Chapter – 4

Chapter –4

Cardinal spline analysis

The B-splines originally introduced by Curry & Schoenberg in 1947.

In this chapter we define B-splines of order 1 & m,m $\geq$ 2 and we study derivations of expressions of B-splines . In the last part Cardinal Splines analysis & properties of

B-spline are expressed.

4.1 **Def^n**: The first order B-Spline,denote by $N_1(x)$ is defined as the characteristic function of the interval [0,1] j.e.

$$N_1(x) = x_{[0,1]} \begin{cases} =1 & \text{if } 0 \leq x \leq 1 \\ =0 & 0 \text{ otherwise} \end{cases} \quad \text{--------- I}$$

Def^n : For m$\geq$2,the mth order B-spline $N_m(x)$ is recursively defined by

$$N_m(x)= \int N_{m-1}(x) * N_1(x)= {}_{-\infty}\int^{\infty} N_{m-1}(x-t)N_1(t)dt \text{ -----------II}$$

Thus we have,

$$N_m(x)= {}_{-\infty}\int^{\infty} N_{m-1}(x-t)N_1(t)dt= {}_0\int^1 N_{m-1}(x-t)dt \text{ -----------------III}$$

4.2 **Derivation of expression for B-splines.**

We will derive explicit expressions for the first few B-Splines below,

1. Expressions for $N_2(x)$

We have, By the defination III

$N_2(x)= {}_0\int^1 N_1(x-t)dt= {}_{x-1}\int^x N_1(t)dt$ ------------------------------------I

Case 1) $-\infty \le x \le 0 \quad \Rightarrow \quad -\infty < t \le x \le 0$

$N_1(t) = 0$

$N_2(x) = 0$

Case2) $0 \le x \le 1 \quad \Rightarrow \quad -1 \le t \le x \le 1$

$N_2(x) = {}_{-1}\int^0 N_1(t)dt + {}_0\int^x N_1(t)\, dt$

$= 0 + {}_0\int^x 1.dt = t \,|^x{}_0 = x$ --------------------------II

Case3) $1 \le x \le 2 \quad \Rightarrow \quad x-1 \le t \le 1 \le x \le 2$

$N_2(x)= {}_{x-1}\int^1 N_1(t)dt$

$= {}_{x-1}\int^1 dt = t \;|^1{}_{x-1} = 2-x$ ----------------------------III

Case4) $2 \le x < \infty \quad \Rightarrow \quad 1 \le t \le x \le \infty$

$N_2(x)=0$ as $N_1(t)=0$ for $a \le t < \infty$ --------------------IV

Thus, We have

$$N_2(x) = \begin{cases} x, & \text{if } 0 \le x \le 1 \\ 2-x, & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$ ----------------------V

2) Expression for $N_3(x)$

We have by definition

$N_3(x)= {}_0\int^1 N_2(x-t)dt = {}_{x-1}\int^x N_2(t)dt$, $0 \leq t \leq 2$ --------------------I

Case1) $\{-\infty < x \leq 0\}$ then

$N_3(x) = {}_{-\infty}\int^x N_2(t)\, dt = 0$ -------------------------------------II

Case2) $\{0 \leq x \leq 1\}$ $\Rightarrow$ $-1 \leq t \leq x$ thus

$N_3(x) = {}_0\int^x N_2(t)dt = {}_0\int^x t.\, dt = t^2/2\ |^x{}_0 = 1/2\ x^2$ ------------III

Case 3) $\{1 \leq x \leq 2\}$ $\Rightarrow$ $x-1 \leq t \leq 1 \leq x \leq 2$

$N_3(x) = {}_{x-1}\int^1 N_2(t)dt + {}_1\int^x N_2(t)dt$

$= {}_{x-1}\int^1 t\, dt + {}_1\int^x (2t)\, dt$

$= (t^2/2)|^1{}_{x-1} + (2t - t^2/2)\ |^x{}_1$

$= ½ - ½(x^2-2x+1) + 2x-2-x^2/2 + ½$

$= -x^2+3x-3/2$ ---IV

Case4) $\{2 \leq x \leq 3\}$ $\Rightarrow$ $x-1 \leq t \leq x \leq 3$

$N_3(x) = {}_{x-1}\int^2 N_2(t)dt + {}_2\int^x N_2(t)\, dt$

$= {}_{x-1}\int^2 (2-t)dt + 0$ $\qquad$ $N_2(t)=0$ for $t>=2$

$=(2t-t^2/2)\ |^2{}_{x-1}$

$=(4-2)-2(x-1)+(x-1)^2/2$

$=2-2x+2+1/2(x^2-2x+1)$

$=x^2/2-3x+9/2$ ---V

Case 5) $\{x \geq 3\}$ $\Rightarrow$ $t \geq x-1 \geq 2$ $\Rightarrow$ $N_2(t) = 0$ $\Rightarrow$ $N_3(x)= 0$

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Thus we have

$$\left.\begin{aligned} N_3(x) &= \tfrac{1}{2} x^2, & 0 \leq x \leq 1 \\ &= -x^2+3x-3/2, & 1 \leq x \leq 2 \end{aligned}\right\} \text{ ---------------VII}$$

$$\begin{aligned} &=1/2x^2-3x+9/2, & 2 \leq x \leq 3 \\ &= 0, & \text{otherwise} \end{aligned}$$

i.e. $N_3(x) = 0$ if $x < 0$

$N_3(x) \neq 0$ if $x \leq x \leq 3$

$N_3(x) = 0$ if $x > 3$

3) Expression for $N_4(x)$

We have by definition

$$N_4(x) = {}_0\int^1 N_3(x-t)\, dt = {}_{x-1}\int^x N_3(t)dt \text{ ------------------------------I}$$

Case 1) $\{-\infty < x \leq 0\}$ $\Rightarrow$ $\{ -\infty < t < 0 \}$ $\Rightarrow$ $N_3(t) = 0$

$N_4(x)=0 \quad \text{if} \quad -\infty < x \le 0$ ----------------------------------II

Case2) $\{0 \le x \le 1\} \Longrightarrow x-1 < 0 \le t \le x \le 1$

$N_4(x) = {}_0\int^x (t^2/2)dt = (t^3/6)\,|^x{}_o = x^3/6$ -------------------------III

Case3) $\{1 \le x \le 2\} \Longrightarrow 0 \le x-1 \le t \le x \le 2$

$N_4(x) = {}_{x-1}\int^1 N_3(t)dt + {}_1\int^x N_3(t)dt$

$= {}_{x-1}\int^1 (t^2/2)dt + {}_1\int^x (-t^2 + 3t - 3/2)\, dt$

$= t^3/6|^1{}_{(x-1)} + \{-t^3/3 + 3\, t^2/2 - 3/2t\}\, |^x{}_1$

$= 1/6 - 1/6(x^3-3x^2+3x-1) + (-x^3/3 + 3\, x^2/2 - 3/2x) + 1/3$

$= -1/2x^3 + 2x^2 - 2x + 2/3$ ----------------------------------IV

Case4) $\{2 \le x \le 3\} \Longrightarrow 1 \le x-1 \le t \le x \le 3$

$N_4(x) = {}_{x-1}\int^2 N_3(t)dt + {}_2\int^x N_3(t)dt$

$= {}_{x-1}\int^2 (-t^2+3t-3/2)dt + {}_2\int^x (t^2/2 - 3t + 9/2)dt$

$= (-t^3/3 + 3t^2/2 - 3/2t)|^2{}_{x-1} + (t^3/6 - 3t^2/2 + 9/2t)|^x{}_2$

$= (-8/3+6\,^{-3}) - \{(x-1)^3/3 + 3(x-1)^2/2 - 3/2(x-1)\} + x^3/6 - 3x^2/2 + 9x/2 - (4/3-6+9)$

$= 1/3 - \{-1/3(x^3-3x^2+3x-1) + 3/2(x^2-2x+1) - 3/2x + 3/2\} + x^3/6 -$

$3x^2/2 + 9x/2 - 13/3$

$= x^3/2 - 4x^2 + 10x - 22/3$ -------------------------------------V

$= x^3/2 - 4x^2 + 10x - 22/3$

Case5) $\{3 \le x \le 4\} \Longrightarrow 2 \le x\text{-}1 < t \le x \le 4$

$N_4(x) = {}_{x-1}\int^3 N_3(xt)dt + {}_3\int^x N_3(t)dt$

$= {}_{x-1}\int^3 (t^2/2 - 3t + 9/2)\, dt$

$N_4(t) = {}_{x-1}\int^3 (t^2/2 - 3t + 9/2)dt$

$=(t^3/6-3t^2/2+9t/2)|^3_{x-1}$

$=(9/2-27/2+27/2)-\{1/6(x-1)^3-3/2(x-1)^2+9/2(x-1)\}$

$=9/2-\{1/6(x^3-3x^2+3x-1)-3/2(x^2-2x+1)+9/2x-9/2\}$

$=9/2-\{x^3/6-1/2x^2+1/2x-1/6-3x^2/2+3x-3/2+9x/2-9/2\}$

$=-x^3/6+2x^2-8x+32/3$ --VI

Case6) $\{4 \le x < \infty\} \Longrightarrow 3 \le x\text{-}1 < t < \infty \Longrightarrow N_3(t)= 0$

$N_4(x) = {}_{x-1}\int^x N_3(t)dt \le {}_3\int^\infty N_3(t)dt = 0$

$N_4(x) = 0$ if $x \ge 4$

Thus we have,

$N_4(x) = x^3/6$ if $0 \le x \le 1$

$= -1/2x^3+2x^2-2x+2/3$ if $1<=x<=2$

$=1/2x^3-4x^2+10x-22/3$ if $2<=x<=3$ ----VII

$=-1/6x^3+2x^2-8x+32/3$ if $3<=x<=4$

$= 0$ otherwise

4.3 Cardinal Spline Analysis :-

Cardinal splines are probably the simplest functions with small supports that are most efficient for both software and hardware implementation. These are used in basic Wavelets and approximations.

We use following notations :

p_m :- Denotes the collection of all algebraic polynomials of degree at most n.

$C^n(R)$:- Collection of all functions f such that $f, f^1, \ldots, f^{(n)}$ are continuous everywhere on R.

Def^n :- For each +ve integer m, the space s_m of cardinal splines of order m and with Knot sequence z is the collection of all functions $f \in c^{m-2}$ such that the restrictions of f to any interval $[k,k+1), k \in z$ are in P_{m-1} j.e. $f|_{[k,k+1)} \in P_{m-1}$, $k \in z$

S1 : The space of piecewise constant functions.

The basis for S_1 can be $\{ N_1(x-k): k \in z\}$ where N_1 is the characteristics function of $[0,1)$

$$\text{j.e. } N_1(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{------------------------------------I}$$

$S_m(m \geq 2)$: To get the basis for S_m first consider $S_{m,N}$ consisting of restrictions of $f \in S_m$ to the interval

[-N,N],where N is a +ve integer. Thus, $S_{m,N}$ is a subspace of functions of $f \in S_m$ such that the restrictions of f to

$(-\infty,-N+1)$ and $[N-1, \infty)$ of f are polynomial in π_{m-1}.

To characterize the space $S_{m,N}$. let $f \in S_{m,N}$ be arbitrary function. Let

$P_{m,j}=f|_{[j,j+1)} \in \pi_{m-1}$

For j= -N, -N+1,......,N-1 then since $f \in C^{m-2}$, we have

$$\left[P_{m,j}^{(k)} - P_{m,j-1}^{(k)} \right] (j) = 0 \quad k=0,1,2,\ldots,m-2 \ ; \ m\geq 2$$

The jumps C_j of $f^{(m-1)}$ at the knot sequence Z are then given by

$$C_j = P_{m,j}^{(m-1)}(j+0) - P_{m,j-1}^{(m-1)}(j-0)$$

$$= \lim_{\epsilon \to 0+} \left\{ f^{(m-1)}(j+\epsilon) \ - f^{m-1}(j-\epsilon) \right\} \quad \text{-------------------------------------II}$$

The adjacent polynomial pieces of f are related by

$$P_{m,j}(x) = P_{m,j-1}(x) + C_j/(m-1)! \ (x-j)_+^{m-1} \quad \text{--------------------------------III}$$

Let $\quad x_+ = \max(0,x)$

$$x_+^{m-1} = (x_+)^{m-1}, \ m\geq 2 \quad \text{---IV}$$

Thus we have

$$f(x)= f_{[-N,-N+1)}(x) + \sum_{j=-N+1}^{N-1} C_j/(m-1)!\ (x-j)_+^{m-1} \text{ --------------------V}$$

Eq^n (V) is true for all $f \in S_{m,N}$ with constants Cj given by Eq^n (II) .Therefore ,the collection

$\{1,x,\ldots,x^{m-1},(x+N-1)_+^{m-1},\ldots,(x-N+1)_+^{m-1}\}$-----------------------VI

of m+2N-1 functions is a basis for $S_{m,N}$.

The basis (VI) consist of monomials and truncated powers.We can replace monomials by truncated powers as

$(x+N+m-1)_+^{m-1},\ldots,(x-N+1)_+^{m-1}$ -------------------------------VII

Eqn (VII) can be rewritten as

$T=\{(x-k)_+^{m-1}, k= -N-m+1,\ldots,N-1\}$ -------------------------------VIII

Which are generated by integer translates of a single function x+m-1, as a basis of Sm.n

Basis (VIII) are more attractive than (VI). For

1. Each function (x-j)+ m-1 vanishes to the left of j
2. they are generated by a single function x+m-1 which is independent of N
3. Sin $S_m = \bigcup_{N=1}^{\infty} S_{m,N} \Longrightarrow \Gamma=\{(x-k)_+^{m-1} : k \in Z\}$ -------------IX

   Is a basis for S_m.

However, we must careful when we deal with infinite dimensional spaces. Since we are interested in $L^2(R)$,We want cardinal splines that are in $L^2(R)$.Unfortunately not a single function in Γ is in $L^2(R)$ as each $(x-k)_+^{m-1}$ grows to infinity as $x \to \infty$.we therefore have to create functions in $L^2(R)$ from those in Γ_N ,which can be denoted by controlling their growth. Since only linear combinations are allowed. We can use difference for this purpose Thus let

$$\Delta f(x) = f(x) - f(x-1)$$

$$\Delta^n f(x) = \Delta^{n-1}(\Delta f(x)) \text{ -- X}$$

When f(x) is a polynomial of degree m-1 or less

We have,

$$\Delta^m f = 0 \ , \ f \in \pi_{m-1} \text{ -- XI}$$

Def^n : Let $m_1=N_1$ be the characteristrics function of [0,1) and for m>=2 let

$$M_m(x) = 1/(m-1)! \ \Delta^m x_+^{m-1} \text{ -- XII}$$

Then we have

$$M_m(x) = 1/(m-1)! \sum_{K=0}^{m} (-1)^k \binom{m}{k} (x-k)_+^{m-1} \text{ -------------------------------- XIII}$$

j.e. $M_m(x)$ is a linear combination of functions in Γ

Now $M_m(x) = 0$ for $x<0$ and $M_m(x)=0$ for $x>=m$ .therefore supp $M_m \subset [0,m]$

In fact supp $M_m=[0,m]$

Since M_m has compact support, $M_m(x) \in L^2(R)$

We now show that $B = \{M_m(x-k) : k \in Z\}$ is a basis for S_m

For, consider $S_{m,N}$. the dimension of $S_{m,N}$ is m+2N-1.using the support properly (j.e. supp $M_m(x)=[0,m]$)

We see that each function in the collection

$\{M_m(x-k) : k = -N-m+1,\ldots N-1\}$ --XIV

is non-trivial on the interval [-N,N].also $M_m(x-k)$ vanishes identically on [-N,N] for $k < -N-m+1$ or $k>N-1$. since the set in (XIV) is linearly independent they form a basis for $S_{m,N}$ thus we have an alternative set of basis for $S_{m,n}$.if we take the union of basis in (XIV) for N=1,2,... we get B as the basis for S_m .the advantage of B over Γ is that we can now talk about a spline series.

$$F(x)= \sum_{K=-\infty}^{\infty} C_k M_m(x-k) \text{ ---XV}$$

Without worrying too much about convergence.

Indeed for each fixed $X \in R$ since M_m has compact support, all except a finite number of terms in (XV) are zero.

We are interested only in those cardinal spline that belong to $L^2(R)$,namely $S_m \cap L^2(R)$

Let $V^m{}_0$Denote its closure in $L^2(R)$

j.e. $S_m \cap L^2 (R) = V^m{}_0$ Since M_m has compact Support, we see that B c $V^m{}_0$ In fact B is a Riesz basis of $V^m{}_0$

The cardinal Spline we have considered so far have the Knot sequence Z.if we consider the Knot sequence 2^{-j} Z,then the corresponding space of spline functions is $S^j{}_m$ denoted by $S^j{}_m$.Since for $j_1 < j_2$ we have $z^{-j1}Z$ c $Z^{-j2}Z$, we have $S^{j1}{}_m$ c $S^{j2}{}_m$ thus we have a doubly infinite nested sequence c $S^{-1}{}_m$ c $S^0{}_m$ c $S^1{}_m$ c ...of cardinal spline spaces where $S^0{}_m = S_m$. Analogous to the def^n of $V^m{}_0$, we let $V^m{}_j$ denote the $L^2(R)$- closure of $S^j{}_m \cap L^2(R)$.hence we have a nested sequence

$$\ldots \subset V^m{}_{-1} \subset V^m{}_0 \subset V^m{}_1 \subset V^m{}_2 \subset \ldots \qquad \text{-----XVI}$$

of closed cardinal spline subspaces of L2(R) Then we have

$$\left.\begin{aligned} &\operatorname{clos}_{L^2(R)}\Big[\bigcup_{j \in z} V^m{}_j\Big] = L^2(R) \\ &\bigcap_{j \in z} V^m{}_j = \{0\} \end{aligned}\right\} \qquad \text{-----XVII}$$

Also if B is a Riesz basis of $V^m{}_0$, then for any $j \in z$ the collection

$$\{\, 2^{j/2} M_m(2^j x-k) : k \in z \} \qquad \text{-----XVIII}$$

is also Riesz basis of $V^m j$ with the same Riesz bounds.

4.4 Cardinal B-splines and their properties :-

The mth order B-spline Nm(x) is defined by

$$N_m(x) = (N_{m-1} * N_1)(x) = {}_0\int^1 N_{m-1}(x-t)dt, m \geq 2 \quad \text{-------------------------XIX}$$

Where N_1 (x) is the characteristics function of the interval [0,1).Also if we set $M_1=N_1$, then we get $M_m(x)=N_m(x)$;m>=2 we will now prove some of the properties of $M_m(x)=N_m(x)$.

Th^m: The m^{th} order cardinal B-Spline $N_m(x)(=M_m(x))$ satisfies the following properties,

1. for every f ϵ c,

$$_{-\infty}\int^{\infty} f(x)\, Nm(x)dx = {}_0\int^1 \ldots {}_0\int^1 f(x_1+x_2+\ldots+x_m)dx_1dx_2 \ldots dx_m \quad \text{----XX}$$

2. for every g ϵ c^m

$$_{-\infty}\int^{\infty} g^m(x)N_m(x)dx = \sum_{k=0}^{m} (-1)^{m-k}\begin{bmatrix} m \\ k \end{bmatrix} g(k) \quad \text{---------------------------XXI}$$

3. $N_m(x)=M_m(x)$ for all x ϵ R
4. Supp $N_m=[0,m]$
5. $N_m(x) \geq 0$ for $0 < x < m$

6. $\sum_{k=-\infty}^{\infty} N_m(x-k) = 1$ for all x

7. $N_m(x) = (\Delta N_{m-1})(x) = N_{m-1}(x) - N_{m-1}(x-1)$

8. $N_m(x) = x/m-1\ N_{m-1}(x) + m-x/m-1\ N_{m-1}(x-1)$ ------------------XXII

9. N_m is symmetric w.r.t. the center of its support

   j.e. $N_m(m/2+x) = N_m(m/2-x)$, $x \in R$

Proof :

1. for m = 1, we have

$$\int_{-\infty}^{\infty} f(x)N_1(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 f(x_1)dx_1$$

so that (XX) is true for m=1 Assuming it to be true for m-1 , we have

$$\int_{-\infty}^{\infty} f(x)N_m(x)dx = \int_{-\infty}^{\infty} f(x)\,[\int_0^1 N_{m-1}(x-t)dt]\,dx$$

$$= \int_0^1 \{\int_{-\infty}^{\infty} f(x)N_{m-1}(x-t)dx\}dt$$

$$= \int_0^1 \{\int_{-\infty}^{\infty} f(y+t)N_{m-1}(y)\,dy\}dt$$

$$= \int_0^1 \int_0^1 \ldots \int_0^1 f(x_1+x_2+\ldots+x_{m-1}+t)dx_1dx_2\ldots dx_{m-1}dt$$

$$= {}_0\int^1 \ldots {}_0\int^1 f(x_1+x_2+\ldots+x_m)dx_1dx_2\ldots dx_m$$

2. we have

$$_{-\infty}\int^{\infty} g^m(x)N_m(x)dx = {}_0\int^1 \ldots {}_0\int^1 g^m(x_1+x_2+\ldots+x_m)dx_1\ldots dx_m \quad \text{by 1}$$

By direct integration, we get

$${}_0\int^1 \ldots {}_0\int^1 g^m(x_1+\ldots+x_m)dx_1\ldots dx_m = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g(k)$$

3. fix x ε R let $g(x) = (-1)^m/(m-1)!\ (x-t)_+^{m-1}$

Differentiating m times , we get

$$g^m(t) = \delta\ (x-t)$$

$$N_m(x) = {}_{-\infty}\int^{\infty} g^m(x)\ N_m(x)dx = M_m(x)$$

4. Supp N_m=[0,m]

The assertion is clearly true for m=1 by def^n

Assuming Supp N_{m-1} = [0,m-1]

Then we have ,

$N_m(x) = {}_{-\infty}\int^{\infty} N_{m-1}(x-t)N_1(t)dt = {}_0\int^1 N_{m-1}(x-t)dt$

Since supp $N_{m-1}(x) = [0,m-1], N_{m-1}(x-t) \neq 0$ for $0 <= x-t <= m-1$

$\Rightarrow$ Let $x-t = y$ $-dt=dy$

$N_m(x) = -{}_x\int^{x-1} N_{m-1}(y)dy = {}_{x-1}\int^{x} N_{m-1}(y)dy$

i.e. $x-1 \leq y \leq x \Rightarrow N_{m-1}(y) \neq 0$

$x-1 \leq y \leq m-1 \leq x \Rightarrow N_{m-1}(y) \neq 0$

thus when $y=m-1$ x can be m

or $N_m(x) \neq 0$ for $0 \leq x \leq m$

5. $N_m(x) > 0$ for $0 < x < m$

we have $N_1(x) > 0$ for $0 < x < 1$

Assuming $N_{m-1}(x) > 0$ for $0 < x < m-1$

From IV above we see that

$N_m(x) > 0$ for $0 < x < m$

6. $\sum_{k=-\infty}^{\infty} N_m(x-k) = 1$ for all x

Proof: We have for m=1

$$\sum_{k=-\infty}^{\infty} N_1(x-k) = 1 \qquad \forall x$$

As there is exactly one interval [k, k+1] such that $N_1(x-k) \equiv 1$ if $x \in [k, k+1]$ and $N_1(x-k)=0$ for any fixed k. Assuming the result to be true for m-1 , we have

$$\sum_{k=-\infty}^{\infty} N_{m-1}(x-t-k) = 1$$

$$\sum_{k=-\infty}^{\infty} N_m(x-k) = \sum_{k=-\infty}^{\infty} \int_0^1 N_{m-1}(x-t-k)dt$$

$$= \int_0^1 \sum_{k=-\infty}^{\infty} N_{m-1}(x-t-k)dt = \int_0^1 1dt = 1 \qquad \text{for all x}$$

7. $N^1_m(x) = (\Delta N_{m-1})(x) = N_{m-1}(x) - N_{m-1}(x-1)$

$$N^1_m(x) = \int_0^1 N^1_{m-1}(x-t)dt = (x-t)\,|^1_0$$

$$= N_{m-1}(x) - N_{m-1}(x-1) = (\Delta N_{m-1})(x)$$

8. $N_m(x) = x/m-1\ N_{m-1}(x) + m-x/m-1\ N_{m-1}(x-1)$

from III we have

$$N_m(x) = M_m(x) = \sum_{K=0}^{m} (-1)^k/(m-1)!\ C^m_k\ (x-k)_+^{m-1}$$

Now we have $x_+^{m-1} = x.x_+^{m-2}$

$N_m(x) = M_m(x) = 1/(m-1)!\ \Delta^m\ x_+^{m-1}$

$= 1/(m-1)!\ \{ x\ \Delta^m x_+^{m-2} + m\Delta^{m-1}(x-1)_+^{m-2}\}$

$= 1/(m-1)!\ \{(m-2)!\ x\Delta\{1/(m-2)!\ \Delta^{m-1}(x-1)_+^{m-2}\}+m(m-$

$2)!(1/m-2)!\ \Delta^{m-1}(x-1)_+^{m-2})\}$

$= 1/(m-1)\{x\Delta M_{m-1}(x)+mM_{m-1}(x-1)\}$

$= x/m-1\ M_{m-1}(x) + m-x/m-1\ M_{m-1}(x-1)$

9. $N_m(x)$ is symmetric w.r.t. the center of its support j.e.

$N_m(m/2+x)=N_m(m/2-x) \qquad \forall\ x \in R$

Proof: we prove this result by induction

For m=1 we have

1. $N_1(1/2+x) = 1$ if $0 \leq 1/2+x \leq 1 \implies -1/2 \leq x \leq 1/2$

$= 0$ otherwise

2. $N_1(1/2-x) = 1$ if $0 \leq 1/2-x \leq 1 \implies -1/2 \leq x \leq 1/2$

$=0$ otherwise

From I & II we see that the symmetry property is true for

m=1

Now consider

$$N_{m-1}(m/2+x) = N_{m-1}\{m-1/2+(x+1/2)\}$$

$$=N_{m-1}\{m-1/2-(x+1/2)\}$$

$$=N_{m-1}(m/2-x-1) \text{ --------------------------I}$$

$$N_{m-1}(m/2-x) = N_{m-1}(m-1/2 - x + 1/2)$$

$$=N_{m-1}(m-1/2)-(x-1/2))$$

$$=N_{m-1}(m-1/2+(x-1/2))$$

$$=N_{m-1}(m/2+x-1) \text{ --------------------------II}$$

Now,

$$N_m(m/2+x) = (m/2+x)/(m-1) .N_{m-1}(m/2+x) + m-(m/2+x)/(m-1)N_{m-1}(m/2+x-1)$$

$$=m-(m/2-x)/m-1. N_{m-1}(m/2-x)-1)+(m/2-x/m-1)N_{m-1}(m/2-x)$$

$$=N_m(m/2-x)$$

Our next problem is to show that the cardinal B-spline basis

$B=\{N_m(x-k) : k \in Z\}$ ----------------------------I

Is a Riesz basis for $V^m{}_0$ in the sense that there exist constants A& B

with $0<A<=B<\infty$

Such that for any sequence $\{c_k\} \in l^2(z)$ we have

$$A\|\{c_k\}\|^2{}_{l^2} \leq \| \sum_{k=-\infty}^{\infty} c_k N_m(x-k)\|^2{}_{L^2(R)} \leq B\| \{c_k\}\|^2{}_{l^2} \text{ ---------II}$$

Condition II is equivalent to a frequency domain condition

$$A \leq \sum_{k=-\infty}^{\infty} |\hat{N}_m(\omega+2\pi k)|^2 \leq B \text{ a.e. ---------------------III}$$

We will work with frequency domain condition to ontain A&B

Replacing ω by 2x in III we have

$$A \leq \sum_{k=-\infty}^{\infty} |\hat{N}_m(2x+2\pi k)|^2 \leq B \text{ a.e.}$$

Since $N_m(x)$is an m-fold convolution of $\hat{N}_1(x)=1$ and

since $N_1(\omega)=(1-e^{-2\omega}/2\omega)$, we have

$$|\hat{N}_m(\omega)|^2 = |1-e^{-2\omega}/2\omega|^{2m} = \sin^{2m}(\omega/2)/(\omega/2)^{2m}$$

therefore,

$$\sum_{k=-\infty}^{\infty} |\hat{N}m(2x+2\pi k)|^2 = \sum_{k=-\infty}^{\infty} \sin^{2m}(x+\pi k)/(x+\pi k)^{2m}$$

$$=\sin^{2m}(x)\sum_{k=-\infty}^{\infty} 1/(x+\pi k)^{2m} \text{ -----------------IV}$$

Now for complex analysis ,we have

$$\text{Cot}(x) = \lim_{n\to\infty} \sum_{k=-n}^{n} 1/x+\pi k \text{ ------------------------------------V}$$

Differentiating V w.r.t. (2m-1) times we get

$$\sum_{k=-\infty}^{\infty} 1/(x+\pi k)^{2m} = -1/(2m-1)!\ d^{2m-1}/dx^{2m-1}(\cot x) \text{ ---------------VI}$$

Thus we get

$$\sum_{k=-\infty}^{\infty} |\hat{N}_m(2x+2\pi k)|^2 = -\sin^{2m}(x)/(2m-1)!\ d^{2m-1}/dx^{2m-1}(\cot x) \text{ --- VII}$$

Expression VII is the general expression for the sum on R.H.S.

For the Solution of Approximation problem we can use function from these Spline Approximation Spaces . As the Spaces are constructed for any x in R , we can approximate any given function 'f' defined on R by means of a member from S_m as $m \to \infty$ the approximation function f_m converges to f.