CHAPTER – 1

0-IDEALS

IN

0-DISTRIBUTIVE

LATTICES

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"0-IDEALS IN 0- DISTRIBUTIVE LATTICES"

1.1. Introduction:-

Throughout this chapter we will be concerned with a bounded 0-distributive lattice L. The concept of 0-ideals in bounded distributive lattices is introduced by Cornish [5]. In the same way 0-ideals in 0-distributive lattices are defined. In this chapters help is taken of the results of the Cornish [4] and Jayram [7] to obtain the properties of 0- ideals in 0-distributive lattices. Several examples of 0-ideals in 0- distributive lattices are also provided.

At the outset we prove the result, which is crucial in defining 0-ideal in L.

Theorem:-1.2 0 (F) is an ideal for any filter F in L,

where $0(F)=\{x \in L / x \wedge f=0,$ for some $f \in F\}$

Proof- As $0 \wedge f=0$, for each $f \in F$

We get, $0 \in 0(F)$ and hence $0(F) \neq \phi$, also we have if $x, y \in 0(F)$ then we get,

$x \wedge f_1 = 0$, for some $f_1 \in F$.

and $x \wedge f_2 = 0$, for some $f_2 \in F$.

Hence $x \wedge (f_1 \wedge f_2)=0$ and $y \wedge (f_1 \wedge f_2)=0$

Where $f_1 \wedge f_2 \in F$ [Result 0.2.2]

Therefore by 0- distributivity of L,

$(x \vee y) \wedge (f_1 \wedge f_2)=0$

Therefore $x \vee y \in 0(F)$, by definition of $0(F)$, lastly consider $x \leq y$, $x \in L$ and $y \in 0(F)$.

As $y \in 0(F)$.

It implies that $y \wedge f = 0$, for some $f \in F$.

As $x \leq y$,

we get, $x \wedge f \leq y \wedge f$.

Therefore, $x \wedge f = 0$, for some $f \in F$.

It implies that, $x \in 0(F)$.

This proves that, $0(F)$ is an ideal foe any filter F in L.

Now the next result describes an ideal $0(F)$ in another form.

Theorem 1.3. For any filter F in L.

$0(F) = \cap \{P / P$ is minimal prime ideal such that $P \cap F = \phi\}$.

Proof- suppose $x \in 0(F)$.

Then $x \wedge y = 0$, for some $y \in F$.

We have $0 \in P$, where P is minimal prime ideal such that $P \cap F = \phi$.

As $y \in F$, we get $y \notin p$.

By prime ness of P, it follows that, $x \in P$.

Thus, we have,

$0(F) \subseteq \cap \{P / P$ is minimal prime ideal such that $P \cap F = \phi\}$.

For the reverse inclusion, suppose that,

$X \in \cap \{P / P$ is minimal prime ideal such that $P \cap F = \phi\}$.

and assume if possible that $x \notin 0(F)$.

Claim. $[f \vee [x)]$ is a proper filter.

Proof- Assume that $F \vee [x)=L$. Then $0 \in L$ implies $0 \in F \vee [x)$.

Hence $f \wedge x=0$, for some $f \in F$.

But then $x \in 0\,(F)$, which contradicts the choice of x.

Hence $0 \notin F \vee [x)$.

It means that $F \vee [x)$ is a proper filter.

Since every proper filter is contained in a maximal filter [see 0.2.1], there exists a maximal filter say M in L such that $F \vee [x) \subseteq M$.

Thus as $x \in M$, we get $x \notin L \setminus M$,

where $L \setminus M$ is a minimal prime ideal disjoint with F [see 0.2.4]

This contradicts the choice of x .

Hence $x \in 0\,(F)$. It follows that,

$\cap \{P/P$ is minimal prime ideal such that $P \cap F=\phi\} \subseteq 0\,(F)$,

combining both the inclusions, for any filter F in L, we get,

$0\,(F)= \cap \{P/P$ is minimal prime ideal such that $P \cap F=\phi\}$.

The definition of an 0-ideal in bounded 0-distributive lattice is as Follows.

Definition 1.4.

Let J be an ideal of a lattice L .Then J is called an 0-ideal if $J=0(F)$, for some filter F, where $0\,(F) = \{x \in L \,/\, x \wedge f =0$, for some $f \in F\}$

Example 1.5

Let L={0,a, b, c, d, e, f, g, 1}.

Consider the 0-distributive bounded lattice <L, ∧, ∨ > ,

whose diagrammatic representation is as shown in figure 1

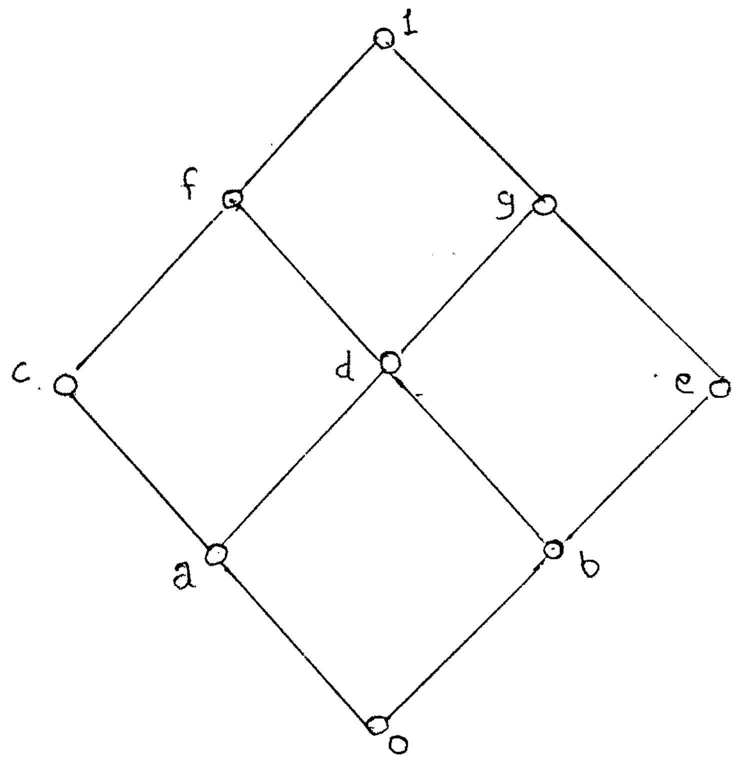

Figure 1

J= {0,a, c} is an ideal in L.

F={b, d, f, e, g, 1} is a filter in L.

0 (F) = {x ∈ L / x ∧ f =0, for some f ∈ F}

= {0,a, c}

As J is an o-ideal,

0 (F) = J

In the following Example 1.6, we establish that every ideal need not an 0-ideal

Example 1.6

Consider the 0-distributive bounded lattice $< L, \wedge, \vee >$,

whose diagrammatic representation is as shown in figure 2

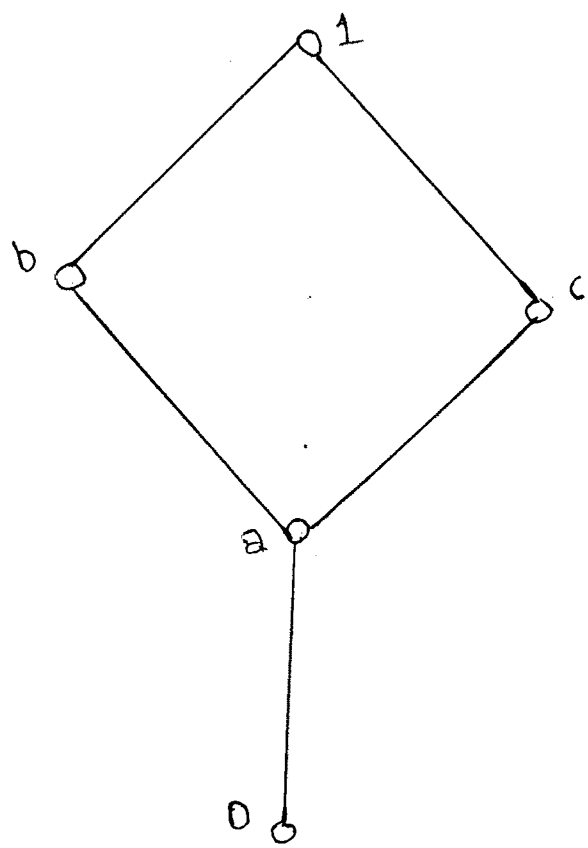

Figure 2

Then $J = \{0, a, b\}$ is an ideal in L also $F=\{c, 1\}$ is a filter in L.

As $0(F) = \{0\} \neq J$,

J is not an 0-ideal.

Thus using the definition of an o-ideal, characterization of o-ideal in o-distributive lattice is given as follows.

Theorem 1.7.

For each $x \in L$, $(x]^*$ is an 0-ideal.

Proof – Claim $(x]^* = 0([x))$.

Here firstly we know that $(x]^*$ is an 0-ideal.

As L is 0-distributive, we have,

i) Let $y \in (x]^*$.

Then $y \wedge x=0$, as $x \in [x)$ we get , $y \in 0([x))$.

Thus $(x]^* \subseteq 0([x))$.

ii) Let $y \in 0([x))$.

It implies that $y \wedge t=0$, for some $t \in [x)$.

We know that, as $t \in [x)$, $t \geqslant x$.

Thus $y \wedge t=0$ and $t \geqslant x$ imply $o= y \wedge t \geqslant y \wedge x$.

Therefore, $x \wedge y=0$.

It follows that $y \in (x]^*$.

Therefore, $0([x)) \subseteq (x]^*$.

From i) and ii) $0([x)) = (x]^*$,

it shows that $(x]^*$ is an 0-ideal foe each $x \in L$.

For a Prime ideal P in L,

we define $\overline{P} = \{x \in L / x \wedge y=0,$ for some $y \in L\backslash P\}$

An interesting property of $\overline{\underline{P}}$ is proved in the following theorem.

Theorem. 1.8 For each prime ideal P, $\overline{P}$ is an 0-ideal,

where $\overline{P} = \{ x \in L / x \wedge y = 0, \text{ for some } y \in L \setminus P\}$

Proof- claim 1 $\overline{P}$ is an ideal in L.

Proof- i) As $0 \in L$, $0 \wedge y = 0$ for $y \notin \overline{P}$

It implies that $\overline{P} \neq \phi$

ii) Let $x, y \in \overline{P}$.
we get $x \wedge t_1 = 0$, for some $t_1 \notin P$

$y \wedge t_2 = 0$, for some $t_2 \notin P$

Now the definition of prime ideal $t_1 \wedge t_2 \notin P$

As $x \wedge (t_1 \wedge t_2) = 0$

and $y \wedge (t_1 \wedge t_2) = 0$, for $t_1 \wedge t_2 \in L \setminus P$.

By 0-distributive of L,

We get $(x \vee y) \wedge (t_1 \wedge t_2) = 0$, for $t_1 \wedge t_2 \in L \setminus P$.

It means that $x \vee y \in \overline{P}$

iii) Let $x \leq y$, $x \in L$ and $y \in \overline{P}$

Hence we write,

$y \wedge t = 0$, for some $t \in L \setminus P$

Since $x \leq y$

We get, $x \wedge t \leq y \wedge t$.

Therefore, $x \wedge t = 0$, for $t \in L \setminus P$.

It shows that $x \in \bar{P}$. Thus from i), ii) and iii),

we get $\bar{P}$ is an ideal in L.

As P is prime ideal then L\P is a filter. [Gratzer [6]]

We have,

$\bar{P} = \{x \in L / x \wedge y=0, \text{ for some } y \notin P\}$

$= \{x \in L / x \wedge y=0, \text{ for some } y \in L\backslash P\}$

$= 0\ (L\backslash P)$.

As L\P is a filter,

$\bar{P} = 0\ (L\backslash P)$, we get $\bar{P}$ is an 0-ideal for each prime ideal P.

It is well known that intersection of an ideal in L is an ideal in L but also intersection of any two 0-ideals is an 0-ideal.

This follows from Theorem 1.9.

Theorem 1.9 The intersection of any two 0-ideals in L is an 0-ideal.

Proof Let J_1 and J_2 be two 0-ideals. Since there are two filter F_1, F_2 such that $J_1 = 0(F_1)$ and $J_2 = 0(F_2)$. Obviously $J_1 \cap J_2$ is an ideal in L.

To prove $J_1 \cap J_2$ is a 0-ideal. It is enough to show that

$0(F_1) \cap 0\ (F_2) = 0\ (F_1 \cap F_2)$ [By Result [0.2.3]

Let $x \in 0(F_1) \cap 0\ (F_2)$

Then $x \wedge f_1 = 0$, for some $f_1 \in F$

$x \wedge f_2 = 0$, for some $f_2 \in F$

Thus $x \wedge (F_1 \vee F_2) = 0$.

Also as $f_1 \in F_1$, $f_1 \leq f_1 \vee f_2$ we get $f_1 \vee f_2 \in F_1$.

and as $f_2 \in F_2$, $f_2 \leq f_1 \vee f_2$, we get $f_1 \vee f_2 \in F_2$. By Result [0.2.2]

$f_1 \vee f_2 \in F_1 \cap F_2$

Thus $x \wedge (f_1 \vee f_2)=0$, for some $f_1 \vee f_2 \in F_1 \cap F_2$

Hence $0(F_1) \cap 0(F_2) \subseteq 0(F_1 \cap F_2)$.

Also let $y \in 0(F_1 \cap F_2)$.

Then it means that $y \wedge f = 0$, for some $f \in F_1 \cap F_2$

Hence $y \wedge f = 0$, for some $f \in F_1$ and $f \in F_2$

Therefore, we have, $y \in 0(F_1)$ and $y \in 0(F_2)$

Therefore, $y \in 0(F_1) \cap 0(F_2)$.

Thus $0(F_1 \cap F_2) \subseteq 0(F_1) \cap 0(F_2)$.

Hence $0(F_1 \cap F_2) = 0(F_1) \cap 0(F_2)$.

This shows that it is an 0-ideal.

Remark 1.10

By generalizing this theorem we say that intersection of any number of 0-ideals in L is an 0-ideal. As $\cap_\alpha 0(F_\alpha) = 0(\cap_\alpha F_\alpha)$.

Using the definition of Moore Family, we get, (Definition 0.1.25)

Remark 1.11

The set of all 0-ideals forms a Moore family.

Proof Let $K = \{I/I \text{ is a 0-ideal}\}$.

As $L=0([1])$, we get $L \in K$

$I_\alpha \in K$ imply $\cap_\alpha \in K$

Necessary and sufficient condition for a nondense ideal I to be prime given in Theorem 1.14 will be obtained from following Theorems viz. Theorem 1.12 and Theorem 1.13.

Theorem 1.12

If I is prime ideal in L then I satisfies the condition, for any $x \in L$, $x \notin I$ implies $I^* \cap \{x\}^* = \{0\}$.

Proof- claim. If $x \notin I$ then $\{x\}^* \subseteq I$

Here $x \notin I$

Let $y \in \{x\}^*$ then we get $y \wedge x=0 \in I$

As I is prime $x \in I$ or $y \in I$

But give that $x \notin I$

hence $y \in I$

Thus $\{x\}^* \subseteq I$

Consider $y \in I^* \cap \{x\}^*$

Then we get $y \in I^*$ and $y \in \{x\}^*$

By the above claim if $x \notin I$, $\{x\}^* \subseteq I$

Therefore, $y \in I$

Now $y \in I^*$ implies that $y \wedge i = 0$, for every $i \in I$

Take particularly $i = y$, it means that $y \wedge y = 0$

Hence $y=0$

Thus $I^* \cap \{x\}^* = \{0\}$.

Using the definition of non- dense ideal converse of the Theorem 1.12 is true for non-dense ideals.

Theorem 1.13

If non-dense ideal I of L satisfies the condition:

$x \notin I$ implies $I^* \cap \{x\}^*=\{0\}$, for any $x \in L$, then I in prime ideal.

Proof – claim 1- I is nondense and satisfies the condition for any $x \in L$,

$x \notin I$ implies $I^* \cap \{x\}^*=\{0\}$ then $I = I^{**}$.

Proof – we know that $I \subseteq I^{**}$ always .

Let $x \in I^{**}$ such that $x \notin I$.

Then by the given condition, we get, $I^* \cap \{x\}^*=\{0\}$

As I is nondense we get, $I^* \neq \{0\}$

Hence, there exist $y \in I^*$ such that $y \neq 0$.

Thus $y \in I^*$, $I^* \cap \{x\}^*=\{0\}$Therefore, $y \notin \{x\}^*$

Hence $y \wedge x \neq 0$

But $y \in I^*$ and $x \in I^{**}$, hence $x \wedge y =0$.

Which is a contradicition.

Thus $x \in I$, proving that $I^{**} \subseteq I$

Therefore , $I = I^{**}$

Here now only to prove that I is prime ideal,

consider $a \wedge b \in I$ and $a \notin I$, for any two elements a and b of L. By the given condition $I^* \cap a\}^*=\{0\}$,

also for any $c \in I^*$, $c \wedge (a \wedge b)=0$ implies $c \wedge b \in I^* \cap \{a\}^*$.

Hence $c \wedge b = 0$.

Consequently, $b \in I^{**} =I$ (by claim 1)

It shows that I is prime ideal.

With the definition of nondense ideal, (Definition 0.1.26) we write the result as follows

Theorem 1.14 If I is a nondense prime ideal then I is minimal prime ideal.

Proof- We know that I is nondense prime ideal.

Hence $I^* \neq \{0\}$

Therefore there exists $y \in I^*$ such that $y \neq 0$. As I is prime,

[By the Result [0.2.5]],

There exists a minimal prime ideal M such that $M \subseteq I$.

Suppose $M \subset I$

Thus there exists $x \in I$ such that $x \notin M$.

Therefore , for $x \in I$ and $y \in I^*$.

Therefore we get $x \wedge y \in M$, $x \notin M$. Hence $y \in M$.

We know that $M \subseteq I$.

Thus $y \in I$, which is a contradiction.

This shows that there exists no minimal prime ideal M properly contained in I. It means that I is minimal prime ideal.

By definition of an 0-ideal developed by Theorem 1.3 we give the result as below.

Theorem 1.15 Every minimal prime ideal is an 0-ideal.

Proof Let M be a minimal prime ideal,

hence L\M is maximal filter. By Result (0.2.4)

Let F = L\M claim that 0(F) =M. By theorem 1.3, we get

$0\,(F)= \cap \{M/M$ is minimal prime ideal such that $M \cap F=\phi\}$

i.e. $M \cap F = \phi$. It implies that $M \cap (L\backslash M) = \phi$.

Thus $0(F) = M$. Hence by definition,

M is an 0-ideal.

Relation between 0-ideal and nondense prime ideal is described in followng Theorem.

Theorem 1.16 If a prime ideal P is nondense then P is an o-ideal .

Proof Here to show that $P = 0(F)$, for some filter F.

Take $F = L \backslash P$.

Claim $P = 0(L \backslash P)$

Let $x \in P$. As P is nondense $P^* \neq \{0\}$ implies that there exist $y \in P^*$ such that $y \neq 0$. As $x \in P$ and $y \in P^*$.

Therefore $x \wedge y = 0$ then we get $x \wedge y = 0$ for $y \notin P$.

Hence $x \wedge y = 0$, for $y \in L \backslash P$. This shows that $x \in 0(L \backslash P)$.

Therefore $P \subseteq 0(L \backslash P)$.

Let $x \in 0(L \backslash P)$ then $x \wedge y = 0$, for $y \in L \backslash P$.

It means that $x \wedge y = 0$, for $y \notin P$. Hence $x \in P$.

Therefore $0(L \backslash P) \subseteq P$. It follows that $P = 0(L \backslash P)$, here $L \backslash P$ is a filter.

Hence P is an 0-ideal.

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