CHAPTER – 0

DEFINITIONS

AND

RESULTS

CHAPTER 0

"DEFINITIONS AND RESULTS"

This chapter is devoted to the definitions and results which will be used in the subsequent chapters.

♦ 0.1 Definitions:-

§ 0.1.1 Partially ordered set or Poset: ([6], Page 2)

Let P be a nonvoid set. Define a relation '$\leq$' on P satisfying the following.

(1) $a \leq a$ (Reflexivity)

(2) If $a \leq b$ and $b \leq a$ then $a = b$ (Antisymmetry)

(3) If $a \leq b$ and $b \leq c$ then $a \leq c$ (Transitivity)

for all a, b, c, $\in$ P,

then the ordered pair $<P, \leq>$ is called Poset or a Partially ordered set.

A Poset $<P, \leq>$ is called chain (or totally ordered set or linearly ordered set) if it satisfies following condition,

(4) $a \leq b$ or $b \leq a$, for all a, b, c $\in$ P (Linearity)

Let $<P, \leq>$ be a Poset and $H \subseteq P$, $a \in P$ is an upper bound of H if $h \leq a$ for all $h \in H$. An upper bound a of H is the least upper bound of H or supremum of H (join) if for any upper bound b of H, we have $a \leq b$.0

We shall write a = Sup H or $a = \vee H$.

The concept of lower bound or infimum is similarly defined. The latter is denoted by Inf H or $\wedge$ H

§ 0.1.2 Zero element in a Poset ([6], Page 56)

Let $< P, \leq >$ be a Poset.

If there exist 0 in P such that $0 \leq x$, for all x in P, then 0 is called Zero element in Poset P.

§ 0.1.3 Unit element In a Poset ([6] Page 56)

Let $< P, \leq >$ be a Poset.

If there exist 1 in P such that $1 \geq x$, for all x in P, then is called Unit element in poset P.

§ 0.1.4 Bounded Poset ([6] Page 56)

A Poset with Zero element and the Unit element is called bounded Poset.

§ 0.1.5 Lattice (as a Poset) ([6] Page 43)

A poset $< L, \leq >$ is called lattice if sup {a,b} and inf {a,b} exists for all a and b in L.

§ 0.1.6 Lattice (as an algebra) ([6] Page 5)

Let L be any nonempty set. If '$\wedge$' and '$\vee$' are binary operations defined on L then $< L, \wedge, \vee >$ is called Lattice if the following conditions hold for all a, b, c in L.

(1) $a \wedge b = b \wedge a$

(commutativity)

$a \vee b = b \vee a$

(2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(Associativity)

$a \vee (b \vee c) = (a \vee b) \vee c$

(3) $a \wedge a = a$

(Idempotency)

$a \vee a = a$

(4) $a \wedge (a \vee b) = a$

(Absorption law)

$a \vee (a \wedge b) = a$

§ 0.1.7 Ideal in a lattice. ([6] Page 21)

A nonempty subset I of a lattice L is called an ideal

(1) If $x \in I$, $y \in I$, then $x \vee y \in I$

(2) If $x \leq y$, $x \in L$, $y \in I$ then $x \in I$

§ 0.1.8 Proper ideal in a lattice. ([6] Page 21)

An ideal I which is different from Lattice L is called proper ideal.

§ 0.1.9 Prime ideal in a lattice. ([6] Page 21)

A proper ideal I in a lattice L is called a prime ideal if for all x and y in L, $x \wedge y \in I$ implies that $x \in I$ or $y \in I$.

§ 0.1.10 Principal ideal in a lattice. ([6] Page 21)

Given an element a in L, the ideal generated by {a} denoted by (a]; where $(a] = \{x \in L / x \leq a\}$ is called Principal ideal of Lattice L.

§ 0.1.11 Maximal ideal in a lattice. ([1] Page 28)

A proper ideal M in a lattice L is called maximal ideal in L if there do not exist any proper ideal J in L such that $M \subset J \subset L$.

§ 0.1.12 Minimal prime ideal in a lattice. ([6] Page 169)

A minimal element in the set of all prime ideals in L is a minimal prime ideal in a lattice.

§ 0.1.13 Filter in a lattice. ([6] Page 23)

A nonempty subset F of a lattice L is called Filter

(1) if $x \in F$ and $y \in F$ then $x \wedge y \in F$,

(2) if $x \leq y$, $y \in L$, $x \in F$ then $y \in F$.

§ 0.1.14 Proper filter in a lattice. ([6] Page 23)

A filter F which is different from Lattice L is called proper filter.

§ 0.1.15 Prime filter in a lattice. ([6] Page 23)

A proper filter F in a lattice L is called a prime filter if for all x and y in L. $x \vee y \in F$ imply that $x \in F$ or $y \in F$.

§ 0.1.16 Principal filter in a lattice. ([6] Page 23)

Given an element a in L, the filter generated by {a} denoted by[a), where $[a) = \{ x \in L / x \geq a \}$ is called principal filter of L.

§ 0.1.17 Maximal filter in a lattice. ([1] Page 28)

A proper filter in a lattice L is called maximal filter in L if there do not exist any proper filter J in L such that $M \supset J \supset L$.

§ 0.1.18 Distributive lattice. ([6] Page 36)

A lattice $< L, \wedge, \vee >$ is said to be distributive if '$\wedge$' is distributive over '$\vee$' or dually.

In other words,

For all x, y, z in L

D1: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

OR D2: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

§ 0.1.19 0-distributive lattice. ([12])

Let L be a lattice with 0. L is said to be 0- distributive if $a \wedge b = 0$, $a \wedge c = 0$ then $a \wedge (b \vee c) = 0$, for all $a, b, c \in L$.

§ 0.1.20 Complemented lattice. ([11] Page 58)

A lattice L with Zero 0 and Unit 1 in which for any element 'a' there is an element 'b' (complement of a) such that

$a \vee b = 1$ and $a \wedge b = 0$

§ 0.1.21 Pseudocomplemented lattice. ([6] Page 58)

A lattice L with a Zero '0' is called a Pseudocomplemented lattice if for any $a \in L$, there is an element a^* such that $a \wedge x = 0$ if and only if $x \leq a^*$.

§ 0.1.22 Quasicomplemented lattice. ([7] Page 41)

A lattice L with zero '0' is called a Quasicomplemented lattice if for any $x \in L$ there is an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is a dense element.

OR

A lattice L is Quasicomplemented if for each $x \in L$, $\exists\, y \in L$ such that $(x]^{**} = (y]^*$.

§ 0.1.23 0- ideal. ([5]Page 1059)

Let J be an ideal of a lattice L, then J is called an 0- ideal if $J = 0(F)$, for some filter F, where $0(F) = \{x \in L / x \wedge f = 0,$ for some $f \in F\}$

§ 0.1.24 $\propto$ - ideal. ([5] Page 1060)

An ideal J is an $\propto$-ideal if and only if $x \in J$ implies $(x]^{**} \subseteq J$.

§ 0.1.25 Moore family in lattice. ([1] Page 111)

Let X be any nonempty set and $\mathcal{F} \subseteq \mathcal{P}(X)$.

$\mathcal{F}$ is said to form a Moore family of subsets of X if

1) $X \in \mathcal{F}$.

2) $\bigcap_{F_\alpha \in \mathcal{F}} F_\alpha \in \mathcal{F}$

§ 0.1.26 Nondense ideal. ([9] Mane)

An ideal $I \neq \phi$ of a lattice L is said to be dense (nondense) if

$(a]^* = \{0\}$ $\left((a]^* \neq \{0\} \right)$.

♦ 0.2 Results

In the following results we are concerned with bounded lattice L with o

§ 0.2.1: In a lattice L with 0, every proper filter is contained in a maximal filter.

§ 0.2.2: Intersection of any number of filters in L is a filter in L

([6], page 21)

§ 0.2.3: Intersection of any number of ideals in L is a ideal in L

([6], page 21)

§ 0.2.4: Complement of a maximal filter is a minimal prime ideal in a 0-distributive lattice.

([9], page -437)

§ 0.2.5: In a lattice L every prime ideal contains minimal prime ideal.

([6], page 79)

§ 0.2.6: In a lattice L a proper filter M in L is maximal if and only if for any element $a \notin M$ $(a \in L)$ there exist an element b in M such that. $a \wedge b = 0$.

(Vankatanarsimhan, [13])

§ 0.2.7: $\{x\}^* \wedge \{y\}^* = \{x \vee y\}^*$

$\{x\}^* \vee \{y\}^* = \{x \wedge y\}^*$ for all $x, y \in L$

([9], page 441)

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