Chapter – 2

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Piecewise Approximation

2.1 Limitation of Polynomial Approximation

The Polynomial interpolation is very sensitive to the choice of interpolation points . The polynomial interpolation at approximately chosen points (e.g.the chebyshev points) produces an approximation which , for all practical purposes , differs very little from the best possible approximant by polynomials of the same order . so we illustrate the essential limitation of the Polynomial approximation . If the function to be approximated is badly behaved anywhere in the interval of approximation . This global dependence on local properties can be avoided by using piecewise polynomial approximants.

Uniform Spacing of data can have bad Consequences :-

e.g. Runge,s Example :-

Consider the polynomial Pn of order n which agrees with the function $y(x) = 1 / d\ (1+ 25\ x^2)$ at the following n uniformly spaced points in [-1,1]

$\tau_i = (i = 1)h-1$ $\quad$ $i = 1,2,\ldots\ldots n$ where $h=2/n-1$

The function g being without apparent defect (g is analytic in a neighbourhood of the interval of approximation [-1,1]

Maximum error $|| e_n || = \max | g (x) - P_n (x) |$

To increase toward zero as n increases . if we estimate the maximum error $|| e_n ||$ by the maximum value obtained when evaluating the absolute value o g- P_n at 20 points in each of (n-1) interval (τ_{i-1}, τ_i) $i = 1,2,\ldots\ldots n$

We find that as a function of n , $|| e_n ||$ decreases to Zero like $B\, n^{\alpha}$ for some constant β and some (negative) –ve constant α.

If $|| e_n || \sim B\, n^{\alpha}$, the

$$\frac{|| e_n ||}{|| e_m ||} \sim (n / m)^{\alpha}$$

and we can estimate the decay exponent α from two error $|| e_n ||$ and $|| e_m ||$ by

$$\alpha^n \sim \frac{\log || e_m || - \log || e_m ||}{\log (n / m)}$$

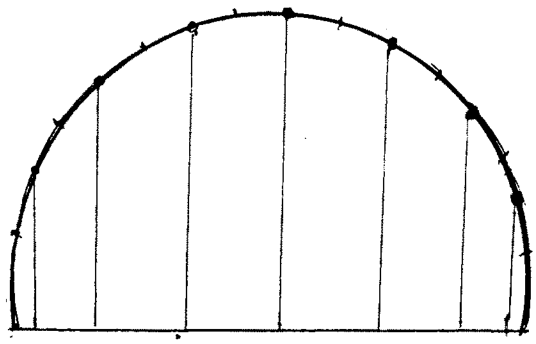
if we calculate decay exponent

N	Max. error	Decay exponent
2	0.9615+00	0.00
4	0.7070+00	-0.44
6	0.4327+00	-1.21
8	0.2474+00	-1.94
10	0.2994+00	0.86
12	0.5567+00	3.40
14	0.1069+01	4.23
16	0.2099+01	5.05
18	0.4214+01	5.92
20	0.8573+01	6.74

we find that, contrary to expectations , the interpolation error for this example actually increase with n . In fact , our estimates for the decay exponent become eventually positive and growing so that $\| e_n \|$ grows with n at an ever increasing rate.

Chebyshev Points :-

The Chebyshev points for the interval [a, b] are obtained by subdividing the semicircle over it into n equal arcs and then projecting the midpoint of each arc onto the interval.



Chebyshev points are good

Theorem :- If τ_1 , τ_2, ...τ_n are chosen as the zero of the chebshev polynomials of degree 'n' for the interval [a,b]

i.e. $\tau_j = \tau^c_j = [\ a+b - (a-b)\cos\{2j-1)\pi/2n]/2$

then with j = 1,2,................n,

$\lambda^c{}_n$ the corresponding Lebesgue function

we have

$|| \lambda^c{}_n || < 2/\pi\,(l_n\, n) + 4$

Runge's Example wit hChebyshev points :-

If we solve Runge's example using chebyshev points & we calculate decay exponent which are as follows

N	Max. error	Decay exponent
2	0.9259+00	0.00
4	0.7503+00	-0.30
6	0.5559+00	-0.74
8	0.3917+00	-1.22
10	0.2692+00	-1.68
12	0.1828+00	-2.12

14	0.1234+00	-2.55
16	0.8311-01	-2.96
18	0.5591-01	-3.37
20	0.3759-01	-3.77

This is quite satisfactory.

2.2 Piecewise Linear Approximation

Though Piecewise Linear Approximation are not having the practical significance of cubic Spline or higher order approximation but it shows most of the essential features of piecewise polynomial approximation in a simple and easily understandable setting .

Broken Line Interpolation / Piecewise Linear Approximation

We consider Broken line interpolation to g at points $\tau_i \ldots\ldots \tau_n$ with interval [a, b]divided as ,

$a = \tau_1 < \ldots\ldots < \tau_n = b$ by $I_2\, g$

The interpolant is given by ,

$$I_2\, g\,(x) = g\,(\tau_i) + (x - \tau_i)\,[\tau_i\,,\,\tau_{i+1}]g \text{ --------------------------------i}$$

On $\tau_i \leq x \leq \tau_{i+1}$ $i = 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots n-1$

but

$$g(x) = g(\tau_i) + (x - \tau_i)\,[\tau_i\,,\,\tau_{i+1}]g + (x - \tau_i)\,(x - \tau_{i+1})\,[\tau_i\,,\,\tau_{i+1}\,,\,x]g \text{ --------------------ii}$$

For $\tau_i \leq x \leq \tau_{i+1}$ eqn. ii – i gives

$$g(x) - I_2\, g\,(x) = (x - \tau_i)\,(x - \tau_{i+1})\,[\tau_i\,,\,\tau_{i+1}\,,\,x]g$$

then error estimated is,

$$|\,g(x) - I_2\, g\,(x)\,| \leq \left(\frac{\partial \tau_i}{2}\right)^2 \max \frac{|\,g''(\xi)\,|}{2}$$

$$\tau_i \le \xi \le \tau_{i+1}$$

where $\partial \tau_i = \tau_{i+1} - \tau_I$

g has two continuous derivatives with $|\ \tau\ | = \max_i \Delta \tau i$

$$||\ g - I_2 g\ || \le \frac{1}{8}\ |\ \tau\ |^2\ ||\ g''\ ||$$

making $\partial \tau_i$ small for all i we can minimize this error bound as small as we like . here we can increase the no. of parameters needed to describe the approximation function $I_2\ g = f$ without increasing complexity of f locally. Since f is always straight line.

2.3 Broken line Interpolation is nearly optimal

Let $\$_2$ = linear space of all continuous broken lines on $[\tau_1\ ,\ \tau_n]$ with breaks at $\tau_2\ ,\ \tau_{n-1}$ (which are splines of order 2 . This concept is introduced later)

We consider that ,

$I_2\ f = f \qquad \forall\ f\ \$_2$ ---------------------------------(1)

On each interval $[\ \tau_i\ ,\ \tau_{i+1}\ ]$, I_2 f agrees with the straight line which interpolates f at τ_i & τ_{i+1} but if f $\in\ \$_2$

The f itself is a straight line on $[\ \tau_i\ ,\ \tau_{i+1}\ ]$

Therefore $I_2(f) = f$ on $[\ \tau_i\ ,\ \tau_{i+1}\ ]$ uniqueness property of polynomial interpolation.

We observe that ,

$$\| I_2 g \| = \max_i | I_2(g)(\tau_i) |$$

$$= \max | g(\tau_i) | \leq \| g \|$$

therefore

$$\| I_2 g \| \leq \| g \| \quad \forall g \in \tau [a, b] \text{ ------------------------(2)}$$

combining (1) and (2) appropriately we get ,

$$\| g-I_2 g \| = \| (g - f) - I_2 (g-f) \| \leq \| (g - f) \| + \| (g - f) \| \quad \forall f \in \$_2$$

Here we minimize the right side of the inequality over all f $\in \$_2$ so we get the second inequality in

$$\text{dist}(g, \$_2) \leq \| g-I_2 g \| \leq 2 \text{ dist}(g, \$_2) \text{------------------------------(3)}$$

this shows that we get the possible approximation to g by broken lines.

2.4 Least squares approximation by broken lines or L_2 Approximation by broken lines.

For this we need a convenient basis for $\$_2$

Let $\tau_0 = \tau_1$; $\tau_{n+1} = \tau_n$ and set

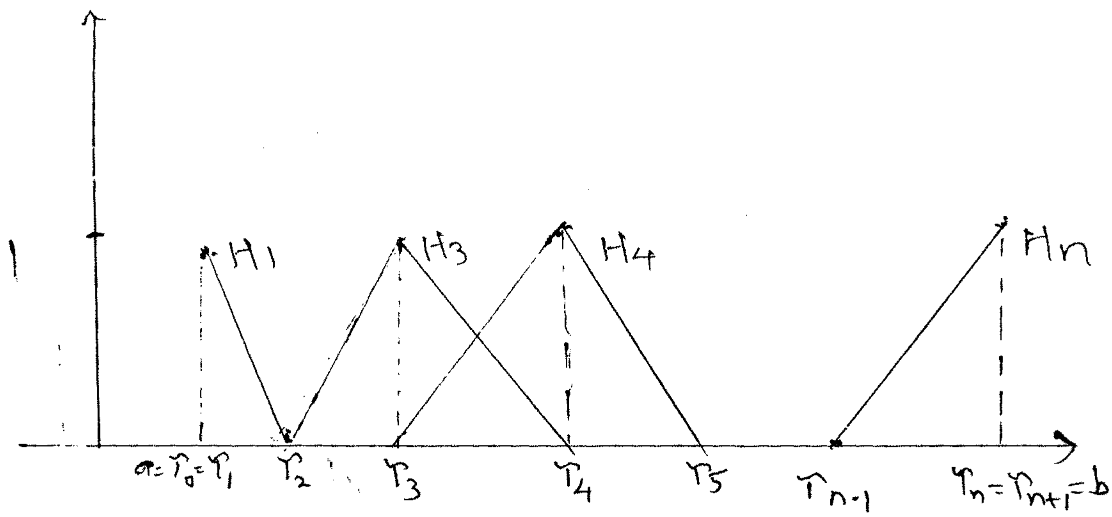
$$Hi(x) = \frac{(x = \tau_{i-1})}{\tau_i - \tau_{i-1}} \qquad \tau_{i-1} < x \leq \tau_i$$

$$= \frac{\tau_{i-1} - x}{\tau_{i+1} - \tau_i} \qquad \tau_{i-} < x \le \tau_{i+1}$$

$= 0$ otherwise -----------------

----(4)

fig :-



These basis functions for $\$_2$ have been called hat functions or Chapeau functions

$H_i \in \$_2$ all i

$H_i(\tau_i) = \delta_{ij} = 1 \quad i = j$

$\forall$ i, j

$$= 0 \quad i \neq j$$

This shows that $\sum_{i=1}^{n} g(\tau_i) H_i \in \$_2$

which agrees with g at $\tau_1, \tau_2 \ldots \ldots \tau_n$

by (3) $[I_2 f = f]$

$I_2 g$ is the only element of $\$_2$.

Which gives ,

$$I_2(g) = \sum_{i=1}^{n} g(\tau_i) H_i \quad \text{the Lagrange form}$$

for broken line interpolant . It also implies that $(H_i)_1^n$ is a basis for $\$_2$.

every line on $[\tau_1, \tau_n]$ with breaks at $\tau_2 \ldots \ldots \tau_{n-1}$ can written in exactly one way as a linear combination of the H_i's

The ordinates of given $f \in \$_2$ considering the basis $(H_i)_1^n$ consists simply of its values $f(\tau_1)$ ----------------$f(\tau_n)$ at the breakpoints i.e.

$$f = \sum_{i=1}^{n} f(\tau_i) H_i \qquad \forall f \in \$_2 \text{----------------} (6)$$

$L_2 g$ be Least – squares approximation to g in $\$_2$

i.c.

$$\int |g(x) - L_2 g(x)|^2 dx = \min_{f \in \$_2} \int |g(x) - f(x)|^2 dx$$

and $L_2 g \in \$_2$. we determine $L_2 g$ using the normal eqns. i.e. we find a minimum of

$$\int \left| g(x) - \sum_{j=1}^{n} \alpha_j H_i(x) \right|^2 dx$$

by setting its first partial derivative with respect to $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots \alpha_n$ to zero , this gives the Linear system

$$\sum_{j=1}^{n} \int [H_i(x) H_j(x) dx] \alpha_j\hat{} = \int H_i(x) g(x) dx \qquad i = 1,2, \ldots\ldots\ldots\ldots n$$

for the coefficient $(\alpha_i\hat{})_1^n$ of L_2 broken line approximation to g

$$L_2 g = \sum_{j=1}^{n} \alpha_j\hat{} H_j$$

The intervals involved are easily evaluated we get more explicitly the Linear system ,

$$\left(\frac{\partial \tau_{i-1}}{6}\right) \alpha\hat{}_{i-1} + \frac{(\tau_{i+1} - \tau_{i-1})}{3} \alpha_i\hat{} + \left(\frac{\partial \tau_i}{6}\right) \alpha\hat{}_{i+1} = \beta_i \qquad \text{--------------------------(7)}$$

$$= \int H_i(x)\, g(x)\, dx \qquad i = 1,2,\ldots\ldots\ldots n$$

The coefficient Matrix is tridiagonal and strictly diagonally dominant . so the system has exactly one solution which can be obtain by Gauss Elimination Method .

Theorem :- The L_2 approximation $L_2\, g$ to $g \in C[a, b]$ i.e. to a continuous function g on [a, b] by elements of $\$_2$ satisfies $\| L_2\, g \| \leq 3 \| g \|$.

Hence , since L_2 is additive and $L_2\, f = f$ for all $f \in \$_2$,

We have , $| g - L_2\, g \| \leq 4 \| g, \$_2 \|$.