CHAPTER - IV

FIXED POINTS OF REASONABLE WANDERER AND ASYMPTOTICALLY REGULAR MAPS IN GENERALIZED HILBERT SPACE

CHAPTER-IV

4.1 INTRODUCTION :

The notions of 'REASONABLE WANDERER MAPS" and 'ASYMPTOTICALLY REGULAR MAPS ' were first introduced by Browder and Petryshyn [8] in Banach spaces. Further the authors [8] extended these concepts in Hilbert spaces and obtained the results (1.5.2, 1.5.3). Hicks and Huffman [14] have generalized these results in generalized Hilbert spaces (1.6.2). Boyte and others [2] extended the results of Hicks and Huffman [14] by considering generalized nonexpansive mappings in genralized Hilbert spaces. In this chapter we have proved some results about reasonable wandererality and asymptotic regularity for strictly pseudocontractive selfmappings of a closed convex subset of a GHS X. Further the results concerning weak and strong convergence of the sequence of iterates for the same mapping are also obtained. Lastly the above results about reasonabe Wanderer and asymptotically regular maps are generalized by considering the generalized contraction mappings. The result is as follows :

Theorem (4.1.1) : Let C be a closed convex subset of a generalized Hilbert space X and T be a strictly pseudocontractive selfmapping of C with contraction coefficient $k, 0 < k < 1$ such that the fixed point set F(T) of T is nonempty. Then for $0 < \lambda < 1$ and $k < 1-\lambda$ the mapping defined by

$$T_\lambda = \lambda I + (1-\lambda)T \qquad .....(4.1.2)$$

is a reasonable wanderer map from C into C with the same fixed points as T.

Proof : First we prove that $F(T) = F(T_\lambda)$.

Let $p \varepsilon F(T)$. Then from (4.1.2) , we have

$$T_\lambda (p) = \lambda p + (1-\lambda) Tp$$
$$= p$$

Hence p $F(T_\lambda)$. Also if $p \varepsilon F(T_\lambda)$ then obviously $p \varepsilon F(T)$. Thus T and T_λ have the same fixed points.

For any x in C, set $x_n = T_\lambda^n x$, $0 < \lambda < 1$. Also

$$x_{n+1} = \lambda Tx_n + (1-\lambda) x_n \qquad \text{.....(4.1.3)}$$

Now

$$\rho^2(x_{n+1} - p) = \rho^2(\lambda Tx_n + (1-\lambda) x_n - p). \qquad \text{.....(4.1.4)}$$

By using the Technique (1.2.20), (4.1.4) takes the form

$$\rho^2(x_{n+1} - p) = \lambda \rho^2(Tx_n - p) + (1-\lambda) \rho^2(x_n - p) - \lambda(1-\lambda) \rho^2(Tx_n - x_n). \qquad \text{......(4.1.5)}$$

Since T is strictly pseudocontractive, for any $x_n \varepsilon$ C, $y \varepsilon$ **C** and $0 < k < 1$,

$$\rho^2(Tx_n - Ty) \leqslant \rho^2(x_n - y) + k \rho^2((I-T)x_n - (I-T)y).$$

Hence for $p=y \varepsilon F(T)$, the above inequality reduces to

$$\rho^2(Tx_n - Tp) \leqslant \rho^2 (x_n - p) + k \rho^2(x_n - Tx_n) \qquad \text{.....(4.1.6)}$$

Introducing (4.1.6) in (4.1.5), we obtain

$$\rho^2(x_{n+1} - p) \leqslant \rho^2(x_n - p) - \lambda(1-\lambda-k) \rho^2(Tx_n - x_n),$$

where $k < 1-\lambda$. Summing these inequalities from n=0 to n=j, j being a positive integer, we have for each $\alpha \varepsilon \Delta$

$$\sum_{n=0}^{j} \rho_\alpha^2 (x_{n+1}-p) \leqslant \sum_{n=0}^{j} \rho_\alpha^2 (x_n-p) - \lambda (1-\lambda - k) \sum_{n=0}^{j} \rho_\alpha^2 (Tx_n-x_n)$$

or

$$\lambda (1-\lambda - k) \sum_{n=0}^{j} \rho_\alpha^2 (Tx_n-x_n) \leqslant \sum_{n=0}^{j} [\rho_\alpha^2 (x_n-p) - \rho_\alpha^2 (x_{n+1}-p)]$$

$$= \rho_\alpha^2 (x_0-p) - \rho_\alpha^2 (x_{j+1}-p)$$

$$\leqslant \rho_\alpha^2 (x_0-p)$$

or

$$\sum_{n=0}^{j} \rho_\alpha^2 (Tx_n-x_n) \leqslant \frac{1}{\lambda (1-\lambda - k)} \rho_\alpha^2 (x_0-p) \qquad .....(4.1.7)$$

which implies that

$$\sum_{n=0}^{\infty} \rho_\alpha^2 (Tx_n-x_n) < \infty .$$

Now from (4.1.3) , it follows that

$$x_{n+1} - x_n = \lambda (Tx_n-x_n).$$

Hence (4.1.7) takes the form

$$\sum_{n=0}^{\infty} \rho_\alpha^2 (x_{n+1}-x_n) = \lambda^2 \sum_{n=0}^{\infty} \rho_\alpha^2 (Tx_n-x_n)$$

$$< \frac{\lambda}{(1-\lambda - k)} \rho_\alpha^2 (x_0-p), \qquad k < 1-\lambda \quad ....(4.1.8)$$

$$< \infty$$

Hence $\sum_{n=0}^{\infty} \rho_\alpha^2 (x_{n+1}-x_n) < \infty$

which shows that T_λ is reasonable wanderer in C and the proof is complete.

Remark (4.1.9) As a immediate corollary to our result we have theorem (1.5.2) of Browder and Petryshyn [8] in Hilbert space. Also the same results for demicontractive and hemicontractive mappings have been taken care of. From the note (1.2.14), we have the following result as a corollary to theorem (4.1.1).

Corollary (4.1.10) Let T be a strictly pseudocontractive selfmapping of a closed convex subset C of a GHS X with contraction coefficient k, $0 < k < 1$ such that the fixed point set F(T) of T is nonempty. Then for $0 < \lambda < 1$ and $k < 1-\lambda$, the mapping on C, maps C into itself, is asymptotically regular and has the same fixed points as T.

Proof As T is strictly pseudocontractive selfmap of C, it follows from (4.1.8) that

$$\sum_{n=0}^{j} \rho_\alpha^2 (x_{n+1}-x_n) \leq \frac{\lambda}{(1-\lambda-k)} \rho_\alpha^2 (x_0-p)$$

Now setting $\lambda = 1/n+1$ and $x_n = T_\lambda^n x$ we have

$$\sum_{n=0}^{j} \rho_\alpha^2 (T_\lambda^{n+1}x - T_\lambda^n x) \leq \frac{1}{n-(n+1)k} \rho_\alpha^2 (x_0-p). \qquad .....(4.1.11)$$

Taking the limit as $n \to \infty$ of both sides in (1.4.11)

$$\lim_{n\to\infty} \left\{ \sum_{n=0}^{j} \rho_\alpha^2 (T_\lambda^{n+1}x - T_\lambda^n x) \right\} = 0$$

and hence

$$\rho_\alpha^2 (T_\lambda^{n+1}x - T_\lambda^n x) \to 0 \text{ as } n \to \infty.$$

This shows that T_λ is asymptotically regular.

Remark (4.1.12) The above result (1.4.10) generalizes the corresponding results of Hicks and Huffman [14, Theorem 1.6.7] and Browder and Petryshyn [8 ,Theorem 1.5.3].
we note the following

Note 4.2.1 Using theorem (1.6.1) of Hicks and Huffman [14] and proceeding on the same lines as in theorem (3.3.3) it can be easily proved that the set of fixed points F(T) of a strictly pseudocontractive mapping T is nonempty.

Theorem (4.2.2) If T is strictly pseudocontractive selfmapping of a closed, bounded, convex and weakly sequentially compact subset **C** of a generalized Hilbert space X and moreover if T is demicompact then the fixed point set F(T) of T is nonempty convex set and for any $x_0 \varepsilon$ **C** and any fixed λ with $0 < \lambda < 1$, the sequence $\{x_n\} = \{T_\lambda^n x_0\}$ determined by the process

$$x_n = \lambda T x_{n-1} + (1-\lambda) x_{n-1}, \ n= 1,2,..., \qquad .....(4.2.3)$$

converges strongly to a fixed point of T in **C**.

Proof As T is strictly pseudocontractive selfmapping of **C**, from the note (4.2.1) it follows that T has a fixed point in **C** and hence $F(T) \neq \phi$. Moreover F(T) is convex. Since **C** is bounded, the sequence $\{x_n\}$ in C is obtained.
Now from (4.2.3) we have

$$x_n - x_{n+1} = \lambda (x_n - Tx_n)$$

or

$$(x_n - Tx_n) = \lambda^{-1} (x_n - x_{n+1}) \qquad .....(4.2.4)$$

Then by (4.1.10), the sequence $\rho_\alpha^2\{x_n - Tx_n\}_{n=0}^{\infty} = \rho_\alpha^2\{\lambda^{-1}(x_n - x_{n+1})\}_{n=0}^{\infty}$ converges strongly to zero .

As T is demicompact and strictly pseudocontractive there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in F(T)$, since T is strictly pseudocontractive, $Tx_{n_j} \to Tp$ and $p=Tp$.

Now

$$\rho^2(x_n-p) = \rho^2(\lambda Tx_{n-1}+(1-\lambda)x_{n-1}-p)$$

$$= \lambda\rho^2(Tx_{n-1}-p) + (1-\lambda)\rho^2(x_{n-1}-p)-$$

$$-\lambda(1-\lambda)\rho^2(Tx_{n-1}-x_{n-1}) \qquad \ldots.(4.2.5)$$

As T is strictly pseudocontractive map (1.2.5), for x_{n-1} and $y=p=Tp$ in C we obtain

$$\rho^2(Tx_{n-1}-Tp) \leq \rho^2(x_{n-1}-p)+ k\,\rho^2(x_{n-1}-Tx_{n-1}),\ 0 < k < 1.$$

Hence for each $\alpha\in\Delta$ and the relation (4.2.5) reduces to

$$\rho_\alpha^2(x_n-p) \leq \rho_\alpha^2(x_{n-1}-p)-\lambda(1-\lambda-k)\ \rho_\alpha^2(x_{n-1}-Tx_{n-1}), \ldots\ldots(4.2.6)$$

where $k < 1 - \lambda$. But $\{\rho_\alpha^2(x_n - Tx_n)\}$ converges strongly to zero, from (4.2.6) it follows that $\{x_n\}$ is monotonically decreasing

i.e. $\rho^2(x_n-p) \leq \rho_\alpha^2(x_{n-1}-p)$,

and which implies that $\{x_n\}$ converges strongly to p and the proof is complete.

Remark (4.2.7)

The theorem (4.2.2) is the generalization of the corresponding results of Browder and Petryshyn [8 theorem (1.5.4)] and Hicks and Huffman [14, theorem (1.6.8)]. Also the hemicontractive and demicontractive mapping have been taken care of.

Now we generalize the results of section 4.1 .By using the generalized contraction mappings we extend the theorem 4.1.1 as follows :

Theorem (4.4.1) Let X be a generalized Hilbert space and T be generalized contraction selfmapping of a closed convex subset **C** of X (i.e. $\rho^2(Tx-Ty) \leq a_1\rho^2(x-y) + a_2\rho^2(Ty-x) + a_3\rho^2(Tx-y) + a_4\rho^2((I-T)x-(I-T)y)$ where $a_i \geq 0$, $\sum_{i=1}^{4} a_i < 1$ for all x,y in **C**) with further assumption that $a_1 + a_2 + a_3 = 1$, $a_3 + a_4 < 1$. Suppose the fixed point set F(T) of T in **C** is nonempty. Then for $0 < \lambda < 1$, the mapping defined by $T_\lambda = \lambda I + (1-\lambda)T$ is reasonable wanderer map from **C** into **C** and has the same fixed points as T.

Proof Let $p \in F(T)$. Then obviously from (4.1.1) it follows that $F(T)=F(T_\lambda)$.For any x in **C**, set $x_n = T_\lambda^n x$, $0 < \lambda < 1$,and $\{x_n\}$ is defined by

$$x_{n+1} = \lambda Tx_n + (1-\lambda)x_n \qquad \text{.....(4.4.2)}$$

Now

$$\rho^2(x_{n+1}-p) = \rho^2(\lambda Tx_n + (1-\lambda)x_n - p) \qquad \text{.....(4.4.3)}$$

By using the Technique (1.2.20),(4.4.3) takes the form

$$\rho^2(x_{n+1}-p)=\lambda\rho^2(Tx_n-p)+(1-\lambda)\rho^2(x_n-p)-$$

$$-\lambda(1-\lambda)\rho^2(Tx_n-x_n). \quad .....(4.4.4)$$

But T is generalized contraction, for any $x_n \varepsilon$ C and y ε C,

$$\rho^2(Tx_n-Ty) \leqslant a_1\rho^2(x_n-y)+a_2\rho^2(Ty-x_n)+a_3\rho^2(Tx_n-y)+$$

$$+a_4\rho^2(I-T)x_n-(I-T)y).$$

Hence for y=p=Tp, above inequality reduces to

$$\rho^2(Tx_n-p) \leqslant a_1\rho^2(x_n-p)+a_2\rho^2(p-x_n)+a_3\rho^2(Tx_n-p)+$$

$$+a_4\rho^2(I-T)x_n-(I-T)p)$$

$$=(a_1+a_2)\rho^2(x_n-p)+a_3\rho^2(Tx_n-p)+a_4\rho^2(x_n-Tx_n).$$

Equivalently

$$\rho^2(Tx_n-p) \leqslant \frac{(a_1+a_2)}{(1-a_3)}\rho^2(x_n-p)+\frac{a_4}{1-a_3}\rho^2(x_n-Tx_n) \quad ....(4.4.5)$$

Introducing (4.4.5) in (4.4.4) we obtain

$$\rho^2(x_{n+1}-p) \leqslant \frac{(a_1+a_2)}{(1-a_3)}\lambda\rho^2(x_n-p)+\frac{(a_4)}{(1-a_3)}\lambda\rho^2(x_n-Tx_n)+$$

$$+(1-\lambda)\rho^2(x_n-p)-\lambda(1-\lambda)\rho^2(Tx_n-x_n)$$

$$=(h\lambda-\lambda+1)\rho^2(x_n-p)-\lambda(1-\lambda-k)\rho^2(Tx_n-x_n),$$

where $h=\frac{(a_1+a_2)}{(1-a_3)}$, $k=(\frac{a_4}{1-a_3})<1$ & $1-h<1$

or

$$\rho^2(x_{n+1}-p) \leqslant (1-\lambda)(1-h)\,\rho^2(x_n-p) - \lambda(1-\lambda-k)\,\rho^2(Tx_n-x_n)$$

$$\leqslant \rho^2(x_n-p) - \lambda(1-\lambda-k)\,\rho^2(Tx_n-x_n) \quad \ldots.(4.4.6)$$

Now letting $k < 1-\lambda$ and proceeding further on the same lines as in Theorem 4.1.1., we get the desired result.

Remark (4.4.7)

Our result (4.4.1) generalizes theorem (1.5.2) of Browder and Petryshyn [8] and theorem (4.1.1) .Also from the table (1.2.11) , obviously the mapping (1.2.3), (1.2.5), (1.2.8), (1.2.9) have been taken care of .

Corollary (4.4.8) Let X be a GHS and C be a closed convex subset of X. If T is a selfmap on C satisfying the conditions of Theorem (4.4.1) and if $F(T) \neq \phi$ then the mapping $T_\lambda = \lambda I+(1-\lambda)T$, $0 < \lambda < 1$ has the same fixed points as T and is asymptotically regular..

Proof The proof is obvious and can be developed on the same lines in the light of Theorem (4.1.1)) and (4.4.1).

Remark (4.4.9) The above corollary (4.4.8) generalizes theorem (1.6.7) of Hicks and Huffman [14], corollary (1.5.3) of Browder and Petryshyn [8] and corollary (4.1.10).