_{;f}H^e) +$$
$$+ A\ [(\ p + 1/2\ \mu H^2)_{;cf}h^{ce} +$$
$$+ (\mu H^c)_{;cf}H^e]\} - \dot{u}_a \dot{u}_b + (\theta^c_a + W^c_a)\times$$
$$\times(\theta_{cb} + W_{cb}) - C_{ab} - K/2\ [(\rho/3 + p)\ h_{ab} -$$
$$- \mu H_a H_b]. \qquad .....(3.10)$$

This is the required form of Strain variation equation in dynamical form as it consist of kinematical parameters as well as dynamical variables of ferrofluid.

4. STRAIN VARIATION EQUATION ALONG SPACE LIKE CONGRUENCE :

In order to obtain the decomposition of $h_{a;b}$ on 2-diamensional projection plane we start with

$$h_{a;b} = \delta^e_a \, \delta^f_b \, h_{e;f}, \qquad \ldots\ldots(4.1)$$

i.e., $h_{a;b} = (p^e_a + u^e u_a - h^e h_a) \, \delta^f_b \, h_{e;f}.$

Where p_{ab} is given by (0.2.10).

Further

$$h_{a;b} = (p^e_a \, \delta^f_b + u^e u_a \, \delta^f_b) \, h_{e;f},$$

$$\text{(Since } h_{e;f} \, h^e = 0).$$

i.e., $h_{a;b} = p^e_a \, p^f_b \, h_{e;f} + p^e_a \, (u^f u_b - h^f h_b) h_{e;f} + u^e u_a h_{e;b} \ldots\ldots(4.2)$

Since $p^e_a \, p^f_b \, h_{e;f} = h_{\hat{a};\hat{b}}$ and $h'_a = h_{a;e} h^e$ we write the equation (4.2)

$$h_{a;b} = h_{\hat{a};\hat{b}} + u_b \dot{h}_a - u^c u_a u_b \dot{h}_c - h_b h'_a + $$
$$+ u^c u_a h_b h'_c + u^c u_a h_{e;b}.$$

Therefore

$$h_{a;b} = h_{\hat{a};\hat{b}} - u_b \dot{h}_a + h_b h'_a + u^c u_a u_b \dot{h}_c -$$
$$- u^e u_a h_b h'_c - u^e u_a h_{e;b}. \qquad \ldots\ldots(4.3)$$

Definition (1) : The Strain tensor for space like unitary congruence $\bar{h}$ is the symmetric part of $h_{\hat{a};\hat{b}}$ (Greenberg, 1970)

i.e. $\overset{*}{\theta}_{ab} = 1/2 \, (h_{\hat{a};\hat{b}} + h_{\hat{b};\hat{a}}).$ $\ldots\ldots(4.4)$

This expression from (4.3) directs

$$\overset{*}{\theta}_{ab} = 1/2\ [(h_{a;b} + h_{b;a}) - u_b\dot{h}_a - u_a\dot{h}_b + h_b h'_a +$$

$$+ h_a h'_b + 2u^c u_a u_b \dot{h}_c - u^c u_a h_b h'_c - u^c u_b h_a h'_c -$$

$$- u^c u_a h_{c;b} - u^c u_b h_{c;a}]. \qquad .....(4.5)$$

2) The rotation Tensor $\overset{*}{W}_{ab}$ is the antisymmetric part of $h_{\hat{a};\hat{b}}$

$$\overset{*}{W}_{ab} = 1/2\ (h_{\hat{a};\hat{b}} - h_{\hat{b};\hat{a}}). \qquad .....(4.6)$$

From equation (4.3) the equation (4.6) becomes

$$\overset{*}{W}_{ab} = 1/2\ [\ (h_{a;b} - h_{b;a}) - u_b\dot{h}_a + u_a\dot{h}_b + h_b h'_a -$$

$$- h_a h'_b - u^c u_a h_b h'_c + u^c u_b h_a h'_c -$$

$$- u^c u_a h_{c;b} + u^c u_b h_{c;a}]. \qquad .....(4.7)$$

Thus equations (4.5) and (4.7) give

$$h_{\hat{a};\hat{b}} = \overset{*}{\theta}_{ab} + \overset{*}{W}_{ab}\ . \qquad .....(4.8)$$

The decomposition of the Strain Tensor (4.5) provides

$$g^{ab}\ \overset{*}{\theta}_{ab} = \overset{*}{\theta}{}^a_a = \overset{*}{\theta} = h^a{}_{;a} - h_{a;b}u^a u^b.$$

3) The Shear Tensor $\overset{*}{\sigma}_{ab}$ is defined as

$$\overset{*}{\sigma}_{ab} = \overset{*}{\theta}_{ab} - 1/2\ \overset{*}{\theta} p_{ab}.$$

So that its expression in terms of $\bar{h}$ is given by

$$\overset{*}{\sigma}_{ab} = 1/2 \, [\, (h_{a;b} + h_{b;a}) - u_b \dot{h}_a - u_a \dot{h}_b + h_b h'_a + h_a h'_b + 2\, u^c u_a u_b \dot{h}_c - u^c u_a h_b h'_c - u^c u_b h_a h'_c - u^c u_a h_{c;b} - u^c u_b h_{c;a}] - 1/2 \, \overset{*}{\theta} p_{ab}. \quad .....(4.9)$$

Note : It follows from the definitions

$$\overset{*}{\theta}_{ab} \, u^a = 0 \text{ and } \overset{*}{\theta}_{ab} h^a = 0. \quad .....(4.10)$$

Hence $\overset{*}{\theta}_{ab}$ is u - orthogonal and can be treated as a material tensor. So also $\overset{*}{\theta}_{ab}$ is h - orthogonal.

According to the definition of Lie derivative along space like congruence $\bar{h}$ we prove the following Lemma for material tensor

Lemma : For the h - orthogonal tensor M_{ab}

$$\underset{\bar{h}}{L} M_{ab} = M_{\hat{a}\hat{b};c} h^{\hat{c}} + M_{ac} h^{\hat{c}}{}_{;\hat{b}} + M_{cb} h^{\hat{c}}{}_{;\hat{a}}. \quad .....(4.11)$$

Proof : We know the expression of Lie derivative of M_{ab} along the space like congruence.

$$\underset{\bar{h}}{L} M_{ab} = M_{ab;c} h^c + M_{ac} h^c{}_{;b} + M_{cb} h^c{}_{;a},$$

i.e., $\underset{\bar{h}}{L} M_{ab} = \delta^e_a \, \delta^f_b \, M_{ef;c} h^c + M_{ac} \, h^c{}_{;b} + M_{cb} \, h^c{}_{;a}.$

On using the defining expression of p_{ab} this equation can be written as

$$\underset{\bar{h}}{L} M_{ab} = (p^e_a + u^e u_a - h^e h_a) (p^f_b + u^f u_b - h^f h_b) \times M_{ef;c} h^c + M_{ac} h^c{}_{;b} + M_{cb} h^c{}_{;a},$$

i.e., $$\underset{h}{L} M_{ab} = p_a^e\, p_b^f\, M_{ef;c} h^c + p_a^e\, u^f u_b\, M_{ef;c}\, h^c -$$

$$- p_a^e\, h^f\, h_b\, M_{ef;c}\, h^c + p_b^f\, u^e\, u_a\, M_{ef;c}\, h^c +$$

$$p_b^f\, h^e h_a\, M_{ef;c}\, h^c + u^e u^f u_a u_b\, M_{ef;c}\, h^c -$$

$$- u^e h^f u_a h_b\, M_{ef;c} h^c -$$

$$- u^f h^e h_a u_b\, M_{ef;c}\, h^c + h^e h^f h_a h_b\, M_{ef;c}\, h^c +$$

$$+ M_{ac} h^c{}_{;b} + M_{cb} h^c{}_{;a}. \qquad \ldots\ldots(4.12)$$

Following the results $p_a^e\, p_b^f\, M_{ef;c} = M_{\hat{a}\hat{b};c}$ and $M_{ef} u^f = M_{ef} h^f = 0$ the equation (4.12) reduces to

$$\underset{h}{L} M_{ab} = M_{\hat{a}\hat{b};c} h^c + p_a^e\, u^f u_b M'_{ef} - p_a^e\, h^f h_b M'_{ef}$$

$$+ p_b^f\, u^e u_a M'_{ef} - p_b^f h^e h_a M'_{ef} +$$

$$+ M_{ac} h^c{}_{;b} + M_{cb} h^c{}_{;a}. \qquad \ldots\ldots(4.13)$$

By substituting the value of $h_{a;b}$ from (4.2) in equation (4.13) we get

$$\underset{h}{L} M_{ab} = M_{\hat{a}\hat{b};c}\, h^c + p_a^e\, u^f u_b\, M'_{ef} - p_a^e\, h^f h_b\, M'_{ef} +$$

$$+ p_b^f\, u^e u_a M'_{ef} - p_b^f\, h^e h_a\, M'_{ef} + M_{ac}\, h^{\hat{c}}{}_{;\hat{b}} + M_{ac} \dot{h}^c u_b$$

$$- M_{ac} h^{c'} h_b + M_{cb} h^{\hat{c}}{}_{;\hat{a}} + M_{cb} \dot{h}^c\, u_a -$$

$$- M_{cb} h^{c'} h_a. \qquad \ldots\ldots(4.14)$$

Further we note the results

$$-p^e{}_a h^f h_b M'_{ef} = p^e{}_a h^{f'} h_b M_{ef'}$$

$$= (\delta^e{}_a + u^e u_a - h^e h_a)\, M_{ef} h^{f'} h_b,$$

$$= h^{e'}\, h_b\, M_{ae'}$$

i.e., $- p^e{}_a\, h^f h_b M'_{ef} = h^{e'}\, h_b\, M_{ae'}$ .....(a)

Similarly $p^e{}_a\, u^f u_b M'_{ef} = -\, p^e{}_a u^{f'} u_b M_{ef'}$

$$= -p^e{}_a \dot{h}^f\, u_b\, M_{ef,} \text{ (vide Co.2.14))}$$

$$= -\, M_{ae} \dot{h}^e u_b,$$

i.e., $p^e{}_a u^f u_b\, M'_{ef} = M_{ae} \dot{h}^e u_b.$ .....(b)

These results (a) and (b) when used in equation (4.14) produces

$$\underset{\bar{h}}{L}\, M_{ab} = M_{\hat{a}\hat{b};c} h^c + M_{ac} h^c{}_{;\hat{b}} + M_{cb} h^c{}_{;\hat{a}}.$$

This is the required Lemma.

This Lemma is the computational aid in obtaining the strain variation equation along the space like congruence $\bar{h}$. The process is given below.

We use the Lemma (4.11) to Strain Tensor $\overset{*}{\theta}_{ab}$ to get

$$\underset{\bar{h}}{L}\, \overset{*}{\theta}_{ab} = p^e{}_a p^f{}_b\, \overset{*}{\theta}_{ef;d}\, h^d + \overset{*}{\theta}_{ad}\, h^d{}_{;\hat{b}} + \overset{*}{\theta}_{db} h^d{}_{;\hat{a}}.$$

This equation with definition of Strain Tensor $\overset{*}{\theta}_{ab}$ (4.5) yields

$$\underset{\bar{h}}{L}\ \overset{*}{\theta}_{ab} = 1/2\ p^e{}_a p^f{}_b [(h_{e;f} + h_{f;e} - u_f \dot{h}_e - u_e \dot{h}_f +$$

$$+ h_f h'_e + h_e h'_f + 2\ u^c u_e u_f \dot{h}_c - u^c u_f h_e h'_c$$

$$- u^c u_e h_f h'_c - u^c u_e h_{c;f} - u^c u_f h_{c;e})_{;d} h^d]$$

$$+ \overset{*}{\theta}_{ad} h^{\hat{d}}{}_{;\hat{b}} + \overset{*}{\theta}_{db} h^{\hat{d}}{}_{;\hat{a}}. \qquad \ldots\ldots(4.15)$$

We know the results $p_{ab}u^a = o$ and $p_{ab}h^a = 0$.

Hence the equation (4.15) becomes

$$\underset{\bar{h}}{L}\ \overset{*}{\theta}_{ab} = 1/2\ p^e{}_a p^f{}_b\ [h_{e;fd}h^d + h_{f;e;d}\ h^d - u_{f;d}\ h^d \dot{h}_e$$

$$-u_{e;d}\ h^d \dot{h}_f + h_{f;d} h^d h'_e + h_{e;d} h^d h'_f] +$$

$$+ \overset{*}{\theta}_{ad} h^{\hat{d}}{}_{;\hat{b}} + \overset{*}{\theta}_{db}\ h^{\hat{d}}{}_{;\hat{a}}\ ,$$

i.e., $\underset{\bar{h}}{L}\ \overset{*}{\theta}_{ab} = 1/2\ p^e{}_a\ p^f{}_b\ [\ h_{e;f;d}\ h^d + h_{f;e;d}\ h^d - u'_f\ \dot{h}_e -$

$$- u'_e\ \dot{h}_f + 2h'_e h'_f - u^c\ u'_e h_{c;f} - u^c\ u'_f\ h_{c;e}] +$$

$$+ \overset{*}{\theta}_{ad}\ h^{\hat{d}}{}_{;\hat{b}} + \overset{*}{\theta}_{db}\ h^{\hat{d}}{}_{;\hat{a}}. \qquad \ldots\ldots(4.16)$$

The Ricci identity for the unitary space like vector $\bar{h}$ is given by

$$h_{e;f;d}\ h^d - h_{e;d;f}\ h^d = h^d\ h^k\ R_{kefd}. \qquad \ldots\ldots(4.17)$$

From equation (4.16) and (4.17) we write

$$\underset{\bar{h}}{L}\ \overset{*}{\theta}_{ab} = 1/2\ p^e{}_a p^f{}_b\ [h_{e;d;f}\ h^d + h_{f;d;e}\ h^d + h^d h^k \times$$

$$\times (R_{kefd} + R_{kfed}) - u'_f \dot{h}_e - u'_e \dot{h}_f + 2h'_e h'_f -$$

$$-u^c u'_e\ h_{c;f} - u^c u'_f h_{c;e}] + \overset{*}{\theta}_{ad} h^{\hat{d}}{}_{;\hat{b}} + \overset{*}{\theta}_{db} h^{\hat{d}}{}_{;\hat{a}}.$$

$$\ldots\ldots(4.18)$$

By adding and substracting the terms $h_{e;d}\, h^{d}{}_{;f}$ and $h_{f;d}\, h^{d}{}_{;e}$ in equation (4.18) we get

$$\frac{L}{h}\ \overset{*}{\theta}_{ab} = 1/2\ p^{e}{}_{a}\ p^{f}{}_{b}[(h_{e;d;f}\ h^{d} + h_{e;d}h^{d}{}_{;f}) +$$

$$+ (h_{f;d;e}\ h^{e} + h_{f;d}\ h^{d}{}_{;e}) - h_{e;d}\ h^{d}{}_{;f} -$$

$$- h_{f;d}\ h^{d}{}_{;e} + 2h^{d}\ h^{k}\ R_{kefd} - u'_{f}\ \dot{h}_{e} -$$

$$- u'_{e}\dot{h}_{f} + 2h'_{e}h'_{f} - u^{c}\ u'_{e}\ h_{c;f} - u^{c}u'_{f}\ h_{c;e}] +$$

$$+ \overset{*}{\theta}_{ad}\ h^{d}{}_{;\hat{b}} + \overset{*}{\theta}_{db}\ h^{d}{}_{;\hat{a}}.$$

Further $h_{e;d;f}\ h^{d} + h_{e;d}\ h^{d}{}_{;f} = (h_{e;d}\ h^{d})_{;f},$

$$= h'_{e;f},$$

i.e., $h_{e;d;f}\ h^{d} + h_{e;d}\ h^{d}{}_{;f} = h'_{e;f}$.

This with the above equation produces

$$\frac{L}{h}\ \overset{*}{\theta}_{ab} = 1/2\ p^{e}{}_{a}p^{f}{}_{b}\ (h'_{e;f} + h'_{f;e}) + 1/2\ h^{e}{}_{a}\ h^{f}{}_{b} \times$$

$$\times[-u'_{f}\dot{h}_{e} - u'_{e}\ \dot{h}_{f} + 2h'_{e}h'_{f} + 2h^{d}h^{k}\ R_{kefd} - h_{e;d}h^{d}{}_{;f} -$$

$$-h_{f;d}\ h^{d}{}_{;e} - u^{c}\ u'_{e}h_{c;f} - u^{c}u'_{f}\ h_{c;e}] +$$

$$+ \overset{*}{\theta}_{ad}\ h^{d}{}_{;\hat{b}} + \overset{*}{\theta}_{db}\ h^{d}{}_{;\hat{a}}. \qquad .....(4.19)$$

From the definition of p_{ab} the equation (4.19) yields

$$\frac{L}{h}\ \overset{*}{\theta}_{ab} = p^{e}{}_{a}\ p^{f}{}_{b}\ h'_{(e;f)} + \frac{1}{2}(\ \delta^{e}{}_{a} - u^{e}u_{a} + h^{e}h_{a}) \times$$

$$\times (\ \delta^{f}{}_{b} - u^{f}u_{b} + h^{f}h_{b})\ [-u'_{f}\dot{h}_{e} - u'_{e}\dot{h}_{f} + 2h'_{e}h'_{f} +$$

$$+ 2h^{d}h^{k}\ R_{kefd} - h_{e;d}h^{d}{}_{;f} - h_{f;d}\ h^{d}{}_{;e} -$$

$$- u^{c}u'_{e}h_{c;f} - u^{c}u'_{f}h_{c;e}] + \overset{*}{\theta}_{ad}h^{d}{}_{;\hat{b}} + \overset{*}{\theta}_{db}h^{d}{}_{;\hat{a}}.$$

$$\text{i.e.,}\ \underset{h}{L}\ \overset{*}{\theta}_{ab} = p^e{}_a p^f{}_b\, h'_{(e;f)} + 1/2\,(\ \delta^e{}_a\,\delta^f{}_b - \delta^e{}_a\, u^f u_b + \delta^e{}_a h^f h_b - \delta^f{}_b\, u^e u_a + \delta^f{}_b\, h^e h_a + u^e u^f u_a u_b - u^e h^f u_a h_b + u^f h^e u_b h_a + h^e h^f h_a h_b)\ (\ -u'_f \dot{h}_e - u'_e \dot{h}_f + 2h'_e h'_f + 2h^d h^k R_{kefd} - h_{e;d} h^d{}_{;f} - h_{f;d}\, h^d{}_{;e} - u^c u'_d\, h_{c;f} - u^c u'_f h_{c;e}] + \overset{*}{\theta}_{ad}\, h^d{}_{;b} + \overset{*}{\theta}_{db}\, h^d{}_{;a}\ . \qquad \ldots\ldots(4.20)$$

After simplication and rearranging the terms in equation (4.20) we get

$$\underset{h}{L}\ \overset{*}{\theta}_{ab} = p^e{}_a\, p^f{}_b\, h'_{(e;f)} + 1/2\,[-u'_b \dot{h}_a - u'_a \dot{h}_b + 2h'_a h'_b + 2h^d h^k R_{kabd} - 2u^f h^d h^k u_b R_{kafd} - 2u^e h^d h^k u_a R_{kebd} + 2u^e u^f h^d h^k u_a u_b R_{kefd} + u^e u_a u'_b \dot{h}_e + u^f u_b u'_a \dot{h}_f - 2u^e u_a h'_b h'_e - 2u^f u_b h'_a h'_f - h^e h_a u'_e \dot{h}_b - h^f h_b u'_f \dot{h}_a + 2u^e u^f u_a u_b h'_e h'_f + u^e h^f u_a h_b \dot{h}_e u'_f + u^f h^e u_b h_a \dot{h}_f u'_e - h_{a;d} h^d{}_{;b} - h_{b;d}\, h^d{}_{;a} + u^e u_a h_{e;d} h^d{}_{;b} + u^e u_a h_{b;d} h^d{}_{;e} + u^f u_b h_{a;d}\, h^d{}_{;f} + u^f u_b h_{f;d}\, h^d{}_{;a} - h^e h_a h_{b;d} h^d{}_{;e} - h^f h_b h_{a;d}\, h^d - u^e u^f\, u_a u_b\, h_{e;d}\, h^d{}_{;f} - u^e u^f u_a u_b h_{f;d} h^d{}_{;e} + u^e h^f u_a h_b h_{e;d} h^d{}_{;f} + u^f h^e u_b h_a h_{f;d}\, h^d{}_{;e} - u^e u'_a\, h_{e;b} - u^f u'_b h_{f;a} +$$

$$+ u^e u_a u'_b \dot{h}_e + u^f u'_a u_b \dot{h}_f - u^e h_a u'_b h'_e -$$
$$- u^f h_b u'_a h'_f + u^e h^f u_a h_b u'_f \dot{h}_e + u^f h^e u_b h_a u'_e \dot{h}_f -$$
$$- u^e h^f h_b u'_f h_{e;a} - u^f h^e h_a u'_e h_{f;b} - u^f h^e h_a h_b u'_e h'_f -$$
$$- u^e h^f h_a h_b u'_f h'_e] + \overset{*}{\theta}_{ad} h^{\overset{\Lambda}{d}}_{;\overset{\Lambda}{b}} + \overset{*}{\theta}_{db} h^{\overset{\Lambda}{d}}_{;\overset{\Lambda}{a}}. \quad \ldots\ldots(4.21)$$

(Since $u^k u^a u^d \quad R_{kabd} = \quad h^k h^a h^d \, {}_* {}_* R_{kahd} = 0$)

Further from the natural transport law (0.2.14) the following terms from the equation (4.21) becomes

$$u^f u_b u'_a \dot{h}_f = 2u^f u_b u'_a \dot{h}_f - u^f u_b u'_a \dot{h}_f$$
$$= 2u^f u_b u'_a \dot{h}_f - u^f u_b \dot{h}_a \dot{h}_f + u^e u^f u_a u_b \dot{h}_e \dot{h}_f -$$
$$- u^e u^f h_a u_b h_e \dot{h}_f,$$

and $- u^f u'_a h_{f;b} = - 2u^f u'_a h_{f;b} + u^f u'_a h_{f;b}$

$$= - 2u^f u'_a h_{f;b} + u^f \dot{h}_a h_{f;b} - u^e u^f u_a \dot{h}_e h_{f;b} +$$
$$+ u^e u^f h_a h'_e h_{f;b}.$$

Similarly for the terms $u^e u_a u'_b \dot{h}_e$ and $- u^e u'_b h_{e;a}$.

By substituting these values in equation(4.21) and after then simplification we get

$$
\begin{aligned}
\frac{L}{h}\, \theta_{ab} = {} & p^{e}{}_{a} p^{f}{}_{b} h'_{(e;f)} + 1/2\, [-u'_{b}\dot{h}_{a} - u'_{a}\dot{h}_{b} + 2h'_{a}h'_{b} + \\
& + 2h^{d}h^{k}(R_{kabd} - u^{e}u_{a} R_{kebd} - u^{f}u_{b} R_{kafd} + \\
& + u^{e}u^{f}u_{a}u_{b}R_{kefd}) + 2u^{e}u_{a}u'_{b}\dot{h}_{e} + 2u^{f}u_{b}u'_{a}\dot{h}_{f} - \\
& -2u^{e}u_{a}h'_{b}h'_{e} - 2u^{f}u_{b}h'_{a}h'_{f} + 2u^{e}u^{f}u_{a}u_{b}h'_{e}h'_{f} + \\
& + 2u^{e}h^{f}u_{a}h_{b}u'_{f}\dot{h}_{e} + 2u^{f}h^{e}u_{b}h_{a}u'_{e}\dot{h}_{f} + u^{e}u'_{a}u_{b}\dot{h}_{e} + \\
& + u^{f}u'_{b}u_{a}\dot{h}_{f} - u^{e}h_{a}u'_{b}\dot{h}_{e} - u^{f}h_{b}u'_{a}\dot{h}_{f} - \\
& - u^{e}h^{f}h_{a}h_{b}u'_{f}h'_{e} - u^{f}h^{e}h_{a}h_{b}\, u'_{e}h'_{f} - \\
& - 2u^{e}u'_{a}h_{e;b} - 2u^{f}u'_{b}h_{f;a} + 2u^{e}u^{f}h_{a}h'_{e}h_{f;b} + \\
& + 2u^{e}u^{f}h_{b}h'_{f}h_{e;a} - h_{a;d}(\dot{h}^{d}{}_{;b} - \dot{h}^{d}u_{b} + h^{d\prime}h_{b} + \\
& + u^{e}u^{d}u_{b}\dot{h}_{e} - u^{e}u^{d}h_{b}h'_{e} - u^{e}u^{d}h_{e;b}) - h_{b;d}\times \\
& \times(h^{d}{}_{;a} - \dot{h}^{d}u_{a} + h^{d\prime}h_{a} + u^{f}u^{d}u_{a}\dot{h}_{f} - u^{f}u^{d}h_{a}h'_{f} - \\
& -u^{f}u^{d}h_{f;a}) + u^{e}u_{a}h_{e;d}(h^{d}{}_{;b} - \dot{h}^{d}u_{b} + h^{d\prime}h_{b} + \\
& + u^{f}u^{d}u_{b}\dot{h}_{f} - u^{f}u^{d}h_{a}h'_{f} - u^{f}u^{d}h_{f;b}) + u^{f}u_{b}h_{f;d} \\
& (h^{d}{}_{;a} - \dot{h}^{d}u_{a} + h^{d\prime}h_{a} + u^{e}u^{d}u_{a}h_{e} - u^{e}u^{d}h_{a}h'_{e} - \\
& - u^{e}u^{d}h_{e;a}) + \overset{*}{\theta}_{ad}\, h^{\hat{d}}{}_{;\hat{b}} + \overset{*}{\theta}_{db} h^{\hat{d}}{}_{;\hat{a}}. \quad \ldots\ldots(4.22)
\end{aligned}
$$

Again using the natural transport law (0.2.14) and rearranging the terms the equation (4.22) reduces to

$$
\begin{aligned}
\frac{L}{h}\, \overset{*}{\theta}_{ab} = {} & p^{e}{}_{a} p^{f}{}_{b}\, h'_{(e;f)} + 1/2\, [-2u'_{a}u'_{b} + 2h'_{a}h'_{b} + \\
& + 2\, h^{d}h^{k}(R_{kabd} - u^{e}u_{a}R_{kebd} - u^{f}u_{b}\, R_{kafd} +
\end{aligned}
$$

$$+ u^e u^f u_a u_b R_{kefd}) + 2u^e u_a u'_b \dot{h}_e + 2u^f u_b u'_a \dot{h}_f -$$

$$- 2u^e u_a h'_b h'_e - 2u^f u_b h'_a h'_f + 2u^e u^f u_a u_b h'_e h'_f +$$

$$+ 2u^e h^f u_a h_b u'_f \dot{h}_e + 2u^f h^e u_b h_a u'_e \dot{h}_f - 2\, u^e h^f h_a h_b u'_f h'_e -$$

$$- 2u^e u'_a h_{e;b} - 2u^f u'_b h_{f;a} + 2u^e u^f h_a h'_e h_{f;b} +$$

$$+ 2u^e u^f h_b h'_f h_{e;a} - h_{a;d} h^{\hat{d}}{}_{;\hat{b}} - h_{b;d} h^{\hat{d}}{}_{;\hat{a}} +$$

$$+ u^e u_a h_{e;d}\, h^{\hat{d}}{}_{;\hat{b}} + u^f u_b h_{f;d} h^{\hat{d}}{}_{;\hat{a}}] +$$

$$+ \overset{*}{\theta}_{ad} h^{\hat{d}}{}_{;\hat{b}} + \overset{*}{\theta}_{db} h^{\hat{d}}{}_{;\hat{a}}.$$

(Since $\dot{u}_a h^a = - \dot{u}^a h_a$ and $u'_a \dot{h}^a = -u_a h^{a\prime}$)

i.e., $$\frac{L}{h} \overset{*}{\theta}_{ab} = p^e{}_a p^f{}_b h'_{(e;f)} + u'_a u'_b + h'_a h'_b + h^d h^k (R_{kabd} -$$

$$- u^e u_a R_{kebd} - u^f u_b R_{kafd} + u^e u^f u_a u_b R_{kefd} + 1/2 \times$$

$$\times [\underbrace{- 4 u'_a u'_b - 2u^e u'_a h_{e;b} - 2u^f u'_b h_{f;a} + 2u^e u_a u'_b \dot{h}_e}_{T_1} +$$

$$+ \underbrace{2u^f u_b u'_a \dot{h}_f - 2u^e u_a h'_b h'_e - 2u^f u_b h'_a h'_f}_{T_2} +$$

$$+ \underbrace{2u^e h^f u_a h_b u'_f \dot{h}_e + 2u^e u^f h_b h'_f h_{e;a} + 2u^f h^e u_b h_a u'_e \dot{h}_f}_{T_3} +$$

$$+ \underbrace{2u^e u^f h_a h'_e h_{f;b} + 2u^e u^f u_a u_b h'_e h'_f - 2u^e h^f h_a h_b u'_f h'_e}_{T_4}$$

$$\underbrace{- h^{\hat{d}}{}_{;\hat{b}}(h_{a;d} - u^e u_a h_{e;d} - 2\, \overset{*}{\theta}_{ad}) - h^{\hat{d}}{}_{;\hat{a}}(h_{b;d} - u^f u_b h_{f;d} - 2\, \overset{*}{\theta}_{bd})}_{T_5}\ ] \qquad .....(4.23)$$

Step I : To simplify T_1 :

$$T_1 = -4u'_a u'_b - 2u^e u'_a h_{e;b} + 2u^f u_b u'_a \dot{h}_f -$$

$$- 2u^f u'_b h_{f;b} + 2u^e u_a u'_b \dot{h}_e ,$$

$$= -4u_{\hat{a};\hat{e}} u_{\hat{b};\hat{f}} h^e h^f + 2h^f u'_a (\dot{u}_{f;b} - \dot{u}_f u_b)$$

$$+ 2h^e u'_b (u_{e;a} - \dot{u}_e u_a),$$

$$(\text{Since } u_{\hat{a};\hat{e}} h^e = (u_{a;e} - \dot{u}_a u_e) h^e = u'_a)$$

$$= -2u_{\hat{a};\hat{e}} u_{\hat{b};\hat{f}} h^e h^f + 2u_{\hat{a};\hat{e}} u_{\hat{f};\hat{b}} h^e h^f -$$

$$- 2u_{\hat{a};\hat{e}} u_{\hat{b};\hat{f}} h^e h^f + 2u_{\hat{b};\hat{f}} u_{\hat{e};\hat{a}} h^e h^f ,$$

$$= -2u_{\hat{a};\hat{e}} h^e h^f (u_{\hat{b};\hat{f}} - u_{\hat{f};\hat{b}}) - 2u_{\hat{b};\hat{f}} h^e h^f \times$$

$$\times (u_{\hat{a};\hat{e}} - u_{\hat{e};\hat{a}}) ,$$

$$= -2(\theta_{ae} + W_{ae}) (2 W_{bf}) h^e h^f - 2 (\theta_{bf} + W_{bf}) \times$$

$$\times (2 W_{ae}) h^e h^f ,$$

(Vide equations (2.3) and (2.5))

i.e., $T_1 = -4h^e h^f [2 W_{ae} W_{bf} + \theta_{ae} W_{bf} + \theta_{ae} W_{ae}]$. .....(4.24)

Step II : To simplify T_2 :

$$T_2 = -2u^e u_a h'_b h'_e - 2 u^f u_b h'_a h'_f ,$$

$$= 2 h^e u'_e (u_a h'_b + u_b h'_a) ,$$

$$= 2 h^e h^f u_{\hat{e};\hat{f}} (u_a h'_b + u_b h'_a) ,$$

$$= 2 h^e h^f (\theta_{ef} + W_{ef}) (u_a h'_b + u_b h'_a)$$

i.e., $T_2 = 2h^e h^f \theta_{ef} (u_a h'_b + u_b h'_a).$ .....(4.25)

Step III : To simplify T_3 :

$$T_3 = 2u^e u^f h_a h'_e h_{f;b} + 2u^f h^e u_b h_a u'_e \dot{h}_f +$$

$$+ 2 u^e u^f h_b h'_f h_{e;b} + 2u^e h^f u_a h_b u'_f \dot{h}_e ,$$

$$= 2 h^e h^f h_a u'_e (u_{f;b} - \dot{u}_f u_b) + 2 h^e h^f h_b u'_f \times$$

$$\times (u_{e;a} - \dot{u}_e u_a) ,$$

$$(\text{by } u_a h^{a'} = -u^{a'} h_a)$$

$$= 2h^d h^e h^f h_a u_{\hat{e};\hat{d}} u_{\hat{f};\hat{b}} + 2 h^d h^e h^f h_b u_{\hat{f};\hat{d}} u_{\hat{e};\hat{a}} ,$$

$$= 2h^d h^e h^f u_{\hat{e};\hat{d}} [h_a (u_{\hat{f};\hat{b}}) + h_b (u_{\hat{f};\hat{a}})] ,$$

$$= 2h^d h^e h^f (\theta_{ed} + W_{ed}) [h_a (\theta_{fb} + W_{fb}) +$$

$$+ h_b (\theta_{fa} + W_{fa})] ,$$

(By definition of $u_{\hat{a};\hat{b}}$).

i.e., $T_3 = 2h^d h^e h^f \theta_{ed} [h_a (\theta_{fb} + W_{fb}) + h_b (\theta_{fa} + W_{fa})].$

.....(4.26)

Step IV : To simplify T_4 :

$$T_4 = 2u^e u^f u_a u_b h'_e h'_f - 2 u^e u^f h_a h_b u'_f h'_e$$

$$= 2u^e u^f h'_e h'_f (u_a u_b + h_a h_b)$$

$$= 2h^e h^f u'_e u'_f (u_a u_b + h_a h_b)$$

$$= 2h^{d}h^{k}h^{e}h^{f}\, u_{\hat{e};\hat{d}}\, u_{\hat{f};\hat{k}}\, (u_a u_b + h_a h_b)$$

$$= 2h^{d}h^{k}h^{e}h^{f}\, (\theta_{ed} + W_{ed})\, (\theta_{fk} + W_{fk}) \times$$

$$\times (u_a u_b + h_a h_b)$$

i.e., $$T_4 = 2h^{d}h^{k}h^{e}h^{f}\, \theta_{ed}\, \theta_{fk}\, (u_a u_b + h_a h_b). \qquad \ldots\ldots(4.27)$$

Step V : To simplify T_5 :

$$T_5 = -\, \hat{h}^{d}{}_{;\hat{b}}\, (h_{a;d} - u^{e}u_{a}h_{e;d} - 2\, \overset{*}{\theta}_{ad}) -$$

$$-\, \hat{h}^{d}{}_{;\hat{a}}\, (h_{b;d} - u^{f}u_{b}h_{f;d} - 2\, \overset{*}{\theta}_{db}) .$$

Since by the definition $h_{\hat{a};\hat{b}}$ (4.8) and $\overset{*}{\theta}{}^{a}{}_{b}\, h_a = \overset{*}{\theta}{}^{a}{}_{b}\, u_a = \overset{*}{W}{}^{a}{}_{b}u_a = \overset{*}{W}{}^{a}{}_{b}h_a = 0$ the above equation becomes

$$T_5 = -\, (\sigma^{d}{}_{b} + \overset{*}{W}{}^{d}{}_{b})(h_{\hat{a};\hat{d}} - 2\, \overset{*}{\theta}_{ad}) - (\overset{*}{\theta}{}^{d}{}_{a} + \overset{*}{W}{}^{d}{}_{a}) \times$$

$$\times (h_{\hat{d};\hat{b}} - 2\, \overset{*}{\theta}_{db})$$

$$= -\, (\overset{*}{\theta}{}^{d}{}_{b} + \overset{*}{W}{}^{d}{}_{b})\, (\overset{*}{W}_{ad} + \overset{*}{\theta}_{ad} - 2\, \overset{*}{\theta}_{ad}) -$$

$$-\, (\overset{*}{\sigma}{}^{d}{}_{a} + \overset{*}{W}{}^{d}{}_{a})(\overset{*}{W}_{db} + \overset{*}{\theta}_{db} - 2\, \overset{*}{\theta}_{db})$$

i.e. $$T_5 = (\overset{*}{\sigma}{}^{d}{}_{b} + \overset{*}{W}{}^{d}{}_{b})\, (\overset{*}{\theta}_{ad} - \overset{*}{W}_{ad}) + (\overset{*}{\sigma}{}^{d}{}_{a} + \overset{*}{W}{}^{d}{}_{a})$$

$$(\overset{*}{\theta}_{bd} - \overset{*}{W}_{bd}) \qquad \ldots\ldots(4.28)$$

By substituting the value from (4.24), (4.25), (4.26) ,(4.27) and (4.28) the equation (4.23) reduces to

$$\underset{\bar{u}}{L}\theta_{ab} = p^{e}_{a}p^{f}_{b}\,h'_{(e;f)} + u'_{a}u'_{b} + h'_{a}h'_{b} + h^{d}h^{k}\times$$

$$\times(R_{kabd} - u^{e}u_{a}R_{kebd}) - u^{f}u_{b}R_{kafd} + u^{e}u^{f}u_{a}u_{b}R_{kefd}$$

$$- 2h^{e}h^{f}\,[\,(2\,W_{ae}W_{bf} + \theta_{ae}W_{bf} + \theta_{bf}\,\theta_{ae}) -$$

$$- 1/2\;\theta_{ef}\;[\,u_{a}h'_{b} + u_{b}h'_{a}\,] + h^{d}h^{e}h^{f}\;\theta_{ed}\times$$

$$\times[\,h_{a}\,(\,\theta_{fb} + W_{fb}) + h_{b}\,(\,\theta_{fa} + W_{fa}\,)\,] +$$

$$+ h^{d}h^{e}h^{f}h^{k}\;\theta_{ed}\;\theta_{fk}\,(\,u_{a}u_{b} + h_{a}h_{b}) +$$

$$+ 1/2\,(\,\overset{*}{\sigma}{}^{d}{}_{b} + \overset{*}{W}{}^{d}{}_{b})\,(\,\overset{*}{\theta}_{ad} - \overset{*}{h}_{ad}) +$$

$$+ 1/2\,(\,\overset{*}{\sigma}{}^{d}{}_{a} + \overset{*}{W}{}^{d}{}_{a})\,(\,\overset{*}{\theta}_{db} - \overset{*}{W}_{db}). \qquad \ldots (4.29)$$

This is the required strain variation equation along the space-like vector $\bar{h}$.

5. DYNAMICAL FORM OF STRAIN VARIATION EQUATION :

To bring the dynamical quantitives in the strain variation equation (4.29) we use the Weyl Tensor expression given by (2.19)

Let us consider the term in strain variation equation (4.29)

$$h^{d}h^{k}(\,R_{kabd} - u^{e}u_{a}\,R_{kebd} - u^{f}u_{b}\,R_{kafd} + u^{e}u^{f}u_{a}u_{b}\,R_{kefd}). \qquad \ldots\ldots(5.1)$$

We write from (5.1)

$$h^d h^k u^e u^f u_a u_b \; R_{kefd} = h^d h^k u^e u^f u_a u_b \; [\; C_{kefd} - 1/2 \; (\; g_{kf} R_{ed} -$$

$$g_{kd} R_{ef} + g_{ed} \; R_{kf} - g_{ef} \; R_{kd}) - R/6 \; (g_{kd} g_{fe} - g_{kf} g_{ed}),$$

i.e., $h^d h^k u^e u^f u_a u_b \; R_{kefd} = h^d h^k \; u^e u^f u_a u_b C_{kefd} - 1/2 \; u_a u_b R_{ef} \times$

$$\times (u^e u^f - h^e h^f) + R/6 \; u_a u_b.$$

This equation with Ricci Tensor expression (0.3.3) gives

$$h^d h^k u^e u^f u_a u_b \; R_{kefd} = h^d h^k u^e u^f u_a u_b \; C_{kefd} + K/2 \; (2p + \mu H^2) \times$$

$$\times u_a u_b + R/6 \; u_a u_b,$$

i.e., $h^d h^k u^e u^f u_a u_b \; R_{kefd} = h^d h^k u^e u^f u_a u_b \; C_{kefd} +$

$$+ \; k/6 (\rho + 3p + 3 \; \mu H^2) \; u_a u_b.$$

(Since $R = K \; (\rho - 3p)$) .....(5.2)

Similarly we get

$$h^d h^k u^e u_a \; R_{kebd} = h^d h^k u^e u_a \; C_{kebd} + K/6 (\rho + 3p + \mu H^2) u_a u_b,$$

.....(5.3)

$$h^d h^k u^f u_b \; R_{kafd} = h^d h^k u^f u_b \; C_{kafd} + K/ 6 (\rho + 3p + \mu H^2) u_a u_b,$$

.....(5.4)

and

$$h^d h^k R_{kabd} = h^d h^k \; C_{kabd} + K/2 \; (\rho + p + \mu H^2) u_a u_b -$$

$$- \; 1/3 \; K\rho \; (\; h_a h_b + g_{ab}).$$ .....(5.5)

It follows from equations (5.1), (5.2), (5.3), (5.4) and (5.5)

$$h^d h^k \left(R_{kabd} - u^e u_a R_{kebd} - u^f u_b R_{kafd} + \right.$$

$$\left. + u^e u^f u_a u_b R_{kefd} \right) = h^d h^k \left(C_{kabd} - u^e u_a C_{kebd} - \right.$$

$$\left. - u^f u_b C_{kafd} + u^e u^f u_a u_b C_{kefd} \right)$$

$$+ K/2 \; (\rho + p + \mu H^2) \; u_a u_b -$$

$$- 1/3 \; K \rho (h_a h_b + g_{ab}) -$$

$$-2K/6 \; (\rho + 3p + 3 \mu H^2) \; u_a u_b +$$

$$+ K/6 \; (\rho + 3p + 3 \mu H^2) \; u_a u_b,$$

i.e., $$h^d h^k \left(R_{kabd} - u^e u_a R_{kebd} - u^f u_b R_{kafd} + \right.$$

$$\left. + u^e u^f u_a u_b R_{kefd} \right) = h^d h^k \left(C_{kabd} - u^e u_a C_{kebd} - \right.$$

$$\left. - u^f u_b C_{kafd} + u^e u^f u_a u_b C_{kefd} \right) -$$

$$-1/3 \; K \rho P_{ab} \qquad \ldots\ldots(5.6)$$

This in terms of tensor $\bar{C}_{ab} = h^d h^k C_{adbk}$, produces

$$h^d h^k \left(R_{kabd} - u^e u_a r_{kebd} - u^f u_b R_{kafd} + u^e u^f u_a u_b R_{kefd} \right) =$$

$$= -\bar{C}_{ab} + u^e u_a \bar{C}_{be} + u^f u_b \bar{C}_{af} - u^e u^f u_a u_b \bar{C}_{ef} - 1/3 K \rho \, p_{ab}.$$

$$\ldots\ldots(5.7)$$

Hence from equation (5.7) and (4.29) we have

$$\frac{L}{h} \overset{*}{\theta}_{ab} = p^e{}_a p^f{}_b h'_{(e;f)} + u'_a u'_b + h'_a h'_b - \bar{C}_{ab} +$$

$$+ u^e u_a \bar{C}_{be} + u^f u_b \bar{C}_{af} - u^e u^f u_a u_b \bar{C}_{ef} - 2h^e h^f \times$$

$$\times [2 \; (W_{ae} W_{bf} + \theta_{ae} W_{bf} + \theta_{bf} W_{ae} - 1/2 \; \theta_{ef} (u_a h'_b +$$

$$u_b h'_a) +$$

$$+ h^d h^e h^f{}_{ed} \, [\, h_a (\theta_{fb} + W_{fb}) + h_b (\theta_{fa} + W_{fa})] \, +$$

$$+ h^d h^e h^f h^k \, \theta_{ed} \, \theta_{fk} \, (u_a u_b + h_a h_b) + 1/2 \, (\overset{*}{\sigma}{}^d{}_b + \overset{*}{W}{}^d{}_b) \times$$

$$\times (\overset{*}{\theta}_{ad} - \overset{*}{W}_{ad}) + 1/2 \, (\overset{*}{\sigma}{}^d{}_a + \overset{*}{W}{}^d{}_a) \, \overset{*}{\theta}_{db} - \overset{*}{W}_{db}) \, -$$

$$- 1/3 \, K \rho \, p_{ab}. \qquad \qquad .....(5.8)$$

This is the required dynamical form of strain variation equation along the space like congruence involving the kinematical parameters and parameters associated with the space like congruences.

I. INERTIAL REFERENCE FRAME (IRF) :

A cloud of free test particles which moves rigidly and without rotation represents an inertial reference frame (Audretsch, 1971). So it is characterized by

$$u_{a;b} = 0 \text{ (covariently constant flow)}$$

$$\Rightarrow \quad \dot{u}_a = \theta = W_{ab} = \theta_{ab} = 0 \, . \qquad \qquad .....(5.9)$$

From Ricci identity (2.15) we have

$$u_{a;bc} - u_{a;cb} = u^k \, R_{kabc}.$$

This implies that

$$R_{abcd} \, u^a = 0 \, .$$

Consequently

$$R_{ab} u^a = 0. \qquad \qquad .....(5.10)$$

In case of the space-time admiting IRF, the strain variation equation (4.29) becomes

$$\underset{h}{L}\ \theta_{ab} = p^e{}_a p^f{}_b h'_{(e;f)} + h'_a h'_b + h^d h^k R_{kabd} + $$
$$+ 1/2\ (\overset{*}{\sigma}{}^d{}_b + \overset{*}{W}{}^d{}_b)\ (\overset{*}{\theta}_{ad} - \overset{*}{W}_{ad}) + $$
$$+ 1/2\ (\overset{*}{\sigma}{}^d{}_a + \overset{*}{W}{}^d{}_a)\ (\overset{*}{\theta}_{db} - \overset{*}{W}_{db}). \qquad .....(5.11)$$

UNIFORM MAGNETIC FIELD (UMF)

This is described through the condition $h_{a;b} = 0$.

$$\text{i.e.,}\quad h_{a;b}\, h^b = h'_a = 0,\ h^a{}_{;a} = 0 , \qquad .....(5.12)$$

$$\text{i.e.,}\quad \overset{*}{\theta} = \overset{*}{W}_{ab} = \overset{*}{\theta}_{ab} = 0 .$$

Hence the Ricci identities for space like congruence given by

$$h_{a;b;c} - h_{a;c;b} = h^k\ R_{kabc},$$

yields

$$R_{ab} h^a = 0. \qquad .....(5.13)$$

Thus the strain variation equation (4.29) in UMF is in the form

$$\underset{h}{L}\ \theta_{ab} = p^e{}_a\ p^f{}_b\ h'_{(e;f)} + u'_a u'_b - 2h^e h^f [\ 2\ W_{ae} W_{bf}) +$$
$$+ \theta_{ac} W_{bf} + \theta_{ae}\ \theta_{bf}\ ] + h^d h^e h^f\ \theta_{ed} \times$$
$$\times [\ h_a\ (\ \theta_{fb} + W_{fb}) + h_b\ (\ \theta_{fa} + W_{fa})\ ] +$$
$$+ h^d h^e h^f h^k \theta_{ed}\ \theta_{fk}\ (u_a u_b + h_a h_b). \qquad .....(5.14)$$

1. INTRODUCTION :

The general transport equation in classical rheology is due to Fredrickson (1964). There exists several types of transport equations available in the literature of classical continuum mechanics. The well-known types of transport equations deals with material transport of entities like mass, energy and momentum. The extension of these transport equations in relativistic domain are due to Narlikar (1978). He has given a concept of parallel transport facilitating the difinition of constant congruences. The astrophysical applications of optical transport equation have been given by Dyer (1977). The transport equations of the kinematical parameters are found by Greenberg (1970). The consequences of Lie transport of a 3-space projection operator leading to Born-rigid motion has examined by Pirani and Williams (1962). Lie transport of different types of tensor fields have been extensialy studied by Davis (1977), Oliver and Davis (1976), Radhakrishna and Rao (1976), Asgekar and Date (1978). The concept of convective transport for non-geodesic congruences is introduced by Oldroyd (1950)and further developed by Carter and Quintana (1977). The relavance of Truesdell transport is explained by Radhakrishna and Walwadkar (1984).

In this chapter we examine the special types of transport equations concomitant with ferrofluid space-time. Our efforts will be directed to evaluate the effects of these transport conditions on the geometrical structure of the space time. Especially we examine how strain and stress and their rates of change are affected by these transport equations.

The section 2 includes the study of self similarity of space time corresponding to ferro-fluid system. The contributors in this area are Taub (1971), Eardely (1974), Asgekar and Date (1977) and Wilson (1986). Our aim in section 3 is to expose the rheological properties of ferrofluid under Jaumann transport. The section 4 includes the Truesdell stress rate and stress field of ferrofluid system. The last section is devoted to study of Fermi-Walker and Convective transport along space like congruences compatible with ferrofluid space-time.

2. RHEOLOGY OF FERROFLUID SPACE-TIME :

We consider two special types of spaces in this article.

I) A Self Similar Space-time :

Definition : This space is characterized by the following mathematical expressions (Davis; 1984).

$$\underset{\bar{X}}{L}\, g_{ab} = \lambda_1\, g_{ab}. \qquad \qquad .....(2.1)$$

Where $\bar{X}$ is any arbitrary vector and λ_1 is any scalar function of Co-ordinates.

We present two special cases of condition (2.1)

Case (1) : Let $\bar{X} = X.\bar{u}$,

Where $\bar{u}$ is time like unit vector.

For this choice equation (2.1) produces

$$Xu_{a;b} + X_{;b}u_a + Xu_{b;a} + X_{;a}u_b = \lambda_1 g_{ab}. \qquad .....(2.2)$$

If we contract these equations with g^{ab}, $u^a u^b$ and $H^a H^b$ we get the following results

$$X\theta + \dot{X} = 2\lambda_1 , \qquad .....(2.3)$$

$$2\dot{X} = \lambda_1 , \qquad .....(2.4)$$

$$X_{;b}H^b + X\dot{u}_bH^b = 0 , \qquad .....(2.5)$$

and

$$\lambda_1 H^2 = -(2X)\theta_{ab}H^aH^b . \qquad .....(2.6)$$

INTERPRETATIONS : We conclude from the above results that the scalar λ_1 and the deformation tensor field θ_{ab} depend explicitely on the magnitude of vector $\bar{X}$. [Vide equation (2.3) and (2.6)].

REMARK : It follows from (2.1) that $\lambda_1 = 0$ for unit vector $\bar{X}$. This states that, the self similar transformations are not compatible with time like unit flow.

Case (2) : Let $\bar{X} = \bar{H}$ (Magnetic field vector). Then the equation (2.1) becomes

$$\underset{H}{£}\, g_{ab} = \lambda_1 g_{ab} , \qquad .....(2.7)$$

Where λ_1 is scalar function.

i.e., $H_{a;b} + H_{b;a} = \lambda_1 g_{ab}$. .....(2.8)

These conditions when contracted with g^{ab}, u^au^b, u^aH^b and H^aH^b give rise to following results respectively.

$$\lambda_1 = 1/2\ H^a{}_{;a} , \qquad .....(2.9)$$

$$2\dot{H}_a u^a = \lambda_1 , \qquad .....(2.10)$$

$$\theta_{ab}H^aH^b + 1/2\ \dot{H}^2 = 0 , \qquad .....(2.11)$$

$$(H^2)_{;b}H^b = \lambda_1 H^2 . \qquad .....(2.12)$$

From equation (2.11), the equation (0.4.3) reduces to

$$\mu\theta + \dot{\mu} = 0. \qquad .....(2.13)$$

This implies that

$$\theta = 0 \Leftrightarrow \dot{\mu} = 0 . \qquad .....(2.14)$$

We infer from this that the magnetic permeability of ferrofluid is preserved along the expansion free flow.

Further we obtain from equations (2.13), (0.4.3) and (0.4.8)

$$\dot{\rho} + (\rho + p + 1/2\ \mu H^2)\theta = 0. \qquad .....(2.15)$$

This produces the result

$$\dot{\rho} = 0 \Leftrightarrow \theta = 0. \qquad .....(2.16)$$

Hence we conclude that the energy density of the ferrofluid remains constant along the expansion free flow.

On using the results (2.9) and (2.10) in Maxwell equation (0.4.4) we get

$$\mu \lambda_1 = -2/3\ \mu_{;b}H^b . \qquad ......(2.17)$$

From the local conservation laws $T^{ab}{}_{;b} = 0$, we get (0.4.10)

$$(\rho + p)\ \dot{u}^a H_a - p_{;b}H^b + H^2[\mu(\dot{u}^a\ H_a + H^b{}_{;b}) +$$
$$+ 1/2\ \mu_{;b}H^b] = 0. \qquad .....(2.18)$$

Further by substituting equations (2.9) and (2.10) in equation (2.18) we derive

$$\lambda_1 = -4\,(2\rho + 2p - 3\,\mu H^2)^{-1} p_{;b} H^b. \qquad .....(2.19)$$

This shows that the explicit dependence of isotropic pressure p on λ_1.

Thus it follows from (2.17) and (2.19) that

$$\mu_{;b} H^b = 6\mu\,(2\rho + 2p - 3\mu H^2)^{-1}\, p_{;b} H^b. \qquad .....(2.20)$$

This provides

$$\mu_{;b} H^b = 0 \Leftrightarrow p_{;b} H^b = 0\ . \qquad .....(2.21)$$

It is observed from this result that the magnetic permeability along the magnetic lines is preserved if and only if the isotropic pressure to ferrofluid remains constant along the magnetic lines.

II) EINSTIEN SPACE :

Definition : The Einstien space is defined through the following expression (Petrov, 1969)

$$R_{ab} = \lambda_2 g_{ab}\ . \qquad .....(2.22)$$

Where R_{ab} is Ricci tensor and λ_2 is Scalar. From the expression (0.3.3) of the Ricci tensor of ferrofluid the equation (2.22) becomes

$$-K\,[(\rho + p + \mu H^2)\, u_a u_b - 1/2\,(\rho - p + \mu H^2) g_{ab} - \mu H_a H_b] = \lambda_2 g_{ab}. \qquad .....(2.23)$$

The transvections of this result (2.23) with $u^a u^b$ and $H^a H^b$ respectively produces the following results

$$-K/2\ (\rho + 3p + \mu H^2) = \lambda_2\ , \qquad \ldots..(2.24)$$

$$K/2\ (\rho - p - \mu H^2) = \lambda_2. \qquad \ldots..(2.25)$$

From the equations (2.24) and (2.25) we get

$$\rho + 2p = 0\ ,$$

This provides the results

$$\rho = 0,\ p = 0, \qquad \ldots..(2.26)$$

and

$$(\mu H^2) = 0 \Rightarrow \mu = 0 \text{ or } H^2 = 0. \qquad \ldots..(2.27)$$

The equations (2.26) and (2.27) states that the ferrofluid space-time cannot be Einstien space.

3. JAUMANN TRANSPORT WITH HYPOELASTIC MEDIA :

I) Jaumann Transport of Ferrofluid :

From the definition of Jumann stress rate (0.5.3) and the stress energy tensor expression (0.3.1) for ferrofluid we get

$$\underset{u}{J}\ T_{ab} = (\rho + p + \mu H^2)^{\cdot} u_a u_b + (\rho + p + \mu H^2)(\dot{u}_a u_b + u_a \dot{u}_b) - (p + 1/2\, \mu H^2)^{\cdot} g_{ab} - (\mu H_a H_b)^{\cdot} + \mu H_c H_b\ W^c{}_{.a} + \mu H_c H_a W^c{}_{.b}. \qquad \ldots..(3.1)$$

The condition $\underset{u}{J}\ T_{ab} = 0$ characterizes Jaumann transport of ferrofluid. We write from equation (3.1)

$$(\rho + p + \mu H^2)^{\cdot} u_a u_b + (\rho + p + \mu H^2)(\dot{u}_a u_b + u_a \dot{u}_b) - (p + 1/2\, \mu H^2)^{\cdot} g_{ab} - (\mu H_a H_b)^{\cdot} + \mu H_c H_b W^c_a + \mu H_c H_a\ W^c{}_{.b} = 0 \qquad \ldots..(3.2)$$

From this we derive the following results

$$(\rho - 3p)\dot{=} 0 \,, \qquad \ldots.(3.3)$$

$$(\rho + p + \mu H^2)^{\cdot} u_b + (\rho + p + \mu H^2)\, \dot{u}_b - $$
$$- (p + 1/2\, \mu H^2)^{\cdot} u_b - \mu \dot{H}_a u^a H_b = 0, \qquad \ldots..(3.4)$$

$$(\rho + 1/2\, \mu H^2)^{\cdot} = 0 \,, \qquad \ldots..(3.5)$$

$$(\rho + p + \mu H^2)\, \dot{u}_a H^a u_b - (p + 1/2\, \mu H^2)^{\cdot} H_b +$$
$$+ \dot{\mu} H^2 H_b + \dot{\mu}/2\, H^2 H_b + \mu H^2 \dot{H}_b -$$
$$- \mu H^2 H_c W^c_b = 0 \,, \qquad \ldots..(3.6)$$

$$(p - 1/2 \mu H^2)^{\cdot} = 0 \,. \qquad \ldots..(3.7)$$

Again by inner multiplication of (3.4) with H^b we get

$$(\rho + p)\, \dot{u}_b H^b = 0 \,,$$

$$\dot{u}_b H^b = 0 \,, \text{ as } \rho + p \neq 0. \qquad \ldots..(3.8)$$

This shows that the acceleration is normal to the magnetic lines.
From equations (3.5) and (3.7) we have

$$\dot{\rho} + \dot{p} = 0 \qquad \ldots..(3.9)$$

Hence we deduce

$$\dot{\rho} = 0 \,, \dot{p} = 0,$$

and

$$\dot{\mu} = 0,\ \dot{H}^2 = 0,\ (\text{by } (3.4)) \qquad \ldots..(3.9')$$

On using the above results in equation (3.6) we get

$$\dot{H}_b - H_c W^c_b = 0. \qquad \ldots..(3.10)$$

From (3.8),(3.9) and (3.4) we get

$$\dot{u}_a = 0 \qquad \ldots..(3.10')$$

$$\underset{u}{\overset{J}{}} T_{ab} = 0 \Leftrightarrow \dot{\rho} = \dot{p} = \dot{\mu} = \dot{H}^2 = \dot{u}_a = o,$$

$$\dot{H}_b - H_c W^c_b = 0. \qquad .....(3.11)$$

INTERPRETATION : If the Jaumann stress rate of ferrofluid system vanishes then the density, pressure, magnetic permeability and magnitude of the magnetic field are preserved with respect to the geodesic flow (vide (3.11).

II) J-Type Hypoelasticity : Hypoelstic material is characterized through the constitutive relation (Prager, 1961)

$$\bar{T}_{ab} = \lambda_{abcd}\, \theta^{cd} \,. \qquad .....(3.12)$$

Where $\bar{T}_{ab}$ is stress rate of T_{ab} and λ_{abcd} is the responce tensor which is the function of deformation tensor as well as stress energy tensor.

There are twelve choices of λ_{abcd} (Synge's Cuckoo Process) out of these we have considereed only four cases with T_{ab} as the stress energy tensor of ferrofluid system. In case of Jaumann relativistic stress rate, constitutive relation (3.12) becomes (Walwadkar, 1983)

$$\underset{u}{\overset{J}{}} T_{ab} = \lambda_{abcd}\, \theta^{cd} \qquad .....(3.13)$$

We study the four cases due to the choice of the response tensor λ_{abcd} in the form

Case (1) $\lambda_{abcd} = h_{ab}h_{cd}$, ...(3.14)

Case (2) $\lambda_{abcd} = h_{ab}T_{cd}$, ...(3.15)

Case (3) $\lambda_{abcd} = T_{ab}h_{cd}$, ...(3.16)

Case (4) $\lambda_{abcd} = h_{ac}h_{bd} + h_{ad}h_{bc}$. .....(3.17)

Case (1) : Let us choose $\lambda_{abcd} = h_{ab}h_{cd}$.

Theorem (1) : For the Jumann stress rate of ferrofluid system with $\lambda_{abcd} = h_{ab}h_{cd}$ the following relations are equivalent.

A) $\underset{\bar{u}}{J}\, T_{ab} - \lambda_{abcd}\,\theta^{cd} = 0,$

B) $\dot{\rho} = \dot{p} = \dot{u}_a = \dot{\mu} = \dot{H}^2 = \theta = 0,\ \dot{H}_a - H_c W^c_{\cdot a} = 0.$

Proof : To prove (A) $\Rightarrow$ (B). We use the constitutive equation (3.14) for a general hypoelastic material

$$\underset{u}{J}\, T_{ab} = h_{ab}h_{cd}\,\theta^{cd},$$

i.e. $\underset{\bar{u}}{J}\, T_{ab} = h_{ab}\theta$ . .....(3.18)

By using equations (3.1) and (3.18) we have

$$(\rho + p + \mu H^2)^{\cdot}\, u_a u_b + (\rho + p + \mu H^2)(\dot{u}_a u_b + u_a \dot{u}_b) -$$

$$-(p + 1/2\,\mu H^2)^{\cdot} g_{ab} - (\mu H_a H_b)^{\cdot} + \mu H_c H_b W^c_{\cdot a} +$$

$$+ \mu H_c H_a W^c_{\cdot b} - h_{ab}\theta = 0. \quad .....(3.19)$$

The various contractions of equation (3.19) with $g^{ab}, u^a_{,} u^a u^b_{,} H^a$ and $H^a H^b$ yields

$$(\rho - 3p)^{\cdot} - 3\theta = 0, \quad .....(3.20)$$

$$(\rho + p + \mu H^2)^{\cdot} u_b + (\rho + p + \mu H^2)\, \dot{u}_b -$$

$$- (p + 1/2\, \mu H^2)^{\cdot} u_b - \mu \dot{H}_a u^a H_b = 0\ , \qquad \text{.....(3.21)}$$

$$(\rho + 1/2\, \mu H^2)^{\cdot} = 0\ , \qquad \text{.....(3.22)}$$

$$(\rho + p + \mu H^2)\, \dot{u}_a H^a u_b - (p + 1/2\, \mu H^2)^{\cdot} +$$

$$+ \dot{\mu}\, H^2 H_b + \mu_{/2} \dot{H}^2 H_b + H^2 \dot{H}_b - \mu H^2 H_c W^c_b - H_b \theta = 0, \qquad \text{.....(3.23)}$$

$$(p - 1/2\, \mu H^2)^{\cdot} + \theta = 0\ . \qquad \text{.....(3.24)}$$

From equations (3.20) , (3.22) and (3.24) we obtain

$$\dot{\rho} = 0,\ \dot{p} = 0,\ \dot{\mu} = 0 \text{ and } \dot{H}^2 = 0. \qquad \text{.....(.3.25)}$$

Also from the equations of continuity (0.4.8) and (3.24) we get

$$\theta = 0\ ,\ \text{and}\ \dot{u}_a = 0 \text{ (vide equations (3.21) and (3.25)} \qquad \text{.....(3.26)}$$

So also (3.23) and (3.25) provide

$$\dot{H}_a - H_c\, W^c_a = 0. \qquad \text{.....(3.27)}$$

Hence the equations (3.25), (3.26) and (3.27) proves (A)$\Rightarrow$(B).

To prove (B) $\Rightarrow$ (A) We substitute (3.25), (3.26), (3.27) and $\theta = 0$ in equation (3.18).

Hence the proof is complet.

Thus (A) and (B) are equivalent conditions.

Case (2) : Let us write $\lambda_{abcd} = h_{ab} T_{cd}$.

For this the constitutive equation (3.12) of hypoelasticity yields

$$\lambda_{abcd}\ \theta^{cd} = h_{ab} T_{cd}\ \theta^{cd}, \qquad .....(3.28)$$

i.e., $\lambda_{abcd}\ \theta^{cd} = h_{ab}\ [(\rho + p + \mu H^2)\ u_c u_d -$

$- (p + 1/2\ \mu H^2) g_{cd} - \mu H_c H_d]\ \theta^{cd}$, vide (0.3.1)

i.e., $$\lambda_{abcd}\ \theta^{cd} = - h_{ab}[\ (p + 1/2\ \mu H^2)\theta + \mu H_c H_d\ \theta^{cd}]. \qquad .....(3.29)$$

This equation together with equation (3.1) produces

$$\frac{J}{u} T_{ab} = \lambda_{abcd}\theta^{cd} \Leftrightarrow \dot{\rho} = \dot{p} = \dot{\mu} = \dot{H}^2 = 0,$$
$$\theta = 0\ ,\ \sigma_{ab} = \theta_{ab}. \qquad .....(3.30)$$

Hence we conclude that if the constitutive relation (3.28) holds then the pressure, density, magnetic permeability, magnitude of the magnetic field remains constant along deformation free flow.

Case (3) : Let us take $\lambda_{abcd} = T_{ab} h_{cd}$.

Theorem (II) : If for the ferrofluid distribution with uniform magnitude of magnetic field the response tenser is governed by $\lambda_{abcd} = T_{ab} h_{cd}$, then

$$\frac{J}{u} T_{ab} - \lambda_{abcd}\theta^{d} = 0 \Leftrightarrow \dot{\rho} = \dot{p} = \dot{\mu} = \dot{u}_a = 0,$$
$$\theta = 0 \text{ and } \theta_{ab} = \sigma_{ab}.$$

Proof : For hypoelastic material we write

$$\frac{J}{u} T_{ab} = T_{ab} h_{cd}\ \theta^{cd},$$

i.e., $\frac{J}{u} T_{ab} = T_{ab}\theta$,

i.e., $\frac{J}{u} T_{ab} = [(\rho + p + \mu H^2) u_a u_b - (p + 1/2\ \mu H^2)\ g_{ab} -$

$$- \mu H_a H_b]\ \theta. \qquad ...(3.31)$$

From equation (3.1) and (3.31) we have

$$(\rho + p + \mu H^2)^{\cdot} u_a u_b + (\rho + p + \mu H^2)(\dot{u}_a u_b + u_a \dot{u}_b)$$
$$-(p + 1/2\mu H^2)^{\cdot} g_{ab} - (\mu H_a H_b)^{\cdot} + \mu H_c H_b W^c_a +$$
$$+ \mu H_c H_a W^c{}_b - (\rho + p + \mu H^2) u_a u_b \theta + (p + 1/2\,\mu H^2) g_{ab} \theta$$
$$+ \mu H_a H_b \theta = 0. \qquad \ldots\ldots(3.32)$$

For the uniform magnitude of magnetic field ($H_{a;b} = 0$), We obtain the following results from (3.32)

$$(\rho - 3p)^{\cdot} - (\rho - 3p)\theta = 0, \qquad \ldots\ldots(3.33)$$

$$(\rho + 1/2\,\mu H^2)^{\cdot} u_b + (\rho + p + \mu H^2)\dot{u}_b - \mu \dot{H}_a u^a H_b -$$
$$- (\rho + 1/2\,\mu H^2)\, u_b \theta = 0, \qquad \ldots\ldots(3.34)$$

$$(\rho + 1/2\,\mu H^2)^{\cdot} - (\rho + 1/2\,\mu H^2)\theta = 0, \qquad \ldots\ldots(3.35)$$

$$(\rho + p + \mu H^2)\dot{u}_a H^a u_b + \dot{\mu}\, H^2 H_b + \mu_{/2} \dot{H}^2 u_b +$$
$$+ \mu H^2 \dot{H}_b - (p + 1/2\,\mu H^2)^{\cdot} H_b - \mu\, H^2 H_c W^c{}_b -$$
$$- (p + 1/2\,\mu H^2) H_b \theta + \mu H^2 H_b \theta = 0, \qquad \ldots\ldots(3.36)$$

$$(p - 1/2\,\mu H^2)^{\cdot} - (p - 1/2\,\mu H^2)\theta = 0, \qquad \ldots\ldots(3.37)$$

and $\dot{u}_a H^a = 0.$ $\qquad \ldots\ldots(3.38)$

From Maxwell equation (0.4.4) and (3.38) we get

$$\dot{\mu} = 0 \;.\; (\text{since } \dot{H}^2 = 0) \qquad \ldots\ldots(3.39)$$

Further the equations (3.33), (3.35) and (3.37) yield

$$\dot{p} = 0, \dot{\rho} = 0. \qquad \text{.....(3.40)}$$

and

$$\dot{u}_a = 0, \text{ (vide 3.34)}$$

$$\dot{H}_a - H_c W^c{}_a = 0. \text{ (by 3.36)} \qquad \text{.....(3.41)}$$

Consequently, $\theta = 0, \ \theta_{ab} = \sigma_{ab}$. .....(3.42)

Finally the equations (3.39), (3.40), (3.41) and (3.42) imply

$$\underset{\bar{u}}{J} T_{ab} - \lambda_{abcd}\,\theta^{cd} = 0 \Leftrightarrow \dot{\rho} = \dot{p} = \dot{\mu} = \dot{u}_a = \dot{H}^2 = 0,$$

$$\theta = 0 \text{ and } \theta_{ab} = \sigma_{ab}.$$

Case (4) : Let us consider $\lambda_{abcd} = h_{ac}h_{bd} + h_{ad}h_{bc}$.

This with the constitutive relation (3.12) of hypoelastic material produces

$$\lambda_{abcd}\ \theta^{cd} = (h_{ac}h_{bd} + h_{ad}h_{bc})\ \theta^{cd},$$

i.e., $\lambda_{abcd}\,\theta^{cd} = 2\ \theta_{ab}$.

This equation together with equation (3.1)and (3.12) with expansion free flow yields

$$\underset{\bar{u}}{J} T_{ab} = \lambda_{abcd}\ \theta^{cd} \Leftrightarrow \dot{\rho} = \dot{p} = \dot{\mu} = \dot{H}^2 = 0,$$

and $\sigma_{ab} = \theta_{ab}$

III) RELATIVISTIC- LAME'S PARAMETERS :

The isotropic elastic material characterized by the relation (Walwadkar, 1983)

$$\lambda_{abcd} = [A \; h_{ab}h_{cd} + B(h_{ac}h_{bd} + h_{ad}h_{ac})]. \qquad .....(3.43)$$

Here the constants A and B are the Lame's parameters.

We recall the J-type constitutive equation for hypoelastic material

$$\underset{u}{\overset{J}{}} T_{ab} = \lambda_{abcd} \; \theta^{cd},$$

$$\text{i.e.,} \; \underset{u}{\overset{J}{}} T_{ab} = [Ah_{ab}h_{cd} + B(h_{ac}h_{bd} + h_{bd}h_{ac})] \; \theta^{cd}, \quad \text{vide (3.43)}$$

$$\text{i.e.,} \; \underset{u}{\overset{J}{}} T_{ab} = Ah_{ab}\theta + 2B\theta_{ab}. \qquad .....(3.44)$$

From equation (3.1) and (3.44) we have

$$(\rho + p + \mu H^2)^{\cdot} u_a u_b + (\rho + p + \mu H^2)(\dot{u}_a u_b + u_a \dot{u}_b) -$$
$$- (p + 1/2 \, \mu H^2) g_{ab} - \mu (HH_a H_b)^{\cdot} + \mu H_c H_a W^c{}_{.b} +$$
$$+ \mu H_c H_b W^c{}_{.a} = Ah_{ab}\theta + 2B \, \theta_{ab}. \qquad .....(3.45)$$

By contracting equation (3.45) with g^{ab}, $\mu^a H^a$ and $u^a u^b$ respectively we get the results

$$(\rho - 3p)^{\cdot} = 3\theta A + 2B\theta , \qquad .....(3.46)$$

$$(p - 1/2 \, \mu H^2)^{\cdot} H^2 = -A\theta H^2 + 2B \, \theta_{ab} H^a H^b. \qquad .....(3.47)$$

$$(\rho + 1/2 \, \mu H^2) = 0$$

After solving the above two results simulteniously we obtain the value of A and B

$$A = \frac{(\rho - 3p)^{\cdot}}{3\theta} + \frac{2/3(\mu H^2)^{\cdot} H^2}{\theta H^2 + 3\theta_{ab} H^a H^b},$$

$$B = \frac{-(\mu H^2)^{\cdot} H^2}{\theta H^2 + 3\theta_{ab} H^a H^b} . \qquad \text{(Vide 3.47}')) \qquad .....(3.48)$$

Remark : This equations give the Lame's coefficients in terms of kinematical and dynamical variables.

4. **THE TRUESDELL STRESS RATE :**

Truesdell transport of a vector field $\bar{X}$ has the defining expression (Walwadkar, 1983)

$$\underset{\bar{u}}{T} X^a = \dot{X}^a - X^c u^a_{;c} + \Omega X^a \theta . \qquad .....(4.1)$$

By using the Leibnitz rule the value of Scalar Ω is 1/2

Hence if the field vector $\bar{X}$ along the magnetic lines then we write

$$\underset{\bar{u}}{T} H^a = \dot{H}^a - H^c u^a_{;c} + 1/2\, \theta H^a . \qquad .....(4.2)$$

The magnetic field vector $\bar{H}$ is Truesdell rate free if

$$\underset{\bar{u}}{T} H^a = 0. \qquad .....(4.2')$$

The contractions of this results (4.2$'$) with u_a and H_a produces respectively the results

$$\dot{u}_a H^a = 0, \qquad .....(4.3)$$

$$1/2\,(H^2)^{\cdot} - u_{a;b} H^a H^b + 1/2\, \theta H^2 = 0. \qquad .....(4.4)$$

It follows from (4.3) and the Maxwell equation (0.4.3)

$$\mu_{;b}H^b + \mu H^b_{;b} = 0 . \qquad .....(4.5)$$

Further by using equation (4.4) in Maxwell equation (0.4.3) we have

$$\mu_{/2}\,\theta + \dot{\mu} = 0. \qquad .....(4.6)$$

OBSERVATIONS : We note the following results

1) $\underset{\bar{u}}{T}\, H^a = 0 \Rightarrow \dot{\mu} = o \Leftrightarrow \theta = 0,$ .....(4.7)

2) $\underset{\bar{u}}{T}\, H^a = 0 \Rightarrow \mu_{;b}H^b \Leftrightarrow H^b_{;b} = o.$ .....(4.8)

The equation (4.7) states that the magnetic permeability is invariant along the expansion free flow and (4.8) shows that the magnetic permeability is invariant along divergence free magnetic lines.

TRUESDELL TRANSPORT TO FERROFLUID :

Theorem : The necessary and sufficient conditions that the stress energy tensor of a ferrofluid is Truesdell transported with expansion free flow that the matter density, isotropic pressure, magnetic permeability and magnitude of magnetic vecter are conserved along $\bar{u}$ with respect to killing flow ($\dot{u}_a = \vartheta_{ab} = 0$).

Proof : (Necessary) :

The Truesdell transport of ferrofluid is given by

$$\underset{\bar{u}}{T}\, T^{ab} = (\rho + p + \mu H^2)^{\cdot}\, u^a u^b - (p + 1/2\,\mu H^2)^{\cdot} g^{ab} -$$

$$- (\mu H^a H^b)^{\cdot} + (p + 1/2\,\mu H^2)(u^{a;b} + u^{b;a}) +$$

$$+ \mu H^c (H^a u^b{}_{;c} + H^b u^a{}_{;c}) + (\rho + p + \mu H^2) u^a u^b \theta -$$

$$(p + 1/2\,\mu H^2)\, g^{ab}\,\theta - \mu H^a H^b \theta . \qquad .....(4.9)$$

Now we impose the condition of $\underset{\bar{u}}{T} T^{ab} = 0$ and the medium is expansion free with suitable chosen contractions, we obtain the equations

$$(\rho - 3p)^{\cdot} + 2\mu H_a H_b u^{a;b} = 0, \qquad \text{.....(4.10)}$$

$$(\rho + 1/2\ \mu H^2)^{\cdot} u^b - (p + 1/2\ \mu H^2)\dot{u}^b - \mu \dot{H}^a u_a H^b = 0, \qquad \text{.....(4.11)}$$

$$(\rho + 1/2\ \mu H^2)^{\cdot} = 0, \qquad \text{.....(4.12)}$$

$$- (p + 1/2\ \mu H^2)^{\cdot} u^b + \mu_{/2} \dot{H}^2 H^b + \mu H^2 \dot{H}^b + \mu_{/2}$$

$$\dot{} H^2 H^b + (p + 1/2\ \mu H^2)(u^{a;b} H_a + u^{b;a} H_a) = 0, \qquad \text{.....(4.13)}$$

$$(p - 1/2\ \mu H^2)^{\cdot} H^2 + 2(p + 1/2\ \mu H^2) u^{a;b} H_a H_b = 0, \qquad \text{.....(4.14)}$$

$$\dot{u}_a H^a = 0. \qquad \text{.....(4.15)}$$

We recall the equation of continuity

$$(\rho + 1/2\ \mu H^2)^{\cdot} + (\rho + p + \mu H^2)\theta +$$

$$+ \mu u^{a;b} H_a H_b = 0.$$

This with above deduction supply

$$u^{a;b} H_a H_b = 0. \ (\text{by}\ \theta = 0) \qquad \text{.....(4.16)}$$

From equation (4.10), (4.12) and (4.14) we get

$$\dot{\rho} = \dot{p} = \dot{\mu} = \dot{H}^2 = 0. \qquad \text{.....(4.17)}$$

Also $\dot{u}_a = 0$. (vide equation (4.11) .....(4.18)

From Maxwell equations (0.4.2), (0.4.4) and (4.15) we get

$$\dot{H}^{a} - u^{a}_{;b} H^{b} = 0. \qquad .....(4..19)$$

By substituting the equations (4.17), (4.18) and (4.19) in equation (4.9) we obtaining the result.

$$u^{a;b} + u^{b;a} = 0,$$

i.e., $\theta^{ab} = 0$ and $\sigma^{ab} = 0$. .....(4.20)

Sufficient Part : If we substitutes the conditions (4.17), (4,18) and (4.20) in equation (4.9) implies

$$\underset{u}{\underline{T}}\, T^{ab} = 0. \qquad .....(4.21)$$

Hence the proof of the theorem is complete.

5. FERMI-WALKER,CONVECTIVE TRANSPORTS AND FERROFLUID RHEOLOGY :

Fermi-Walker Transport along a space like congruence :

A Fermi-Walker (F-W) transport along space like vecter is physically much less similar than F-W transport along time like congruence. To discuss the physical interpretation of F-W transport along a space like vector, We can use the same argument of F-W transport along time-like vecter with making the suitable changes in sign (Synge 1960).

According the F-W transport of tensors along the time like vector is studied by Radhakrishna and Bhosale (1976). Adopting the same procedure and taking into account the sign changes we define the F-W transport equation along $\bar{h}$ as follows

$$\underset{\bar{h}}{F} X_a = X_{a;c}h^{c'} - X_c(h_a h^{c'} - h'_a h^c).$$

and

$$\underset{\bar{h}}{F} X_{ab} = X_{ab;c} h^c - X_{cb} (h_a h^{c'} - h'_a h^c) -$$

$$- X_{ac}(h_b h^{c'} - h'_b h^c). \qquad .....(5.2)$$

It follows from (5.1) and (5.2)

$$\underset{\bar{h}}{F} h_a = 0 \text{ and } \underset{\bar{h}}{F} g_{ab} = 0 \qquad .....(5.3)$$

Then we observe that the metric tensor and the magnetic field vector are Fermi-Walker transported along the magnetic field vector $\bar{h}$.

FERMI-WALKER TRANSPORT OF FERROFLUID :

Theorem (1) : The stress energy tensor of ferrofluid is Fermi-Walker stress rate free along the magnetic lines if and only if the matter density, isotropic pressure, magnetic permeability and magnitude of the magnetic field are invariant along the Fermi-Walker transported flow.

Proof : (Necessary Part) :

The Fermi-Walker transport of stress energy tensor is given by equation (5.2)

$$\underset{\bar{h}}{F} T_{ab} = T_{ab;c}h^c - T_{cb}(h_a h^{c'} - h'_a h^c) -$$

$$- T_{ac}(h_b h^{c'} - h'_b h^c)$$

This for ferrofluid distribution yields

$$\underset{\bar{h}}{F} T_{ab} = (\rho + p + \mu H^2)' u_a u_b + (\rho + p + \mu H^2) \times (u'_a u_b + u_a u'_b) - (p + 1/2 \mu H^2)' g_{ab} - (\mu H^2)' h_a h_b - (\rho + p + \mu H^2) u_c u_b h_a h^{c'} - (\rho + p + \mu H^2) u_c u_a \times h_b h^{c'} . \quad .....(5.4)$$

If T_{ab} is Fermi-Walker transported along $\bar{h}$ then we have $\underset{\bar{h}}{F} T_{ab} = 0$. Hence we write from (5.4)

$$(\rho + p + \mu H^2)' u_a u_b + (\rho + p + \mu H^2) (u'_a u_b + u'_a u_b) - (p + 1/2 \mu H^2)' g_{ab} - (\mu H^2)' h_a h_b - (\rho + p + \mu H^2) u_c u_b h_a h^{c'} - (\rho + p + \mu H^2) \times u_c u_a h_b h'_c = 0. \quad .....(5.5)$$

It follows easily from this

$$(\rho + 3p)' = 0, \quad .....(5.6)$$

$$(\rho + 1/2 \mu H^2)' u_b + (\rho + p + \mu H^2) u'_b - (\rho + p + \mu H^2) u_c h_b h^{c'} = 0, \quad .....(5.7)$$

$$(\rho + 1/2 \mu H^2)' = 0, \quad .....(5.8)$$

$$(p - 1/2 \mu H^2)' = 0. \quad .....(5.9)$$

From equation (5.6) , (5.8) and (5.9), we obtain

$$\rho' = 0 , p' = 0, \mu' = 0 \text{ and } H^{2'} = 0. \quad .....(5.10)$$

These conditions when used in equation (5.4) provides the result

$$u'_a u_b + u_a u'_b - u_c u_b h_a h^{c'} - u_c u_a h_b h^{c'} = 0,$$

i.e., $u'_a - u_c h_a h^{c'} = 0,$

i.e., $\underset{\bar{h}}{F} u_a = 0.$ .....(5.11)

Hence the flow is F-W transported along $\bar{h}$.

Thus the equations (5.10) and (5.11) are the required necessary conditions of the theorem.

Sufficient Part : Now on substituting the results (5.10) and (5.11) in equation (5.4) then we easily get

$$\underset{\bar{h}}{F} T_{ab} = 0.$$

Here the T_{ab} is F-W transported along $\bar{h}$.

Here the proof is complete.

Theorem (2) : For ferrofluid system

$$\underset{\bar{u}}{F} T_{ab} = 0 \Leftrightarrow \text{(A)} \quad \dot{\rho} = \dot{p} = \dot{\mu} = \dot{H}^2 = 0,$$

$$\text{(B)} \quad \underset{\bar{u}}{F} h_a = 0.$$

Proof : If part :

Let $\underset{\bar{u}}{F} T_{ab} = 0.$ .....(5.12)

From the definition of Fermi-Walker operator (vide (0.5.6), we write the stress energy tensor of ferrofluid as,

$$\underset{u}{F}\, T_{ab} = (\rho + p + \mu H^2)^{\cdot}\, u_a u_b -$$

$$-(p + 1/2\, \mu H^2)^{\cdot}\, g_{ab} - (\mu H^2 h_a h_b)^{\cdot} -$$

$$- \mu H^2 h_c h_b u_a \dot{u}^c - \mu H^2 h_c h_a u_b \dot{u}^c. \qquad .....(5.13)$$

From the equation (5.12), we get

$$(\rho + p + \mu H^2)^{\cdot}\, u_a u_b - (p + 1/2\, \mu H^2)^{\cdot} g_{ab} -$$

$$- (\mu H^2 h_a h_b)^{\cdot} - \mu H^2 h_c h_b u_a \dot{u}^c - \mu H^2 h_c h_a u_b \dot{u}^c = 0. \qquad .....(5.14)$$

On account of contractions of equation (5.12) with $g^{ab}, u^a u^b$, h^a and $h^a h^b$, we obtain

$$(\rho - 3p)^{\cdot} = 0, \qquad .....(5.15)$$

$$(\rho + 1/2\, \mu H^2)^{\cdot} = 0, \qquad .....(5.16)$$

$$- (p + 1/2\, \mu H^2)^{\cdot} h_b + \dot{\mu} H^2 h_b + \mu \dot{H}^2 h_b + \mu H^2 \dot{h}_b +$$

$$+ \mu H^2\, h_c h_b \dot{u}^c = 0, \qquad .....(5.17)$$

$$(p - 1/2\, \mu H^2)^{\cdot} = 0 \qquad .....(5.18)$$

Byusing equations (5.13), (5.14) and (5.16) we deduce that

$$\dot{\rho} = \dot{p} = \dot{\mu} = \dot{H}^2 = 0\,. \qquad .....(5.19)$$

Consequently the equation (5.15) implies that

$$\dot{h}_a + h_c u_a \dot{u}^c = 0,$$

i.e. $\underset{u}{F}\, h_a = 0.$ .....(5.20)

Only if Part : If the conditions (5.19) and (5.20) are satisfied then the equation (5.13) implies

$$\underset{\bar{u}}{F} T_{ab} = 0.$$

Here the proof of the theorem is complete.

Remark : We observe from the result (5.19) and the equation of continuity (0.4.8) that $\theta = 0$ .This means that the flow is expansion free.

CONCLUSION : The equations (5.19) and (5.20) imply that the isotropic pressure, magnetic permeability, matter density and magnitude of magnetic field are conserved along $\bar{u}$ and the magnetic lines are F-W transported along $\bar{u}$.

CONVECTIVE TRANSPORT ALONG SPACE LIKE VECTOR :

The expression for the Convective transport of a tensor field X_{ab} along the flow is given by expression (0.5.5). On similar lines we provide the definition of Convective transport of X_{ab} along the unit space like vector $\bar{h}$ as

$$\begin{aligned} \underset{\bar{h}}{C} X_{ab} = \; & X_{ab;c} + X_{cb}(h^c{}_{;a} - h^c u_c - \dot{h}^{c'} u_a + \\ & + u^d u^c u_a \dot{h}_d - u^d u^c h_a h'_d - h^d u^c h_{d;a}) + \\ & + X_{ac} (h^c{}_{;b} - \dot{h}^c u_b - h^{c'} u_a + u^d u^c u_b \dot{h}_d - \\ & - u^d u^c h_b h'_d - h^d u_c h_{d;b}). \qquad \ldots\ldots(5.21) \end{aligned}$$

This implies that

$$\underset{\bar{h}}{C}\, g_{ab} = 2\,\overset{*}{\theta}_{ab}.$$

CONVECTIVE TRANSPORT AND FERROFLUID SYSTEM :

We recall the result (0.4.10)

$$(\rho + p + \mu H^2)\, \dot{u}^a h_a - (p + 1/2\, \mu H^2)' + \mu' H^2 + $$
$$+ \mu H^{2'} + \mu H^2 h^a{}_{;a} = 0,$$

i.e., $p' + (\rho + p)\, h_{a;b} u^a u^b - \mu H^2 (h^a{}_{;a} -$

$$- h_{a;b} u^a u^b) - 1/2\, (\mu H^2)' = 0,$$

i.e., $p' + (\rho + p)\, h_{a;b} u^a u^b - 2\, H^2 \overset{*}{\theta} - 1/2\, (\mu H^2)' = 0$ .....(5.22)

(Since $\overset{*}{\theta} = 1/2 (h^a{}_{;a} - h_{a;b} u^a u^b)$)

The Convective transport of the stress energy tensor (0.3.1) of ferrofluid along the magnetic field vector $\bar{h}$ gives

$$\underset{\bar{h}}{C}\, T_{ab} = (\rho + p + \mu H^2)'\, u_a u_b + (\rho + p + \mu H^2)(u_a' u_b +$$
$$+ u_a u_b') - (p + 1/2\, \mu H^2)'\, g_{ab} -$$
$$- (\mu H^2 h_a h_b)' - 2\, (p + 1/2\, \mu H^2)\, \overset{*}{\theta}_{ab}. \quad .....(5.23)$$

From equation (5.23) we have

$$g^{ab} \underset{\bar{h}}{C}\, T_{ab} = (\rho - 3p)' - 2\, (p + 1/2\, \mu H^2)\, \overset{*}{\theta}, \quad .....(5.24)$$

$$u^a u^b \underset{\bar{h}}{C}\, T_{ab} = (\rho + 1/2\, \mu H^2)', \quad .....(5.25)$$

$$h^a \underset{\bar{h}}{C} T_{ab} = (\rho + p + \mu H^2)\, u_a{}' u_b h^a - (p + 1/2\ \mu H^2)' h_b + $$
$$+ (\mu H^2 h_b)', \qquad .....(5.26)$$

$$h^a h^b \underset{\bar{h}}{C} T_{ab} = (p - 1/2\, \mu H^2)', \qquad .....(5.27)$$

$$u^a h^b \underset{\bar{h}}{C} T_{ab} = (\rho + p)\, u'_b h^b. \qquad .....(5.28)$$

If for the geodesic flow of ferrofluid the equations (5.22) and (5.27) are used in the transport equation $\underset{\bar{h}}{C} T_{ab} = 0$, then we get

$$\overset{*}{\theta} = 0. \qquad .....(5.29)$$

Further using the results (5.24) , (5.25) and (5.27) we obtain

$$\rho' = 0,\ p' = 0,\ \mu' = 0 \text{ and } H^{2\prime} = 0. \qquad .....(5.30)$$

These results with equation (5.26) gives

$$h'_b = 0. \qquad .....(5.31)$$

Finally by substituting the results (5.30) and (5.31) in equation (5.23) we get

$$\overset{*}{\theta}_{ab} = 0 \text{ and } \overset{*}{\sigma}_{ab} = 0\ . \qquad .....(5.32)$$

REMARK : The condition $\underset{\bar{h}}{C} T_{ab} = 0$ directs that the system is physically constant (Ehlers, 1973).

CONCLUSION : The convective derivative of stress energy tensor of the ferrofluid system is physically constant along $\bar{h}$ if and only if the pressure, density magnetic permeability and magnitude of the magnetic field vector are conserved along $\bar{h}$. Moreover the space like congruence is killing congruence. It means that $\overset{*}{\theta}_{ab} = \overset{*}{\sigma}_{ab} = \overset{*}{\theta} = 0$.