Chapter – 1

Chapter - 1

APPROXIMATION AND INTERPOLATION

1.1 Introduction

The subject of Approximation theory has been studied from last 130 years . It plays very important role in application to many branches of applied sciences and engineering.

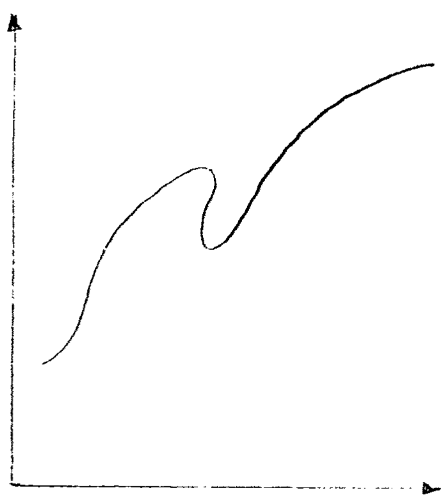
Approximation theory means representing an arbitrary functions in terms of other function which is simpler and nicer than the given. E.g. expansion of function is a power series , representation of the function in terms of polynomials . such a representation always gives us a simple way of obtaining information about the function which would be otherwise intractable. So instead of writing a difficult program to evaluate the function directly and getting an approximate answer in the end , we can use good polynomial approximation to the function obtain an even more accurate answer . Now question arises, what is good class of nicer and simpler functions?

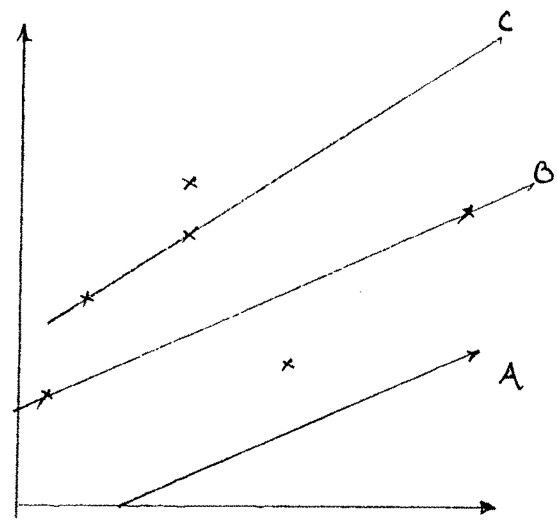
Here we use metric space X of functions to be approximated , a subset VCX of approximates , and the metric d on X to find how good the approximation is.one typical example of approximation theory:- Is V dense in X? i.e. can we approximate elements of X arbitrarily closely by those of V? if V not dense , how close will be $x \in X$ from V? If there exist $v_o \in V$ which is closed to x what special properties will have v_o .

i.e. How we can find explicitly a good , if not exist , the best approximation to x from V? These questions develops a theory of approximations.

Examples of approximation problems

Suppose we want approximate the following curve ① by a straight line





A function to be approximated

Fig①

data to be approximated

Fig 2

To approximate a curve in fig .1 by straight line means we require a straight line fit to the data in fig. 2

We will find that lines B and C are better than line A . This example shows that there are three main ingredients of an approximation and are ,

1 A function (some data) that is to be approximated .

Let it ' f ' (given curve)

2 A set $\mathcal{A}$ of approximations (in above examples a set of straight lines)

3 A means of selecting an approximation from $\mathcal{A}$.

Approximation problems of this type are

1) Estimatinng the solution of a differential equ. by a function of a certain simple form that depends on adjustable parameters.Measure of goodness of the approximation is a scalar quantity that is derived from the residual that occurs when the approximating function is substituted into the diff. equation.
2) The choice of components in electrical circuits . The function f may be the required response from the circuit and the rangre of available components gives a set $\mathcal{A}$

1.2 Approximation in a metric space

One of the property of Metric space is that it has a distance function .

Let β be a metric space d(x,y) is a real valued function , defined for all pairs of points (x,y) in β having property.

1) if $x \neq y$ d(x,y) is positive , d(x,y) = d(y, x) •
2) if x = y the d (x,y)=0 •
3) $d(x,y) \leq d(x,z) + d(z,y)$ must satisfied by x, y, z, ∈ β ⇐ the triangle inequality.

Let there exist metric space which contains f and $\mathcal{A}$ (set of approximations)

Then $a_o \in \mathcal{A}$ is a better approximation than $a_1 \in \mathcal{A}$ if the inequality

$d(a_o, f) < d(a, f)$ is satisfied,

Defn:- We define $a^* \in \mathcal{A}$ to be a best approximation if the condition

$d(a^*, f) < d(a, f)$ holds $\forall\, a \in \mathcal{A}$

Theorem 1:- $\mathcal{A}$ be a compact set in a metric space β , then , for every f in β $\exists$ an element $a^* \in \mathcal{A}$ such that ,

$$d(a^*, f) \leq d(a, f) \text{ holds for all } a \in \mathcal{A}$$

i.e. if $\mathcal{A}$ is a copact set then $\exists$ best approximation for f.

1.3 Approximation in a Normed Linear Space

The Norm is a real valued function $\|x\|$ i.e. defined for $x \in \beta$ (where β is Normed Linear Space) having properties ,

$$d(x, y) = \|x - y\|$$

which have following properties

$$d(x, y) = ||x - y||,$$

$||x + y|| \leq ||x|| + ||y||$ put $z = 0$ in triangle in equation

homogeneity

$||\lambda x|| = ||\lambda||\ ||x||$ condition $\forall\ x \in \beta$ for all scalars λ

Result:- If $\mathcal{A}$ is a finite – dimensional linear space in a normed linear space β then for every $f \in \beta$ there exists an element of $\mathcal{A}$ that is a best approximation from $\mathcal{A}$ to f.

Proof :- Let $\mathcal{A}_0$ C $\mathcal{A}$ containing elements of A which satisfies condition

$$||a|| \leq 2\ ||f|| \text{ -------------------------- } 1$$

$\mathcal{A}_0$ is compact since is a closed and bounced subset of a finite dimensional space

$\mathcal{A} \neq \{\phi\}$ [zero is an element of $\mathcal{A}$]

so by theorem 1 there is a best approximation from $\mathcal{A}_0$ to f.

let it be a_0* by definition

$$||a - f|| \geq ||a_0^* - f|| \text{ holds } \quad \forall \quad a \in A_0$$

again if the element 'a' is in $\mathcal{A}$ and $\mathcal{A}_0$ then because condition 1 is not obtained we have the bound

$$|| a-f|| \geq || a || - || f||$$

$$> || f||$$

$$|| a-f|| \geq || a_0^*-f|| \qquad \text{for } \forall\, a \in A$$

which proves that , a_0* is a best approximation.

1.4 The L_p – Spaces (Norms)

In most approximation problems we consider A , f , ∈, **C**[a, b],(set of continuous real valued functions defined on the interval [a,b] of the real line sometimes we consider A , f , ∈, R^m (set of real m component vectors). τ [a, b], R^m are linear .

L_p - norms P= 1,2,------------∞ for finite P. The L_p - norm in **C**[a, b] is defined to have the value

$$|| f||_p = [\, {}_a\!\int^b | f(x)|^p dx\,]^{1/p} \qquad 1 \leq p < \infty$$

in R^m L_p - norm has the value

$$|| f||_p = [\, \sum_{i=1}^{m} | y_i |^p dx\,]^{1/p} \qquad 1 \leq p < \infty$$

where {y_i ; I=1,2,............n} are the components of f and

$$|| f||_\infty = \underset{a \leq x \leq b}{M_{ax}} | f(x) |$$

and

$$|| f||_\infty = \underset{1 \leq i \leq m}{M_{ax}} | y_i |$$

The 2 – norm , or a weighted 2-norm of the form

$$\| f \|_2 = [{}_a\int^b w(x) | f(x) |^2 dx]^{1/2}$$

where 'w' is a fixed positive function

1- Norm is least used and useful for fitting to discrete data in the case when there are some gross errors in the data due to blunders.

2- Norm is used for data fitting when the errors in the data have normal distribution . In ₰, Linear space the calculation of best approximation in the 2-norm reduces to a system of linear equations gives highly efficient algorithms so in 2-Norm best approximation calculation is straight forward to solve . ∞ - norm provides the foundation of much of appro. theory for computer calculations in complicated mathematical functions , approximations having small errors we use ∞ - norm.

∞ - norm is called the uniform on minimax norm and 2-norm is sometimes called

the least squares or Eucliclean norm

1.5 Uniqueness of best Approximation

Def^n. :- Ball of radius r , center at f is defined to be the set

$$N(f,r) = \{ g ; \| g - f \| \leq r , g \in \beta \}$$

Def^n. :- Convex set

The set ϒ of a linear space is convex if , for all s_o and s_1 in ϒ, the

points

$(\theta s_0 + (1-\theta) s_1 ; 0 < \theta < 1)$ are also in $\mathcal{f}$.

Def^n :- Strictly convex set

The set $\mathcal{f}$ of a linear space is convex if for all s_o and s_1 in $\mathcal{f}$ ($s_o \neq s_1$)the points $\{\theta s_0 + (1-\theta) s_1 ; 0 < \theta < 1\}$ are interior points of $\mathcal{f}$. So boundary of a strictly convex set will not contain segment of straight line .

We use these ideas for the Uniqueness of the best approximation .

Theorem :- Let β be a normed linear space then for any $f \in \beta$ and for any r, $r > 0$ the ball

$$N(f, r) = \{ x : ||x - f|| \leq r , x \in \beta \} \text{ is convex.}$$

Proof :- x_0 , $x_1 \in N(f, r)$ then

$$||\theta x_0 + (1-\theta) x_1 - f|| \leq ||\theta x_0 - \theta f|| + ||(1-\theta) x_1 - (1-\theta) f||$$

$$= |\theta| \, ||x_0 - f|| + |1-\theta| \, ||x_1 - f||$$

$$= r \, \{ |\theta| + |1-\theta| \} \qquad x_0, x_1, \in N f r$$

$$= r \qquad 0 < \theta < 1$$

which is convexity condition .

Theorem :- Let $\mathcal{A}$ be a convex set in a normed Linear space β and let f be any point of β

such that there exists a best approximation from $\mathcal{A}$ to f. Then the set of best approximation is convex.

Proof : - Let h^* = error of the best approximation

$$h^* = \min_{a \in \mathcal{A}} ||a - f||$$

The set of best approximation is the intersection of $\mathcal{A}$ and the ball $N(f, h^*)$

Hence theorem . since intersection of two convex sets is convex .

Theorem :- Let $\mathcal{A}$ be a compact and strictly convex set in a normed Linear space β.

Then for all $f \in \beta$ there is just one best approximation from $\mathcal{A}$ to f .

Theorem :- Let $\mathcal{A}$ be a convex set in a normed Linear space β whose norm is strictly convex .Then for all $f \in \beta$ there is at most one best approximation from $\mathcal{A}$ to f .

1.6 Interpolation

The general problem of interpolation consist of representing a function with the aid of given Values which this function takes for definite values of the independent variable.

Suppose $y = f(x)$ be a function which takes values $y_1, y_2, y_3, \ldots\ldots, y_n$ for values $x_1, x_2, x_3, \ldots\ldots, x_n$ of the independent variable x, and further suppose that $\phi(x)$ represents an arbitrary function constructed in a way such that it takes the values of $f(x)$ for the values of. $x_1, x_2, x_3, \ldots\ldots, x_n$ Then $f(x)$ is replaced by $\phi(x)$ over a given interval. The process constitutes interpolation and the function $\phi(x)$ is a formula of interposition or smoothing function.

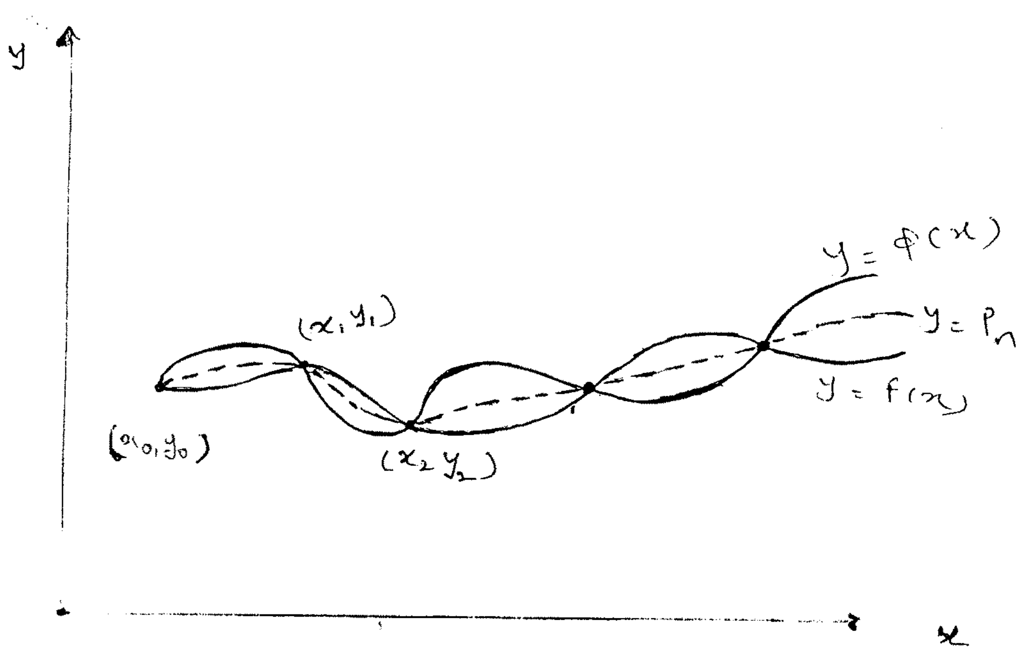
The function $\phi(x)$ can take a variety of forms. When $\phi(x)$ is a polynomial, the process of

Representing $f(x)$ by $\phi(x)$ is called parabolic or polynomial interpolation. When $\phi(x)$ is a finite

trigonometric series, the process is known as trigonometric interpolation. Similarly $\phi(x)$ may be a

Series of exponential functions . Legendre polynomials, Bessel functions etc. In practical problems we always choose $\phi(x)$ to be the simplest function which represents the given functions over the given interval.

Let $y_i = f(x_i)$ where i = 0 ,1, ...,n In many cases we have to find y = f(x) such that $y_i = f(x_i)$ from the given table . This is little difficult because there are infinity of functions $y = \phi(x)$ such that $y_i = \phi(x_i)$. Hence from the given table we cannot find a unique $\phi(x)$ such that $y = \phi(x)$ satisfies the set of values given in the table above. Of the seq. of functions { $\phi(x)$ }, there is a unique n^{th} degree polynomial Pn(x) such that yi = Pn (xi) where i = 0, 1, 2,.........,n



The function ϕ (x) is called interpolating function or smoothing function.

The polynomial function Pn(x) may be taken as an interpolating polynomial where

$y_i = f(x_i)$ = Pn (xi) , i = 0,1,2,......n

Out of other approximating function types the polynomial interpolation is mostly preferred because of 1] They are simple form of functions which can be easily manipulated.

2] Computations for definite values of the argument integration and different of such functions are easy.

3] Polynomials are free from singularities where as rational functions or other types do have singularities.

The basis of finding such Pn(x) polynomial is the fact that there is exactly only polynomial Pn(x) of degree 'n' such that the values of Pn(x) at $x_0, x_1, x_2, \ldots\ldots, x_n$ coincide With the gi ven functional values $y_0, y_1, y_2, \ldots\ldots, y_n$.Here Pn(x) is called polynomial approximation of f(x)

Polynomial approximation serves as a basis for numerical integration and the solution of differential equation. The interpolating polynomial can be used for extrapolation, [if the function value is required at outside of the interval given $[x_0, x_n]$.] .

When a suitable approximating function has to be obtained the two aspects that require Consideration are

1] Whether or not the given data points x_i are equidistant.

2] Whether the interpolation is needed towards the beginning , middle or end of the table.

These aspects determine the choice of suitable functions e.g. If the argument intervals are equidistant, the Newtons forward backward interpolation formula can be used forward formula is used to find interpolation near the beginning and backward for near the end near the centre of a different table, central difference formulae are preferable.

If the argument values are unequal one of the following formula can be used Lagrange formula, Newtons divided diff . formula Aitkenis form , Hermite , Spline which are useful for equidistant interval also.

Out of these we shall discuss first Lagrange's formula and Newton's divided diff formula.

1.7 Lagrange form.

Let $\tau = [\tau_i]_{i=1}^{n}$ be a segment of n distinct points i.e. $\tau = [\tau_1, \tau_2, \ldots\ldots \tau_n]$

Then $$l_i(x) = \prod_{\substack{j=1 \\ i \neq j}}^{n} \frac{(x - \tau_i)}{\tau_i - \tau_j} \quad \text{------------------------------} 1$$

is the i^{th} Lagreange polynonial for τ

Where notation ,

$$\prod_{i=r}^{s} \tau_i = \tau_r, \tau_{r+1} \ldots\ldots\ldots \tau_s \quad \text{if } r \leq s$$
$$= 1 \quad \text{if } r > s$$

it is a polynomial of order n and vanishes at all τ_j's except for τ_i at which it takes the value 1. We write this Lagrange's polynomial with the aid of the Knonecker delta as,

$$l_i(\tau_j) = \delta_{ij} = 0 \quad i \neq j$$

$$= 1 \quad i = j$$

Hence for an arbitrary given function ' g '

$$P = \sum_{i=1}^{n} g(\tau_i) l_i$$

is an element of Pn and satisfies,

$$P(\tau_i) = g(\tau_i), \quad i = 1, 2, \ldots n$$

So we find that the Lagrange polynomials make it possible to write down at once a polynomial interpolant to given g at τ and this is the only interpolant in Pn to g at τ i.e. if $q \in Pn$ is also a polynomial for τ_i then $g(\tau_i) = q(\tau_i)$ $\forall$ all i, then $\gamma = p - q$ is also a polynomial of order n and vanishes at the n distinct points $\tau_1, \tau_2, \ldots \tau_n$.

Theorem :- If $\tau_1, \tau_2, \ldots \tau_n$ are distinct points and $g(\tau_1) \ldots g(\tau_n)$ are given data .Then there exists exactly one polynomial $p \in Pn$ for which $P(\tau_i) = g(\tau_i)$, $i = 1, 2, \ldots n$ .This polynomial can be written in Lagrange form

$$l_i(x) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x - \tau_j)}{\tau_i - \tau_j} \quad \text{for all } i$$

The lagrange form is certainly quite elegant .But compared to other ways of writing and evaluating the interpolating polynomial , it is far from the most efficient.

The lagrange interpolation formula provides some algebraic relations.One of this is "The interpolation process is a projection Operator ''. In particular for $0 \leq i \geq n$ let f be a function

$$f(x) = x^i \qquad a \leq x \leq b$$

By lagrange formula.

$$\sum_{k=0}^{n} x_k^i \, l_k(x) = x^i \qquad a \leq x \leq b \text{ ------------3}$$

The value i=0 gives,

$$\sum_{k=0}^{n} l_k(x) = 1 \qquad a \leq x \leq b \quad *$$

Which is useful for checking the numbers $\{l_k(x), k=0,1,\ldots.n\}$ when the lagrange interpolation method is applied

Using 1 in 3 we get ,

$$\sum_{k=0}^{n} x_k^i \prod_{\substack{j=0 \\ k\neq j}}^{n} \frac{(x - x_j)}{(x_k - x_j)} = x^i \qquad a \leq x \leq b$$

$$\sum_{k=0}^{n} x_k^i \prod_{\substack{j=0 \\ k\neq j}}^{n} \frac{(x - x_j)}{(x_k - x_j)} = x^i \qquad a \le x \le b$$

By considering the coeff. of x^n we get identity.

$$\sum_{k=0}^{n} \frac{x_k^i}{\prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j)} = \delta_{in} \qquad i = 0, 1, \text{------------} n$$

The error in polynomial interpolation, e = error =error function of an approximation.

$$e(x) = f(x) - p(x) \qquad a \le x \le b$$

where $p \in P_n$ for which $p(x_i) = f(x_i)$ $i = 0, 1, \text{----------} n$

If we change f by adding to it an element of P_n then the interpolation process automatically adds the same element to p , which leaves e unchanged.

Theorem :- For any set of distinct interpolation points $\{ x_i ; i = 0, 1, \text{-------} n\}$ in [a, b] and for any $f \in \tau^{(n+1)}[a, b]$, let p be the element of P_n that satifies the eq^n. $f(x_i) = p(x_i)$ $i = 0, 1, \text{-------} n$

Then for any x in [a, b] the error has the value,

$$e(x) = \frac{1}{(n+1)!} \prod_{j=1}^{n} (x-x_j) \; f^{(n+1)}(\xi)$$

Where ξ is a point of [a, b] that depends on x.

1.8 Newton's Divided Differences

Def :- The k^{th} divided difference of a function g at the points $\tau_i, \ldots\ldots\ldots \tau_{i+k}$ is the leading coefficient (i.e. the coefficient of x^k) of the polynomial of order k +1 which agrees with g at the points $\tau_i, \ldots\ldots\ldots \tau_{i+k}$.

It is denoted by ,

$$[\tau_i, \ldots\ldots\ldots \tau_{i+k}]g$$

Formula :- When function values are given at non-equidistant points. The Lagrangion interpolation scheme is not computationally economical.Here divided differences offer better possibilities .

Let $[\tau_i, y_i]$, i = o(1)n be the given points and $y_i = y(\tau_i) = f[\tau_i]$, i = o(1)n where $f(\tau)$ is the function being approximated by the polynomial $y(\tau)$ since $(\tau-\tau_i)$ is zero for $\tau = \tau_i$ by writing the n^{th} degree polynomial $y(\tau)$ as sums of the products of such factors consider the following formula for $y(\tau)$.

$$y(\tau) = \alpha_0 + \alpha_1(\tau-\tau_0) + \alpha_2(\tau-\tau_0)(\tau-\tau_1) + \text{-----} + \alpha_n(\tau-\tau_0)(\tau-\tau_1)\text{----}(\tau-\tau_{n-1}) \text{-------} 1$$

$y(\tau)$ has the property that all the term after α_0 - term are zero for $\tau = \tau_0$, all the terms after the α_1 - term are zero for $\tau = \tau_1$ and so on . $y(\tau)$ is the interpolating polynomial for the actual function $f(\tau)$

$$y(\tau_0) = \alpha_0$$

$$y(\tau_1) = \alpha_0 + \alpha_1(\tau_1-\tau_0)$$

$$y(\tau_2) = \alpha_0 + \alpha_1(\tau_2-\tau_0) + \alpha_2(\tau_2-\tau_0)(\tau_2-\tau_1)$$

.

.

.

$$y(\tau_n) = \alpha_0 + \alpha_1(\tau_n-\tau_0)+\ldots..+ \alpha_n(\tau_n-\tau_0)\ldots\ldots.(\tau_n-\tau_{n-1}) \text{--------}2$$

The system of linear equation is easy to solve because the coefficient matrix is triangular. The coefficients α_0 , α_1,-----, α_n can now be calculated recursively i.e. α_k is computed in terms of α_0 , α_1,α_2,-----α_{k-1} which are already obtained. Here we introduce new notation i.e. divided differencies notation . from 1 of 2

$\alpha_0 = y(\tau_o) = y[\tau_o]$ is zero order divided

Solving 2 eqn. of 2

$$\alpha_1 = \frac{y(\tau_1)-y(\tau_0)}{\tau_1-\tau_0} = \frac{y[\tau_1]-y[\tau_0]}{\tau_1-\tau_0} = y[\tau_1-\tau_0]$$

The third eqn of 2 gives

$$\alpha_2 = \frac{y(\tau_2)-y(\tau_0) - [\tau_2-\tau_0]\, y[\tau_1,\tau_0]}{[\tau_2-\tau_0]\,[\tau_2-\tau_1]}$$

This on rearrangement gives

$$y[\tau_2] - y[\tau_0] = y[\tau_2] - y[\tau_1] + \frac{y(\tau_1)-y(\tau_0)}{\tau_1-\tau_0} \quad [\tau_1-\tau_0]$$

$$\alpha_2 = \frac{y(\tau_2)-y(\tau_1) - [\tau_2-\tau_1]\, y[\tau_1,\tau_0]}{[\tau_2-\tau_0]\,[\tau_2-\tau_1]}$$

$$= \frac{y[\tau_2,\tau_1] - y[\tau_1,\tau_0]}{[\tau_2-\tau_0]}$$

$$\alpha_2 = y[\tau_2, \tau_1, \tau_0] = \frac{y[\tau_2,\tau_1] - y[\tau_1,\tau_0]}{[\tau_2-\tau_0]}$$

is a 2^{nd} order divided diff. By Mathematical Induction .The k^{th} order divided diff. is ,

$\alpha_k = \quad y[\tau_k, \tau_{k-1} \text{---------------} \tau_0]$

$$= \frac{y[\tau_k, \tau_{k-1} \text{---------------} \tau_1] - y[\tau_{k-1}, \tau_{k-2} \text{---------------} \tau_0]}{[\tau_k - \tau_0]}$$

Which provides a convenient algorithm for computing the coefficient from the vector values of τ_i, and the vector value of y_i, from eqn. 1 by putting $\alpha_1, \alpha_2 \ldots\ldots\ldots \alpha_k$

Newton's divided diff. formula can be written as,

$y(\tau) = \{y(\tau_0) + (\tau-\tau_0)\, y\,[\tau_0, \tau_1] + (\tau-\tau_0)\,(\tau-\tau_1) * y[\tau_0, \tau_1, \tau_2] + \text{---------} + (\tau-\tau_0)$

$(\tau-\tau_1)\,(\tau-\tau_2) \text{-------------} (\tau-\tau_{n-1})\, y[\tau_0, \tau_1, \ldots\ldots\ldots \tau_n]\}$

1.9 Divided diff. table,

Let y be function having argument values $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4$ and

$y(\tau)$ having $y(\tau_0)$, $y(\tau_1)$, $y(\tau_2)$, $y(\tau_3)$, $y(\tau_4)$ respectively

τ	$y(\tau)$	1st divided diff.	2nd divided diff.	3rd divided diff.	4th divided diff.
τ_0	$y(\tau_0)$				
		$y[\tau_1, \tau_0]$			
τ_1	$y(\tau_1)$		$y[\tau_2, \tau_1, \tau_0]$		
		$y[\tau_2, \tau_1]$		$y[\tau_3, \tau_2, \tau_1, \tau_0]$	
τ_2	$y(\tau_2)$		$y[\tau_3, \tau_2, \tau_1]$		$y[\tau_4, \tau_3, \tau_2, \tau_1, \tau_0]$
		$y[\tau_3, \tau_2]$		$y[\tau_4, \tau_3, \tau_2, \tau_1]$	
τ_3	$y(\tau_3)$		$y[\tau_4, \tau_3, \tau_2]$		
		$y[\tau_4, \tau_3]$			
τ_4	$y(\tau_4)$				

Where,

$$y[\tau_1, \tau_0] = \frac{y[\tau_1] - y[\tau_0]}{\tau_1 - \tau_0}$$

$$y[\tau_2, \tau_1, \tau_0] = \frac{y[\tau_2, \tau_1] - y[\tau_1, \tau_0]}{\tau_2 - \tau_0}$$

and so on...

1.10 Properties of Divided diff.

i) If $p_i \in P_i$ agrees with g at $\tau_1 \ldots\ldots \tau_i$ for $i = k$ and $k+1$ then,

$P_{k+1}(x) = P_k(x) + (x-\tau_1)\text{--------}(x-\tau_k)(\tau_1\text{---------}\tau_{k+1})g$

We know that $P_{k+1} - P_k$ is a polynomial of order k+1 which vanishes at $\tau_1 \ldots\ldots \tau_k$ and has $[\tau_1 \ldots\ldots \tau_{k+1}]g$ as its leading coefficient therefore must be of the form $P_{k+1}(x)-\ \ Pk(x) = C\,[x-\tau_1]\ldots\ldots\ldots\ldots[x-\tau_k]$

With $C= (\tau_1\text{---------}\tau_{k+1})g$

This property shows that divided differences can be used to build up the interpolating polynomial by adding the interpolation points one at a time. So we obtain,

$$P_n(x) = P_1(n) + P_2(x) - P_1(x) + P_3(x) - P_2(x) + \text{---------} + P_n(x) - Pn\text{-}1(x)$$

$$= [\tau_1]g + (x-\tau_1)[\tau_1, \tau_2]g + (x-\tau_1)(x-\tau_2),$$

$$[\tau_1, \tau_2, \tau_3]g + \ldots\ldots\ldots\ldots + (x-\tau_1)\text{---------}(x-\tau_{n-1})[\tau_1, \tau_2, \ldots\ldots\ldots\ldots, \tau n]g$$

This is the Newton Form.

$$P_n(x) = \sum_{i=1}^{n} (x-\tau_1)\ldots\ldots\ldots\ldots\ldots(x-\tau_{i-1})[\tau_1, \tau_2, \ldots\ldots\ldots\ldots, \tau n]g$$

for the polynomial Pn of order n which agrees with g at $\tau_1 \ldots\ldots \tau_n$

[here $(x-\tau_1)(x-\tau_2)\ldots\ldots\ldots\ldots(x-\tau_j)=1$ if $i > j$]

ii) $[\tau i, \ldots\ldots\ldots\ldots, \tau_{i+k}]g$ is a systematic function of its arguments $\tau i, \ldots\ldots\ldots, \tau_{i+k}$

i.e. it depends only on the number $\tau i, \ldots\ldots\ldots, \tau_{i+k}$ and not on the order on which they occur in the argument list since Inter. Poly. Depends only on the points of Interpolation.

iii) $[\tau i, \ldots\ldots\ldots, \tau_{i+k}]g$ is linear in g I.e. if $f = \alpha g + \beta h$ for some functions g & h and some numbers α & β then ,

$[\tau i, \ldots\ldots\ldots, \tau_{i+k}]f = \alpha\ [\tau_i, \ldots\ldots\ldots, \tau_{i+k}]g+\beta[\tau_i, \ldots\ldots\ldots, \tau_{i+k}]$has follows from the Uniqueness of the interpolating polynomial .

iv) Leibnitz's Formula ,

If $f = gh$ i.e. $f(x) = g(x)\ h(x)\quad \forall\ x$

Then ,

$$[\tau_i, \ldots\ldots\ldots, \tau_{i+k}]f = \sum_{r=i}^{i+k} [\tau_i, \ldots\ldots\ldots, \tau_r]g[\tau_r, \ldots\ldots\ldots, \tau_{i+k}]h$$

v) If g is a polynomial of degree $\leq$ k then $[\tau_i, \ldots\ldots\ldots, \tau_{i+k}]g$ is constant as a function of $[\tau_i, \ldots\ldots\ldots, \tau_{i+k}]g=0 \quad \forall\ \ g \in P_k$

Def^n:- Repeated interpolation at a point is called oscillatory interpolation.

Def^n :- let $\tau = (\ \text{Pi})^n_1$ be a seq. Of points not necessarily distinct . we say that the function P agrees with the function g at τ. Provided that for every point τ which occur in times in the seq. $\tau_1 \ldots\ldots\ldots\ldots \tau_n$ p & g agree m fold at τ i.e.

$P_{(i-1)}(\tau) = g^{(i-1)}(\tau)$ for i=1 , ...m

vi) $[\tau_i, \ldots\ldots, \tau_{i+k}]g$ is a continuous function of its k+1 arguments in case g $\in C^{(k)}$ i.e. g has k continuous derivatives .

vii) If g $\in C^{(k)}$ i.e. g has k continuous derivatives then there exists a point ξ in the smallest interval containing $\tau_i, \ldots\ldots, \tau_{i+k}$ so that ,

$$[\tau_i, \ldots\ldots, \tau_{i+k}]g = \frac{g^{(k)}(\xi)}{k!}$$

viii) For computations it is important to note that ,

$$[\tau_i, \ldots\ldots, \tau_{i+k}]g = \frac{g^{(k)}(\tau_i)}{k!} \quad \text{if } \tau_i \ldots\ldots \tau_{i+k} \quad g \in C^{(k)}$$

$$= \frac{[\tau_i, \ldots\ldots, \tau_{r-1}, \tau_{r+1}, \ldots\ldots \tau_{i+k}]g - [\tau_i, \ldots\ldots, \tau_{s-1}, \tau_{s+1}, \ldots\ldots \tau_{i+k}]g}{\tau_s \qquad\qquad \tau_r}$$

if τ_s & τ_r are any two distinct points in the seq. $\tau_i, \ldots\ldots, \tau_{i+k}$.

1.11 Algorithm for Lagrange's & Newton's Formula :-

" c" Program for Lagrange's form :-

```
        Lagrange (int*x[], int*y[], int n, int t)
{ int i product , float sum
          sum = 0
        for ( i  = 0 , i<n, i++ )
}
        Product  = y [i]
        For (  j =0, j<n,  j+ + )
{
        If ( i_= j )
product = product *  (t – x[i]) / [x(i)=x(j)]
}
        Sum = sum  + product
}
        return (sum);
}
}
```

c" Program for Newton's Interpolation Formula :-

```
        newint ( int * x[ ] int * y[] , int n ,
        int * x,
        int * y ,
        int * ea ,
{
        int fdd [n] [n] ;
        for (i = 0; i < n ; i + + )
        fdd [ i ] [ 0 ] = y [i] ;
        for (j =0 ; j < n ; j + +)
        for (i = 0 ; i < n – j ; i + +)
fdd [ i ] [ j ] = fdd [ i + 1] [ j-1] - fdd [i] [ j - 1 ] / [x(i+j)-x(i)]
        x term = 1;
        y int [ 0] = fdd [0] [0]
        for [ood =1; ood < n ; ood + +]
}
        x term = x term * (xi – x [ood – 1] )
        y int 2 = y int [ood - 1 ] + fdd [0] [ood] * x term ;
```

```
Ea [ ood – 1] = y int 2 – y int [ ood - 1 ]

y int  [ood] = y int 2 ;

}
```