CHAPTER II

MAGNETO-DUST DISTRIBUTION AND A GROUP OF CONHARMONIC CONFORMAL MOTIONS

1. INTRODUCTION :

One of the recent symmetries known as Conformal symmetry plays a key role in obtaining space-time models for relativistic distributions of matter. This symmetry consists of both the earlier symmetries, namely isometry and self similarity (Taub 1971, Eardly 1974, Wilson 1986).

The Conformal symmetry property has been an essential geometric prescription for a good part of physics (Duggal 1989). The Conformal invariance is the root of twisstar programme.

The aim here is to examine the effect of Conformal motion on the dynamical structure of relativistic Magneto-Dust Distribution. Accordingly, section 2 deals with several properties of Conformal motions. The next section 3 includes the study of Conharmonic Conformal motions compatible with Magneto-Dust Distribution.

2. IMPLICATIONS OF CONFORMAL MOTIONS :

We know that the necessary condition for Conformal motion is

$$\underset{\xi}{L}\, g_{ab} = 2\,\psi\, g_{ab}, \qquad 2.1$$

i.e. $$\xi_{a\,;\,b} + \xi_{b\,;\,a} = 2\,\psi\, g_{ab}. \qquad 2.2$$

CASE (I) : Let us choose $\xi = u$.

For this choice equation (2.2) becomes

$$u_{a\,;\,b} + u_{b\,;\,a} = 2\,\psi\, g_{ab}. \qquad 2.3$$

Theorem 1 : If the flow vector u is Conformal Killing vector then

$$\theta = 0, \sigma = 0, \psi = 0 .$$

Proof : On contracting (2.3) with g^{ab}, we obtain

$$g^{ab} u_{a\,;\,b} + g^{ab} u_{b\,;\,a} = 2\,\psi\, g^{ab} g_{ab} ,$$

i.e. $u^{b}{}_{;\,b} + u^{a}{}_{;\,a} = 2\,\psi\,(4)$,

i.e. $2\,\theta = 8\,\psi$, $\quad\because u^{b}{}_{;\,b} = \theta$

i.e. $\theta = 4\,\psi$. 2.4

Transvection of (2.3) with $u^a u^b$, yields

$$u^a u^b u_{a\,;\,b} + u^a u^b u_{b\,;\,a} = 2\,\psi\, u^a u^b g_{ab} ,$$

i.e. $(u_{a\,;\,b}\, u^a)\, u^b + (u_{b\,;\,a}\, u^b)\, u^a = 2\,\psi$,

i.e. $\psi = 0$. 2.5

$\because u_{a\,;\,b}\, u^a = 0$

Hence from (2.4) and (2.5),

$$\theta = 0 \text{ and } \psi = 0 . \qquad 2.6$$

The gradient of the flow vector u^a can be written as

$$u_{a\,;\,b} = \sigma_{ab} + W_{ab} + \dot{u}_a u_b + (1/3)\,\theta\, h_{ab} . \qquad 2.7$$

If we use (2.6) in this, we have

$$u_{a\,;\,b} = \sigma_{ab} + W_{ab} + \dot{u}_a u_b . \qquad 2.8$$

This implies that

$$u_{b\,;\,a} = \sigma_{ba} + W_{ba} + \dot{u}_b u_a \qquad 2.9$$

$$u_{a\,;\,b} + u_{b\,;\,a} = 2\,\sigma_{ab} + \dot{u}_a u_b + \dot{u}_b u_a \qquad 2.10$$

$\because W_{ab} = -W_{ba}$

$\sigma_{ab} = \sigma_{ba}$

i.e. $\theta = \sigma$.

This gives $\sigma = 0$. 2.11

Hence the Kinematical implications of (2.3) can be summarized as

$$\theta = \sigma = \psi = 0. \qquad 2.12$$

Hence the theorem.

- **Remark (1) :** If u^a is Conformal Killing vector then flow lines are expansion and shear free. VIDE (2.6, 2.11)
- **Remark (2) :** (2.12) implies that Conformal Killing vector is changed to Killing vector ($\psi = 0$).

Theorem 2 : If u^a is the Conformal Killing vector then

i) $\dot{u}_b h^b = 0$

ii) $2\,\dot{\mu} h^2 + \mu\,(h^2)^{\bullet} = 0$

iii) $\dot{\rho} = \frac{1}{2}\,\dot{\mu} h^2$

iv) $\mu_{;b} h^b = 0$

v) $h^b{}_{;b} = 0$.

Proof : On contracting (2.3) with $u^a h^b$, we obtain

$$u^a h^b (u_{a;b}) + u^a h^b (u_{b;a}) = 2\,\psi\, u^a h^b\, g_{ab},$$

i.e. $(u_{a;b} u^a) h^b + (u_{b;a} u^a) h^b = 0,\qquad \because h^b u_b = 0$

i.e. $u_{b;a}\, u^a h^b = 0,\qquad \because u_{a;b} u^a = 0$

i.e. $\dot{u}_b h^b = 0 . \qquad 2.13$

Further transvection of (2.3) with $h^a h^b$, gives

$$h^a h^b u_{a;b} + h^a h^b u_{b;a} = 2\,\psi\, h^a h^b\, g_{ab},$$

i.e. $2\, h^a h^b u_{a;b} = -2\,\psi\, h^2. \qquad \because h^a h_a = -h^2$

This with equation (2.5) yields,

$$h^a h^b u_{a;b} = 0 . \qquad 2.14$$

By using equation (2.13) in Maxwell equations I (8.7), we get

$$\mu\,[\,h^2\,\theta + \tfrac{1}{2}\,(h^2)^{\bullet}\,] + \dot{\mu}h^2 = 0\,,$$

i.e. $$2\,\dot{\mu}h^2 + \mu\,(h^2)^{\bullet} = 0\;. \qquad \text{vide (2.6)} \qquad 2.15$$

The continuity equation I (9.10) with (2.6) provides

$$\dot{\rho} = \tfrac{1}{2}\,\dot{\mu}h^2\,. \qquad 2.16$$

If we use the condition (2.13) in stream line equations I (9.16), then we get

$$\mu_{;b}h^b = 0\,, \qquad \because\ \rho \neq 0,\ h^2 \neq 0 \qquad 2.17$$

The Maxwell equations I (8.8), yields $h^b{}_{;b} = 0$ vide (2.13), (2.17)

Hence the theorem.

Interpretation :

i) ρ is invariant iff μ is invariant along flow lines.

vide (2.16)

ii) Magnetic permeability is preserved along magnetic lines.

vide (2.17)

CASE (II) : Let us choose $\xi = h$.

For this choice equation (2.2) takes the form

$$h_{a;b} + h_{b;a} = 2\,\psi\,g_{ab}\,. \qquad 2.18$$

Theorem 3 : If magnetic field vector h is the Conformal Killing vector then

i) $h^b{}_{;b} = 4\,\dot{h}_a u^a = (\,2/h^2\,)\,\underset{h}{L}\,h^2 = 4\,\psi\,,$

ii) $\dot{\mu} + \mu\,\theta = 0\;.$

Proof : On contracting (2.18) with g^{ab}, yields

$$g^{ab} h_{a;b} + g^{ab} h_{b;a} = 2\psi g^{ab} g_{ab},$$

i.e. $$h^{b}{}_{;b} + h^{a}{}_{;a} = 8\psi,$$

i.e. $$h^{b}{}_{;b} = 4\psi, \qquad 2.19$$

Further, if we contract (2.18) with $u^a u^b$, we get

$$u^a u^b h_{a;b} + u^a u^b h_{b;a} = 2\psi u^a u^b g_{ab},$$

i.e. $$2 u^a u^b h_{a;b} = 2\psi, \qquad \because u^a u_a = 1$$

i.e. $$\dot{h}_a u^a = \psi, \qquad \because h_{a;b} u^b = \dot{h}^a \qquad 2.20$$

i.e. $$\dot{u}_a h^a = -\psi,$$

i.e. $$4 \dot{h}_a u^a = 4\psi. \qquad 2.21$$

Also multiplying (2.18) with $h^a h^b$, we get

$$h^a h^b h_{a;b} + h^a h^b h_{b;a} = 2\psi h^a h^b g_{ab},$$

i.e. $$h_{a;b} h^a h^b = -\psi h^2, \qquad \because h^a h_b = -h^2$$

i.e. $$(-\tfrac{1}{2} h^2{}_{;b}) h^b = -\psi h^2,$$

i.e. $$h^2{}_{;b} h^b = 2\psi h^2, \qquad 2.22$$

i.e. $$(2/h^2) h^2{}_{;b} h^b = 4\psi,$$

i.e. $$\underset{h}{L} h^2 = 2 h^2 \psi. \qquad 2.23$$

From equations (2.19), (2.21) and (2.23), we have

$$4\psi = h^{b}{}_{;b} = 4\dot{h}_a u^a = (2/h^2) h^2{}_{;b} h^b. \qquad 2.24$$

On contracting (2.18) with $u^a h^b$, we get

$$u^a h^b h_{a;b} + u^a h^b h_{b;a} = 0, \qquad \because u^a h_a = 0$$

i.e. $$h_{a;b} u^a h^b + h_{a;b} u^b h^a = 0,$$

i.e. $$h_{a;b} u^b h^a - u_{a;b} h^a h^b = 0, \qquad 2.25$$

i.e. $u_{a;b}h^a h^b - h_{a;b}u^b h^a = 0,$

i.e. $u_{a;b}h^a h^b - h^a{}_{;b}h_a u^b = 0,$ 2.26

Hence from (2.26) and Maxwell equation I (8.5), we have

$$\dot{\mu} + \mu\,\theta = 0 . \qquad \because h^2 \neq 0 \qquad 2.27$$

Hence the theorem. vide (2.27) & (2.24)

Theorem 4 : If the magnetic field vector h is the Conformal Killing vector then

i) $\dot{\rho} = 0$ iff $\dot{\mu} = 0,$

ii) $\mu_{;b}h^b = -2\,\rho\,\psi / h^2 .$

Proof : We know the continuity equation I (9.10), viz.

$$\dot{\rho} + \rho\,\theta = \tfrac{1}{2}\,\dot{\mu}h^2 ,$$

But $\dot{\mu} = -\mu\,\theta .$ vide (2.27)

Hence above continuity equation becomes

$$\dot{\rho} + \rho\,\theta = -\tfrac{1}{2}\,\mu\,\theta\,h^2,$$

i.e. $\dot{\rho} + (\rho + \tfrac{1}{2}\,\mu h^2)\,\theta = 0 .$ 2.28

This implies, $\dot{\rho} = 0 \Leftrightarrow \dot{\mu} = 0 .$ $\because \theta = -\dot{\mu}/\mu$

$\because \mu, \rho, h^2 \neq 0$

We know stream line equation I (9.16), viz.

$$\rho\,\dot{u}_b h^b = \tfrac{1}{2}\,\mu_{;b}h^b h^2 ,$$

i.e. $\mu_{;b}h^b = (2/h^2)\,\rho\,\dot{u}_b h^b ,$

$= -(2/h^2)\,\rho\,\dot{h}_b u^b ,$

$= -(2/h^2)\,\rho\,\psi ,$ vide (2.20) 2.29

Hence the theorem.

Theorem 5 : If magnetic field vector h is the Conformal Killing vector then

i) $\psi = 0$ OR

ii) $\rho = (3/2)\,\mu h^2$ OR

iii) $\psi = 0, \quad \rho = (3/2)\,\mu h^2$.

Proof : We know Maxwell equation I (8.8), viz.

$$\mu\,(\dot{u}_b h^b + h^b{}_{;b}) + \mu_{;b} h^b = 0 ,$$

Putting $\dot{u}_b h^b = -\psi$ vide (2.20)

$h^b{}_{;b} = 4\,\psi$ and vide (2.19)

$\mu_{;b}\, h^b = -(2/h^2)\,\rho\,\psi$ vide (2.29) in above Maxwell equations,

we obtain

$$\mu\,(-\psi + 4\,\psi) + -(2/h^2)\,\rho\,\psi = 0 ,$$

i.e. $3\,\mu\,\psi - (2/h^2)\,\rho\,\psi = 0$,

i.e. $\psi\,(3\,\mu - 2\,\rho/h^2) = 0$,

i.e. $\psi\,(3\,\mu h^2 - 2\,\rho) = 0$. 2.30

This implies that

a) $\psi = 0$ OR

b) $\rho = (3/2)\,\mu\, h^2$ OR 2.31

c) $\psi = 0 \ , \ \rho = (3/2)\,\mu h^2$.

Hence the theorem.

Remark : We have $\psi \neq 0$ always. Hence the only possibility is $\rho = (3/2)\,\mu h^2$ is considered.

CASE (III) : Let us choose $u = h = \xi$ be Conformal Killing vectors, then we have the following results :

$\theta = 0,$		vide (2.6)	2.32
$\psi = 0,$		vide (2.5)	2.33
$\sigma = 0,$		vide (2.11)	2.34
$W \neq 0,$		vide (2.8)	2.35
$\dot{u}_a = 0,$		vide (2.44)	2.36

And

$\dot{h}^a u_a = 0 = u^a h_a$	vide (2.13)	2.37
$h^b{}_{;b} = 0 ,$	vide (2.19)	2.38
$\underset{h}{L}\, h^2 = (h^2{}_{;b})\, h^b = 0$	vide (2.23)	2.39
$\dot{\rho} = 0 ,$	vide (2.28)	2.40
$\mu_{;b} h^b = 0$	vide (2.29)	2.41
$\dot{\mu} = \mu_{;b} u^b = 0$	vide (2.27)	2.42
$\mu\, (h^2)^{\bullet} = 0$	vide (2.15)	2.43

From (2.39) and (2.41) stream line equations gives

$$\dot{\mu}_a = 0 . \qquad 2.44$$

- **Conclusions :** When the time-like flow vector u^a and magnetic field vector h^a both are Conformal Killing vectors then we have:

1. Flow lines are expansion free, shear free and geodesic. vide (2.32, 2.34, 2.36).
2. Magnetic lines are divergence free. vide (2.38).
3. Magnetic permeability and magnitude of magnetic field are invariant along magnetic lines and flow lines. vide (2.39, 2.41).
4. Matter energy density is conserved along u^a vide (2.40).

3. CONHARMONIC CONFORMAL MOTIONS AND FIELD EQUATIONS FOR MAGNETO-DUST DISTRIBUTION :

For Conformal Killing vector ξ, we have (Coley & Tupper, 1989)

$$\underset{\xi}{L} R_{ab} = -2\,\psi_{;ab} - (\psi_{;cd})\,g^{cd} g_{ab} \,, \qquad 3.1$$

This for Conharmonic Conformal motion (CCM), yields

$$\underset{\xi}{L} R_{ab} = -2\,\psi_{;ab} \text{ as } \psi_{;cd}g^{cd} = 0 \text{ for CCM .} \qquad 3.2$$

By using field equations in (3.2) gives

$$-2\,\psi_{;ab} = \underset{\xi}{L} T_{ab} - \tfrac{1}{2} \underset{\xi}{L} (T g_{ab}). \qquad 3.3$$

This with I (3.4) and I (4.7) provides

$$-2\,\psi_{;ab} = \underset{\xi}{L} [(\rho + \mu h^2) u_a u_b - \tfrac{1}{2} \mu h^2 g_{ab} - \mu h_a h_b]$$

$$- \tfrac{1}{2} \underset{\xi}{L} (\rho\, g_{ab}) ,$$

i.e.

$$-2\,\psi_{;ab} = \underset{\xi}{L} [A u_a u_b + B g_{ab} + D h_a h_b] . \qquad 3.4$$

where

$$A = \rho + \mu h^2$$

$$B = -\tfrac{1}{2} (\rho + \mu h^2)$$

$$D = -\mu \text{ (variable magnetic permeability) .}$$

We know if a fluid space-time admits a Conformal Killing vector ξ, then (Maartens et al., 1986)

$$\underset{\xi}{L} u_b = \psi u_b + h_b . \qquad 3.5$$

Equation (3.4) on simplifying, gives

$$-2\,\psi_{;ab} = (\underset{\xi}{L} A) u_a u_b + A u_a (\underset{\xi}{L} u_b) + A u_b (\underset{\xi}{L} u_a) .$$

$$+ (\underset{\xi}{L} B)\, g_{ab} + B (\underset{\xi}{L}\, g_{ab}) + \underset{\xi}{L} (D)\, h_a h_b$$

$$+ D\, ha (\underset{\xi}{L}\, h_b) + D\, h_b (\underset{\xi}{L}\, h_a),$$

$$-2\,\psi_{;ab} = (\underset{\xi}{L}(A))\, u_a u_b + A\, u_a (\psi\, u_b + h_b)$$

$$+ A\, u_b (\psi\, u_a + h_a) + (\underset{\xi}{L} B)\, g_{ab} + 2 B \psi\, g_{ab}$$

$$+ (\underset{\xi}{L}(D)\, h_a h_b + D (\underset{\xi}{L}\, h_b)\, h_a$$

$$+ D\, h_b (\underset{\xi}{L}\, h_a). \qquad \text{vide (3.5)} \qquad 3.6$$

? Def^n

Theorem 6 : If Magneto-Dust Distribution admits Conformal Killing vector ξ then

$$2 \underset{\xi}{L}\, \mu h^2 = (4/h^2)\, \psi_{;ab} h^a h^b - 4\, \psi_{;ab} u^a u^b + 2\psi h^2$$

Proof : On transvecting (3.6) with u^a and using $u^a u_a = 1$, $u^a h_a = 0$, we get

$$-2\,\psi_{;ab} u^a = (\underset{\xi}{L} A)\, u_b + A (\psi\, u_b + h_b) + A \psi\, u_b$$

$$+ (\underset{\xi}{L} B)\, u_b + 2 B \psi\, u_b + D\, u^a h_b (\underset{\xi}{L}\, h_a). \qquad 3.7$$

Further contracting (3.7) with u^b, we obtain

$$-2\,\psi_{;ab} u^a u^b = (\underset{\xi}{L} A) + A \psi$$

$$+ \underset{\xi}{L}(B) + 2 B \psi + A \psi,$$

$$\because u^b u_b = 1,\ u^b h_b = 0$$

i.e. $$\underset{\xi}{L}(A + B) = -2 (A + B)\, \psi - 2\, \psi_{;ab} u^a u^b, \qquad 3.8$$

As $A + B = \frac{1}{2}(\rho + \mu h^2)$, we have

$$\tfrac{1}{2} [\underset{\xi}{L} (\rho + \mu h^2)] = -2 [\tfrac{1}{2} (\rho + \mu h^2)]\, \psi - 2\, \psi_{;ab} u^a u^b,$$

i.e. $\underset{\xi}{L}(\rho + \mu h^2) = -2(\rho + \mu h^2)\psi - 4\psi_{;ab} u^a u^b$, 3.9

i.e. $\underset{\xi}{L}(\rho) + \mu \underset{\xi}{L} h^2 + h^2 \underset{\xi}{L} \mu =$

$$= -2(\rho + \mu h^2)\psi - 4\psi_{;ab} u^a u^b, \quad 3.10$$

Multiplying equation (3.6) by h^a, we have

$$-2\psi_{;ab} h^a = -A h^2 u_b + h_b(\underset{\xi}{L} B) + 2B\psi h_b$$
$$- D h^2(\underset{\xi}{L} h_b) - h^2 h_b(\underset{\xi}{L} D)$$
$$+ D(\underset{\xi}{L} h_a) h^a h^b, \quad 3.11$$

$\because u^a h_a = 0, h^a h_a = -h^2$

Further contracting (3.11) with h^b, we get

$$-2\psi_{;ab} h^a h^b = -h^2(\underset{\xi}{L} B) - 2B h^2 \psi - D h^2 h^b(\underset{\xi}{L} h_b)$$
$$+ h^4 \underset{\xi}{L} D - h^2 h^a D(\underset{\xi}{L} h_a) . \quad 3.12$$

$\because u^b u_b = 0, h^b h_b = -h^2$

Putting $B = -\frac{1}{2}(\rho + \mu h^2)$, $D = -\mu$ and simplifying equation (3.12), we get

$$-\tfrac{1}{2} \underset{\xi}{L} \rho - \tfrac{1}{2} \mu \underset{\xi}{L} h^2 + (h^2/2) \underset{\xi}{L} \mu =$$
$$= (2/h^2)[\psi_{;ab} h^a h^b] + (\rho + \mu h^2)\psi$$
$$+ 2\mu h^a(\underset{\xi}{L} h_a), \quad 3.13$$

i.e. $\underset{\xi}{L}(\rho) + \mu \underset{\xi}{L}(h^2) - h^2 \underset{\xi}{L}(\mu) =$

$$= -(4/h^2)[\psi_{;ab} h^a h^b] - 2(\rho + \mu h^2)\psi -$$
$$- 4\mu(-1/2 \underset{\xi}{L} h^2) - 2\psi h^2,$$

$$= -(4/h^2)[\psi_{;ab} h^a h^b] - 2(\rho + \mu h^2)\psi$$
$$- 4\mu(-\tfrac{1}{2} h^2_{;c} \xi^c) - 2\psi h^2$$

$$= -(4/h^2)[\psi_{;ab}h^a h^b] - 2\ (\rho + \mu h^2)\psi$$

$$+ 2\mu \underset{\xi}{L} h^2, - 2\psi h^2 \qquad 3.14$$

$$\because\ h^a \underset{\xi}{L} h_a = -\tfrac{1}{2}\underset{\xi}{L} h^2 - 2\psi h^2 ?$$

i.e. $\underset{\xi}{L}(\rho) - \mu \underset{\xi}{L}(h^2) - h^2 \underset{\xi}{L}(\mu) =$

$$= -2(\rho + \mu h^2)\psi - (4/h^2)[\psi_{;ab}h^a h^b] - 2\psi h^2$$

i.e. $\underset{\xi}{L}(\rho) - \underset{\xi}{L}(\mu h^2) =$

$$= -2(\rho + \mu h^2)\psi - (4/h^2)[\psi_{;ab}h^a h^b] - 2\psi h^2 \qquad 3.15$$

On subtracting (3.9) from (3.14), we get

$$-2\underset{\xi}{L}(\mu h^2) = +4\psi_{;ab}u^a u^b - (4/h^2)[\psi_{;ab}h^a h^b] - 2\psi h^2$$

i.e. $$+2\underset{\xi}{L}(\mu h^2) = (4/h^2)(\psi_{;ab}h^a h^b) - 4\psi_{;ab}u^a u^b + 2\psi h^2 \qquad 3.16$$

Here proof of theorem is complete.

On contracting (3.6) with g^{ab}, we get

$$-2\psi_{;ab}g^{ab} = \underset{\xi}{L}(A) + A\psi + 4\underset{\xi}{L}(B) + 8B\psi + A\psi$$

$$+ D h^a(\underset{\xi}{L} h_a) + D h^b \underset{\xi}{L} h_b - h^2 \underset{\xi}{L}(D), \qquad 3.17$$

$$\therefore\ \underset{\xi}{L}(A + 4B) + 2(A + 4B)\psi +$$

$$+ 2D h^a \underset{\xi}{L}(h_a) - h^2 \underset{\xi}{L}(D) = 0, \qquad 3.18$$

Putting $A + 4B = -(\rho + \mu h^2)$ and $D = -\mu$ in (3.18), we get

$$\underset{\xi}{L}(\rho) + \mu \underset{\xi}{L} h^2 = -2(\rho + \mu h^2)\psi - 2\mu h^a \underset{\xi}{L}(h_a), \quad 3.19$$

Using $h^a \underset{\xi}{L}(h_a) = -\tfrac{1}{2} \underset{\xi}{L} h^2$ in equation (3.19), we get

$$\underset{\xi}{L}(\rho) + \mu \underset{\xi}{L} h^2 = -2(\rho + \mu h^2)\psi + \mu \underset{\xi}{L} h^2$$

i.e. $$\underset{\xi}{L}\rho = -2(\rho + \mu h^2)\psi \quad 3.20$$

Theorem 7 : If Magneto-Dust Distribution admits Conformal Killing vector ξ then

$$\psi_{;ab} u^a h^b = (\rho/2)\, h^2 .$$

Proof : If we transvect (3.7) with h^b, we get

$$-2\psi_{;ab} u^a h^b = -A h^2 - D h^2 u^a \underset{\xi}{L} h_a ,$$

$$\because h^a h_a = -h^2,\ u^a h_a = 0$$

i.e. $$2\psi_{;ab} u_a h^b = A h^2 + D h^2 u^a \underset{\xi}{L} h_a , \quad 3.21$$

i.e. $$(2/h^2)\psi_{;ab} u^a h^b = A + D u^a \underset{\xi}{L} h_a ,$$

$$= (\rho + \mu h^2) - \mu[-h^a \underset{\xi}{L} u_a], \quad 3.22$$

$$\because h^a \underset{\xi}{L} u_a = -u^a \underset{\xi}{L} h_a$$

$$= (\rho + \mu h^2) + \mu h^a [\psi u_a + h_a],$$

$$= (\rho + \mu h^2) + \mu(-h^2),$$

$$(2/h^2)\psi_{;ab} u^a h^b = \rho ,$$

i.e. $$\psi_{;ab} u^a h^b = \rho(h^2/2). \quad 3.23$$

Hence the theorem.

Theorem 8 : If Magneto-Dust Distribution admits Conformal Killing vector ξ then

$$\psi_{;ab} u^a u^b = -\tfrac{1}{4} \underset{\xi}{L} \mu h^2 .$$

Proof : Subtracting result (3.20) from (3.9), we get

$$\underset{\xi}{L}(\mu h^2) = -4 \psi_{;ab} u^a u^b , \qquad 3.24$$

i.e. $$\psi_{;ab} u^a u^b = -1/4 \; L \mu \underset{\xi}{h^2} .$$

Hence the theorem.

Theorem 9 : If Magneto-Dust Distribution admits Conformal Killing vector ξ then

$$\psi_{;ab} h^a h^b = (h^2/4) \underset{\xi}{L}(\mu h^2).$$

Proof : We have equation (3.16), viz

$$+2 \underset{\xi}{L}(\mu h^2) = + (4/h^2) \psi_{;ab} h^a h^b - 4 \psi_{;ab} u^a u^b. \qquad 3.25$$

If $$-4 \psi_{;ab} u_a u^b = \underset{\xi}{L}(\mu h^2), \qquad \text{vide } 2.24$$

Hence the above equation (3.25) reduces to,

$$+2 \underset{\xi}{L}(\mu h^2) - \underset{\xi}{L}(\mu h^2) = (4/h^2) \psi_{;ab} h^a h^b ,$$

i.e. $$(h^2/4) \underset{\xi}{L}(\mu h^2) = \psi_{;ab} h^a h^b.$$

Hence the theorem.

Therefore, from theorems (7), (8) and (9), we get

$$\psi_{;ab} u^a u^b = -1/4 \underset{\xi}{L} (\mu h^2) ,$$

$$\psi_{;ab} h^a h^b = (h^2/4) \underset{\xi}{L} (\mu h^2),$$

$$\psi_{;ab} u^a h^b = (\rho/2) h^2.$$

Sub-case of Case (I) :

Let $\xi = u$ be Conformal Killing vector
Then we have

$\theta = 0$, vide (2.6)

$\psi = 0$, vide (2.5)

$\sigma = 0$, vide (2.11)

$\dot{u}_b h^b = 0$. vide (2.13)

We recall equation (3.16), as

$$+ 2 \underset{u}{L} (\mu h^2) = (4/h^2) (\psi_{;ab} h_a h_b) - 4 \psi_{;ab} u^a u^b ,$$

As $\psi = 0$, this gives

$$\underset{u}{L} \mu h^2 = 0 . \qquad 3.27$$

Again, we recall equation (3.20)

$$\underset{u}{L} \rho = - 2 (\rho + \mu h^2) \psi ,$$

As $\psi = 0$, this gives

$$\underset{u}{L} \rho = 0 . \qquad 3.28$$

Hence we have a claim that matter energy density is ~~preserved~~ conserved.

We have equation (2.16) viz, $\dot{\rho} = \frac{1}{2}\dot{\mu}h^2$

Hence from equation (3.28)

$$\dot{u}h^2 = 0 \quad \Rightarrow \quad \underset{u}{L}\,\mu = 0\ , \qquad 3.29$$

This gives from (3.27),

$$\underset{u}{L}\,h^2 = 0\ . \qquad 3.30$$

Thus we have

$$\underset{u}{L}(\mu) = \underset{u}{L}\,h^2 = \underset{u}{L}\,\rho = 0\ . \qquad 3.31$$

Sub-case of Case (II) :

Let $\xi = h$ be Conformal Killing vector

Then we have a result, (3.20), viz.

$$\underset{h}{L}(\rho) = -2(\rho + \mu h^2)\psi\ .$$

If we put $\psi = (1/2h^2)\underset{h}{L}(h^2)$ in above equation, we get

$$\underset{h}{L}(\rho) = -2\ (\rho + \mu h^2)(1/2h^2)\underset{h}{L}(h^2) \qquad 3.32$$

vide (2.23)

i.e. $$\underset{h}{L}(\rho) = \frac{-(\rho + \mu h^2)}{h^2}\underset{h}{L}(h^2),$$

i.e. $$\underset{h}{L}(\rho) = 0 \text{ iff } \underset{h}{L}\,h^2 = 0). \qquad 3.33$$

This implies that $\underset{h}{L}(\rho) = -(\rho/h^2)$ iff $\underset{h}{L}\,h^2 = 0$.

Sub-case of Case (III) :

Let $\xi = \mu = h$ be Conformal Killing vector

For this choice we have from (3.16)

$$L(\mu h^2) = 0 . \qquad \text{vide (2.33)} \qquad 3.34$$

Also equation (3.20), yields

$$L(\rho) = 0 . \qquad \text{vide (2.33)} \qquad 3.35$$

Equation (2.39) and (3.34) implies that

$$L\mu = 0 , \qquad 3.36$$

$$L h^2 = 0 . \qquad 3.37$$

Thus we have a claim

$$L\mu = L h^2 = L\rho = 0 . \qquad 3.38$$

This implies that,

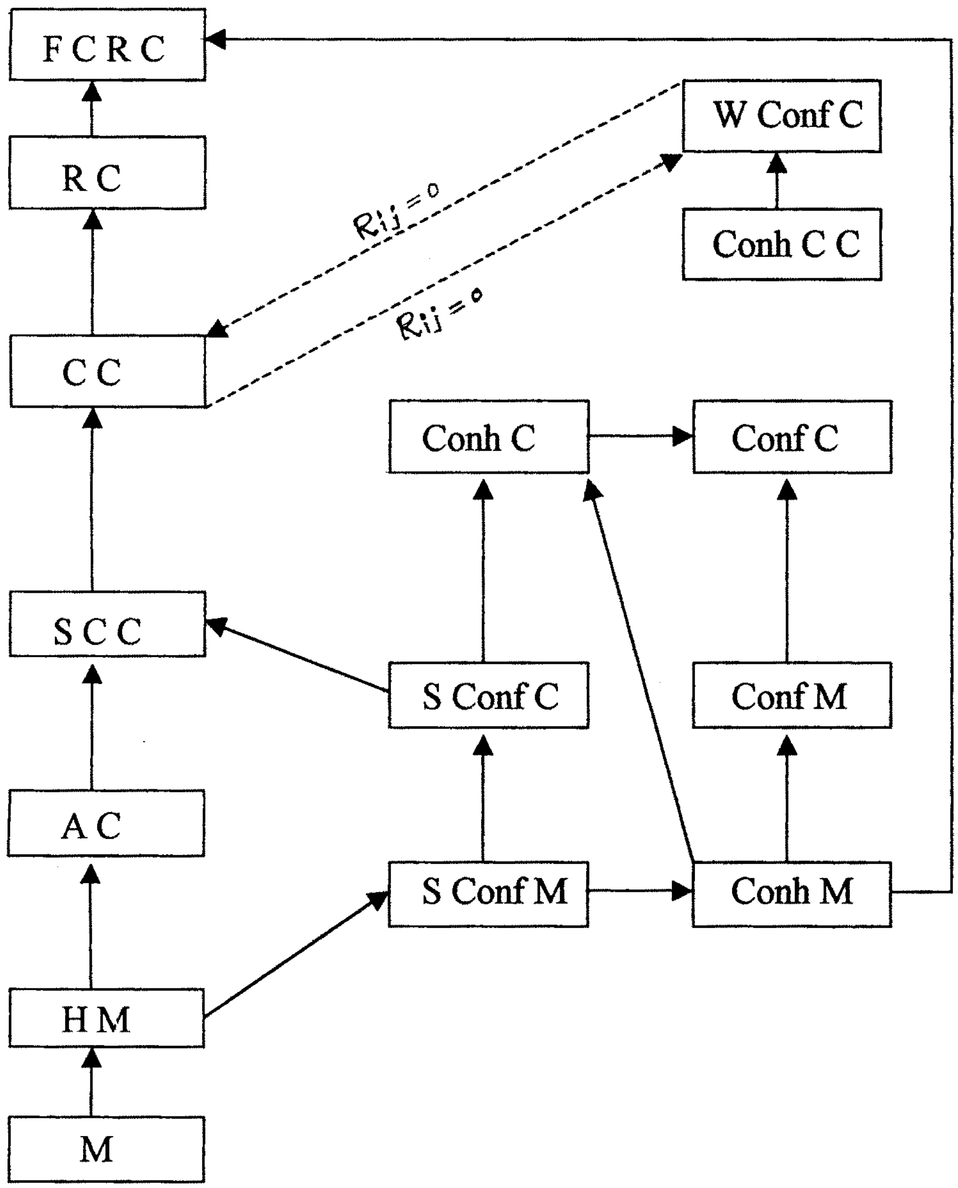
$$L_u \rho = L_h \rho = 0 ,$$

$$L_u h^2 = L_h h^2 = 0 , \qquad 3.39$$

$$L_u \mu = L_h \mu = 0 .$$

i.e. $\rho_{;a} = h^2{}_{;a} = \mu_{;a} = 0$. 3.40

THE RELATIONS BETWEEN CONHARMONIC AND OTHER SPACE-TIME SYMMETRIES.



(Abdussattar and Babita Dwivedi, 1998).