FUZZY IDEALS OF A SEMIGROUP

Introduction

In chapter 2, we have discussed different types of ideals of a semigroup and some results about them. In this chapter we have fuzzified the classical concepts of Ideals and related structures.

In first section, different types of fuzzy ideals are defined with examples and preliminary results about them are discussed.

In second section theorems characterizing these fuzzy ideals are proved.

In third section prime fuzzy ideals and semiprimality is discussed in detail.

In fourth section Green's relations are discussed in case of fuzzy ideals and theorems characterizing them are proved.

In all that follows S denotes Semigroup.

Section 1

Definition 3.1.1

Fuzzy subsemigroup : Fuzzy set δ in S is called fuzzy subsemigroup of S if $\delta(x,y) > \min\{\delta(x), \delta(y)\} \forall x,y \in S$

Example 1 : (N,X) is a semigroup, where N is a set of natural numbers

Define $\delta : N \rightarrow [0,1]$ as follows

$\delta(x) = 1$ if $x \in (4n)$

$= 1/2$ if $x \in (2n) - (4n)$

$= 0$ if $x \in (2n)$

Then it can be easily seen that

$\delta(xy) \geqslant \min\{\delta(x), \delta(y)\} \ \forall\, x,y \in N$

So δ is fuzzy subsemigroup of N.

Definition 3.1.2

Fuzzy Left ideal : Fuzzy set δ in S is called fuzzy left ideal of S iff $\delta(xy) \geqslant \delta(y) \ \forall\, x,y \in S$

Example 2 :

Let $S = \{1,2,3,4\}$ .Define binary operation * on S by $a*b = b \ \forall\, a.b \in S$

Then $(S,*)$ is semigroup

Define $\delta : S \to [0,1]$ by

$\delta(1) = 1 \quad \delta(3) = 1/3$

$\delta(2) = 1/2 \quad \delta(4) = 0$

Then it can be seen that

So $\delta(x*y) \geqslant \delta(y) \quad x,y \in S$

i.e. δ is fuzzy left ideal of S.

Definition 3.1.3 :

Fuzzy Right Ideal : Fuzzy set δ in S is called fuzzy right ideal of S iff $\delta(xy) \geqslant \delta(x) \ \forall\, x,y \in S$

Example 3 :

Let $S = \{1,2,3,4\}$. Define binary operation * on S by $a * b = a \ \forall\, a,b \in S$

Define $\delta : S \to [0,1]$ by

$\delta(1) = 1 \; \delta(2) = 1/2 \quad \delta(3) = 1/3 \quad \delta(4) = 0$

Then it can be seen that

$\delta(x * y) \geqslant \delta(x) \; \forall \; x, y \in S$

So δ is fuzzy right ideal of S.

This is fuzzy right ideal but not fuzzy left ideal.

Definition 3.1.4 :

Fuzzy set δ in S is called fuzzy ideal iff it is both fuzzy left and fuzzy right ideal.

In other words fuzzy set δ in S is called fuzzy ideal iff

$\delta(xy) \geqslant \max\{\delta(x), \delta(y)\}$.

Fuzzy set δ in example 1 is fuzzy ideal.

Remark :

Fuzzy Left (Right, two sided) ideal is a fuzzy subsemigroup of S.

Trivially true. For

If δ is a fuzzy ideal of S

Then $\delta(xy) \geqslant \max\{\delta(x), \delta(y)\}$

$\geqslant \min\{\delta(x), \delta(y)\}$

So δ is fuzzy subsemigroup, of S

Similarly other things can be proved.

But every fuzzy subsemigroup is not ideal.

Fuzzy set δ defined in example-3 is fuzzy subsemigroup but not fuzzy ideal.

Definition 3.1.5 :

Fuzzy Bi-ideal : Fuzzy subsemigroup δ of semigroup S is called fuzzy bi-ideal of S if

$$\delta(xyz) \geq \min\{\delta(x), \delta(z)\} \forall x,y,z \in S.$$

Fuzzy set δ, defined in example-1 is fuzzy bi-ideal.

Remarks :

1) Every fuzzy left (Right, two sided) ideal is fuzzy bi-ideal of S

Proof :

Let δ be fuzzy left ideal of S

i.e. $\delta(xy) \geq \delta(y) \forall x,y \in S$

consider

$$\delta(xyz) \geq \delta(yz) \geq \delta(z) \geq \min\{\delta(x), \delta(z)\}$$

Thus δ is fuzzy bi-ideal of S.

Similarly, other results can be proved.

2) Every fuzzy bi-ideal of S need not be fuzzy ideal of S.

e.g Fuzzy set δ in example-3 is fuzzy bi-ideal but it is not fuzzy ideal.

3) Union and Intersection of two fuzzy Left (Right, two sided) ideals of a semigroup S are fuzzy Left (Right, Two sided) ideal.

Proof :

Let δ_1 and δ_2 be any two fuzzy left ideals. $\forall x,y \in S$

i) $(\delta_1 \cup \delta_2)(xy) = \max\{\delta_1(xy), \delta_2(xy)\}$

$\geq \max\{\delta_1(y), \delta_2(y)\}$

$= (\delta_1 \cup \delta_2)(y)$.

So $\delta_1 \cup \delta_2$ is fuzzy left ideal.

ii) $(\delta_1 \cap \delta_2)(xy) = \min\{\delta_1(xy), \delta_2(xy)\}$

$\geq \min\{\delta_1(y), \delta_2(y)\}$

$= (\delta_1 \cap \delta_2)(y)$.

So $\delta_1 \cap \delta_2$ is fuzzy left ideal.

Similarly result can be proved for fuzzy Right and fuzzy two sided ideals.

4) If $f: S \to S'$ is an epimorphism of semigroup S and S'. If A and B are fuzzy ideals of S and S' respectively. Then (1) f (A) is fuzzy ideal of S' (2) $f^{-1}(B)$ is fuzzy ideal of S.

Proof :

Let A be fuzzy ideal of S

Defined $f(A) : S' \to [0,1]$

By $f(A)(y) = \sup\limits_{x \in f^{-1}(y)} A(x)$

To prove that f(A) is fuzzy ideal of S'

Let $u \in S'$ and $v \in S'$

Since f is onto $\exists$ $x, y \in S$

s.t. $f(x) = u$ and $f(y) = v$

So $f(xy) = uv$

Thus

$f(A)(uv) = \sup\limits_{z \in f^{-1}(uv)} A(z)$

$= \operatorname{Sup}_{xy \in f^{-1}(uv)} A(xy)$

$\geqslant \operatorname{Sup}_{\substack{x \in f^{-1}(u) \\ y \in f^{-1}(v)}} (A(x) \vee A(y))$

$= (\bigvee_{x \in f^{-1}(u)} A(x)) \vee (\bigvee_{y \in f^{-1}(v)} A(y))$

$= f(A)(u) \vee f(A)(v)$

So f(A) is fuzzy ideal of S'

On the other hand if

$f : S \rightarrow S'$ is homomorphism of S onto S' and B is fuzzy ideal of S'

Then define

$A : S \rightarrow [0,1]$

By $A(x) = B(f(x)) \; \forall \; x \in S$

To prove that A is fuzzy ideal of S

$A(xy) = B(f(xy))$

$= B(f(x)f(y))$

$\geqslant \max (B(f(x)), B(f(y)))$

$= \max \{ A(x), A(y) \}$

So A is fuzzy ideal of S.

5) If $f : S \rightarrow S'$ is not epimorphism, then image of fuzzy ideal of S need not be fuzzy ideal of S'.

6) Let S and S' be two semigroups

Let $f : S \rightarrow S'$ be an epimorphism of S onto S'.

If δ_1 and δ_2 are fuzzy ideals of S,

then

1) $f(\delta_1 \cap \delta_2) = f(\delta_1) \cap f(\delta_2)$

2) $f(\delta_1 \cup \;_2) = f(\delta_1) \cup f(\delta_2)$

Proof :

1) Let $f(\delta_1)$ and $f(\delta_2)$ be fuzzy ideals of S'

clearly $f(\delta_1) \cap f(\delta_2)$ is fuzzy ideal of S'

Let $x' \in S'$ be any element.

$$(f(\delta_1) \cap f(\delta_2)(x') = f(\delta_1)(x') \wedge f(\delta_2)(x')$$

$$= (\bigvee_{x \in f^{-1}(x')} \delta_1(x)) \wedge (\bigvee_{x \in f^{-1}(x')} \delta_2(x))$$

$$= \bigvee_{x \in f^{-1}(x')} (\delta_1(x) \wedge \delta_2(x))$$

$$= \bigvee_{x \in f^{-1}(x')} (\delta_1 \cap \delta_2)(x)$$

$$= f(\delta_1 \cap \delta_2)(x)$$

So $f(\delta_1 \cap \delta_2) = f(\delta_1) \cap f(\delta_2)$

2) Let $x' \in S'$ be any element.

$$(f(\delta_1) \cup f(\delta_2)(x')$$

$$= f(\delta_1)(x') \vee f(\delta_2)(x')$$

$$= \bigvee_{x \in f^{-1}(x')} f(\delta_1(x) \vee \delta_2(x))$$

$$= f(\delta_1 \cup \delta_2)(x')$$

So $f(\delta_1 \cup \delta_2) = f(\delta_1) \cup f(\delta_2)$

7) Let $f : S \to S'$ be homomorphism of semigroups S into S'. If δ_1' and δ_2' are fuzzy ideals of S' then

1) $f^{-1}(\delta'_1 \cdot \delta'_2) = f^{-1}(\delta'_1) \cdot f^{-1}(\delta'_2)$

2) $f^{-1}(\delta'_1 \cup \delta'_2) = f^{-1}(\delta'_1) \cup f^{-1}(\delta'_2)$

Proof : Let x be any element of S'

1) $f^{-1}(\delta'_1 \cap \delta'_2)(x) = (\delta'_1 \cap \delta'_2)f(x)$

$= \delta'_1(f(x)) \wedge \delta'_2(f(x))$

$= f^{-1}(\delta'_1(x)) \wedge f^{-1}(\delta'_2(x))$

$= (f^{-1}(\delta'_1) \cap f^{-1}(\delta'_2))(x)$

Thus $f^{-1}(\delta'_1 \cap \delta'_2) = f^{-1}(\delta'_1) \cap f^{-1}(\delta'_2)$

2) Let $x \in S'$ be any element

$f^{-1}(\delta'_1 \cup \delta'_2)(x) = (\delta'_1 \cup \delta'_2)(f(x))$

$= \delta'_1(f(x)) \vee \delta'_2(f(x))$

$= f^{-1}(\delta'_1)(x) \vee f^{-1}(\delta'_2(x))$

$= (f^{-1}(\delta'_1) \cup f^{-1}(\delta'_2))(x)$

So $f^{-1}(\delta'_1 \cup \delta'_2) = f^{-1}(\delta'_1) \cup f^{-1}(\delta'_2)$

8) If $f : S \to S'$ be epimorphism of semigroup S onto S' and if δ_1 and δ_2 are fuzzy ideals of S, then $\delta_1 \subseteq \delta_2 = f(\delta_1) \subseteq f(\delta_2)$

Proof :

Let x' be any element of S'

$$f(\delta_1)(x') = \bigvee_{x \in f^{-1}(x')} \delta_1(x)$$

$$\leq \bigvee_{x \in f^{-1}(x')} \delta_2(x)$$

$= f(\delta_2)(x')$

Thus,

$f(\delta_1) \subseteq f(\delta_2)$

9) If $f : S \to S'$ be homomorphism of semigroup S into S' and if δ_1' and δ_2' be fuzzy ideals of S' then

$\delta_1' \subseteq \delta_2' = f^{-1}(\delta_1') \subseteq f^{-1}(\delta_2')$

Proof :

Let $x \in S$ be any element

$f^{-1}(\delta_1')(x) = \delta_1'(f(x))$

$\subseteq \delta_2'(f(x))$

$= f^{-1}(\delta_2'(x))$

So $f^{-1}(\delta_1') \subseteq f^{-1}(\delta_2')$

10) Let δ_1 be a fuzzy ideal of semigroup S.

Two level ideals δ_{t_1}, δ_{t_2} with $t_1 < t_2$ are equal iff $\exists$ no $x \in S$ s.t. $t_1 \leq \delta(x) \leq t_2$

Proof :

Let $\delta_{t_1} = \delta_{t_2}$

If $\exists$ $x \in S$ s.t. $t_1 \leq \delta(x) \leq t_2$

Then $x \in \delta_{t_1}$ but $x \notin \delta_{t_2}$

This is contradiction.

Conversly, suppose there is no x s.t.

$t_1 \leq \delta(x) \leq t_2$

Since $t_1 < t_2$ .Then $\delta_{t_2} \subseteq \delta_{t_1}$

and if $x \in \delta_{t_1}$ then $\delta(x) \geq t_1$

So by condition it follows that

$\delta(x) \geqslant t_2$

i.e. $x \varepsilon \delta_{t_2}$

So $\delta_{t_1} = \delta_{t_2}$

Section 2

Lemma 1

Let A be nonempty subset of semigroup S and δ_A be characteristic function of A. Then

1) A is subsemigroup of S iff δ_A is fuzzy subsemigroup of S.

2) A is left (Right) ideal of S iff δ_A is fuzzy Left (Right) ideal of S.

3) A is an ideal of S iff δ_A is fuzzy ideal of S.

Proof :

1) Let A be subsemigroup of S, i.e. $A^2 \subseteq A$.

i.e. $\forall$ $x,y \in A$ we have $xy \in A$.

To prove that δ_A is fuzzy subsemigroup of S.

i.e. To prove that

$\delta A(xy) \geqslant \min\{ \delta_A(x), \delta_A(y) \; \forall \; x,y \in S$

Let $x \in A$ $y \in A$, $\delta_A(xy) = 1 = \min\{\delta_A(x), \delta_A(y)\}$

$= \min\{1,1\}$

Let $x \in A$, $y \notin A$

$\delta_A(xy) \geqslant \min\{\delta_A(x), \delta_A(y)\} = \min\{1,0\} = 0$

similarly other cases can be proved.

Conversly

Let δ_A be fuzzy subsemigroup of S,

i.e. $\delta_A(xy) \geqslant \min\{\delta_A(x), \delta_A(y)\} \; \forall \; x,y \in S$

To prove that A is subsemigroup of S

i.e. $\forall\, x, y \in A = xy \in A$

If x and $y \in A$, we have

$\delta_A(x) = 1 = \delta_A(y)$

$\delta_A(xy) \geq \min\{\delta_A(x), \delta_A(y)\} = 1$

So $\delta_A(xy) = 1$

$\Rightarrow xy \in A$

i.e. $A^2 \subseteq A$

So A is subsemigroup of S.

2) Let A be left ideal of S, i.e. $SA \subseteq A$,

i.e. $\forall\, x \in S$ and $y \in A$ we have $xy \in A$

To prove that δ_A is fuzzy left ideal of S

i.e. To prove that $\delta_A(xy) \geq \delta_A(y)\ \forall\, x, y \in S$

If $x \in S$, $y \in A$ $xy \in A \Rightarrow \delta_A(xy) = \delta_A(y) = 1$

If $x \in S$, $y \notin A$ $\delta_A(xy) \geq 0 = \delta_A(y)$

So proved.

Conversly,

Let δ_A be fuzzy left ideal of S

i.e. $\delta_A(xy) \geq \delta_A(y)\ \forall\, x, y \in S$

To prove that A is left ideal of S,

i.e. To prove that $\forall\, x \in S$, $y \in A$ we have $xy \in A$

Let $x \in S$ and $y \in A$

$\delta_A(xy) \geq \delta_A(y) = 1$

$\Rightarrow \delta_A(xy) = 1$

$\Rightarrow xy \in A$

So A is left ideal of S. Proved

Similar result can be proved for right ideals.

3) A be an ideal of S

$\Rightarrow$ A is both left and Right ideal.

$\Rightarrow \delta_A$ is both fuzzy left and fuzzy right ideal.

$\Rightarrow \delta_A$ is fuzzy ideal.

Lemma 2 :

Let A be nonempty subset of a semigroup S. δ_A be characteristic function of A, then A is bi-ideal of S iff δ_A is fuzzy bi-ideal of S

Proof :

Let $A \subseteq S$ be any bi-ideal of S.

i.e. $ASA \subseteq A$ and A is subsemigroup of S

i.e. $xyz \in A \ \forall \ y \in S$ and x and $z \in A$

To show that δ_A is fuzzy bi-ideal of S

i.e. To prove that $\delta_A(xyz) \geq \min\{\delta_A(x), \delta_A(z)\} \forall x,y,z \in S$

As A is subsemigroup of S, δ_A is fexxy semigroup of S.

Case 1 :

Let x and $z \in A$, $y \in S$

$xyz \in A \Rightarrow \delta_A(xyz) \geq \min\{\delta_A(x), \delta_A(z)\}$

Case 2 : x and $z \in A$ $y \in S$

then $\delta_A(xyz) \geq 0 = \min\{\delta_A(x), \delta_A(z)\}$

similarly, in other cases it can be proved that

$\delta_A(xyz) \geq \min\{\delta_A(x), \delta_A(z)\}$

So δ_A is fuzzy bi-ideal.

Conversly

Let δ_A be fuzzy bi-ideal of S

To prove that A is bi-ideal of S

Given $\delta_A(xyz) \geq \min\{\delta_A(x), \delta_A(z)\}$

and δ_A is fuzzy subsemigroup of S.

To prove that

A is subsemigroup of S and $\forall$ $x,z \in A$ and $y \in S$. we have $x\ y\ z \in A$ as δ_A is fuzzy subsemigroup of S we have A is subsemigroup.

Now $\delta_A(xyz) \geq \min\{\delta_A(x), \delta_A(z)\} = 1$

$\Rightarrow \delta_A(xyz) = 1$

$\Rightarrow xyz \in A$

$\Rightarrow ASA \subseteq A$

$\Rightarrow$ A is bi-ideal of S.

Definitions 3.2.1

i) A semigroup S is call <u>Left(Right) duo</u> if every Left (Right) ideal of S is a two sided ideal of S

ii) Semigroup S is <u>duo</u> iff it is both Left and Right duo.

iii) Semigroup S is called <u>fuzzy Left (Right)duo</u> iff every fuzzy left (Right) ideal of S is a fuzzy ideal of S.

iv) Semigroup S is called <u>fuzzy duo</u> iff it is both fuzzy left and fuzzy right duo.

Theorem 1 :

For a regular semigroup S, following conditions are equivalent.

1) S is left duo

2) S is fuzzy left duo.

Proof :

First assume that S is left duo.

Let δ be any fuzzy left ideal of S and a,b be two elements of S

Then as left ideal Sa is two sided ideal of S and since S is regular we have

$ab \in (aSa)b \subseteq (Sa)\ S \subseteq Sa$

i.e. $\exists$ x in S s.t. ab=xa

$\delta\ (ab) = \delta(xa) \geq \delta\ (a)$

i.e. δ is fuzzy right ideal of S.

So δ is fuzzy ideal of S

i.e. S is fuzzy left duo.

Conversly,

Let S be fuzzy left duo

Let A be any left ideal of S

Then characteristic function δ_A is fuzzy left ideal

But by assumption δ_A is fuzzy two sided ideal of S. So A is fuzzy ideal of S.
So S is left duo.

Theorem 2 :

For a regular semigroup S, following conditions are equivalent.

1) S is right duo

2) S is fuzzy right duo.

This can be proved on the same line as proved in theorem 1.

Combining these two theorems we get

Theorem 3 :

For a regular semigroup S, following conditions are equivalent.

1) S is duo

2) S is fuzzy duo.

Theorem 4 :

For a regular semigroup S, following conditions are equivalent.

1) Every bi-ideal of S is left ideal of S

2) Every fuzzy bi-ideal of S is fuzzy left ideal of S.

Proof :

Let (1) hold.

Let δ be any fuzzy bi-ideal of S.

Let a and b ε S, then aSa is bi-ideal of S.

By assumption, aSa is left ideal of S.

AS S is regular

ba ε S(a Sa) $\subseteq$ aSa

$\Rightarrow \exists$ x ε S, s.t. ba = axa

Since, δ is fuzzy bi-ideal of S

δ (ba) = δ (axa) $\geqslant$ min{ δ (a) , δ(a)} = (a)

So δ is fuzzy left ideal of S

So (1) $\Rightarrow$ (2)

Conversly, let (2) hold

Let A be any bi-ideal of S,

$\Rightarrow$ δA is fuzzy bi-ideal of S.

$\Rightarrow$ δA is fuzzy left ideal of S

So as A is nonempty we have A is left ideal of S

So (2) $\Rightarrow$ (1) Hence proved.

Similarly, it can be proved that

Theorem 5 :

For a regular semigroup S, following conditions are equivalent.

1) Every bi-ideal of S is Right (two sided) ideal of S.

2) Every fuzzy bi-ideal of S is fuzzy right (two sided) ideal of S.

We denote

L(a) (J(a)) the principal left (two sided) ideal of semigroup S, generated by $a \in S$

i.e. $L(a) = \{a\} \cup Sa$

$J(a) = \{a\} \cup Sa \cup aS \cup SaS$ $\langle a \rangle$ P. 22

Remark :

It S is a regular semigroup, then $L(a) = Sa \; \forall a \in S$.

Proof : $L(a) = \{a\} \cup Sa$

So $Sa \subseteq L(a)$ .....(1)

To show that $\{a\} \cup Sa \subseteq Sa$

i.e. $a \in Sa$

As S is regular $\exists$ $x \varepsilon S$ s.t.

$a = axa = (ax)\ a \varepsilon Sa$

So $\{a\} \cup Sa \subseteq S(a)$

i.e. $L(a) \subseteq Sa$ .....(2)

From (1) and (2) $L(a) = Sa$.

Definition 3.2.2

Semigroup S is called right (Left) zero if $xy=y(xy=x) \forall x,y \varepsilon S$

Theorem 6 :

For a regular semigroup S, following conditions are equivalent.

1) The set of idempotent elements of S forms a left zero subsemigroup of S.

2) For every fuzzy left ideal δ Of S, $\delta(e)= \delta(f)$ $\forall$ idempotents e and f of S.

Proof : Let Es i.e. set of all idempotent elements of S forms a left zero subsemigroup of S.

Let e and f be any two elements of Es.

δ be any fuzzy left ideal of S.

as $ef = e$ and $fe = f$

$\delta(e) = \delta(ef) \geqslant \delta(f) = \delta(fe) \geqslant \delta(e)$

So $\delta(e) = \delta(f)$

So (1) $\Rightarrow$ (2)

Conversly, Let (2) hold.

Since S is regular, Es is non empty.

Let e and f be any two elements of Es.

L(f) principal left ideal of semigroup S,

generated by f

L(f) = Sf as S is regular.

Characteristic function $\delta L(f)$ of L(f) is fuzzy left ideal of S

So $\delta_{L(f)}(e) = \delta_{L(f)}(f) = 1$

So $e \varepsilon L(f) = Sf$

So $\exists\ x \varepsilon S$ s.t. $e = xf$

$e = xf = xff = ef$

i.e. Es is left zero subsemigroup.

So (2) $\Rightarrow$ (1)

Corollary 7

For an idempotent semigroup S, following conditions are equivalent

1) S is left zero.

2) For every fuzzy left ideal δ of S

$\delta(e) = \delta(f)\ \forall\ e,f \varepsilon S$

Proof follows from theorem 6.

Remark :

Regular semigroup containing exactly one idempotent e, is a group

Proof : Let S be regular semigroup.

Let $a \varepsilon S$ be any arbitrary element.

As S is regular, $\exists\ x \in S$ such that $a = axa$

But xa is idempotent element

Now as S contains exactly one idempotent

$xa = e$

i.e. $ae = a$ So identity exists in S.

and $xa = e \Rightarrow$ each element $a \in S$ has an inverse

So S is group. Proved.

To prove Theorem 8, we use following result.

"For a semigroup S, following conditions are equivalent

1) S is group

2) Every fuzzy bi-ideal of S is a constant function"

Theorem 8 :

For a regular semigroup S, following conditions are equivalent.

1) S is group

2) For every fuzzy bi-ideal δ of S

$\delta(e) = \delta(f)\ \forall$ idempotents e and f of S.

Proof :

Assume (1) holds

δ be fuzzy bi-ideal of S.

Then by result quoted above, δ is constant.

So $\delta(e) = \delta(f)\ \forall$ idempotents e and f of S.

So $(1) \Rightarrow (2)$

Conversly, assume (2) holds

Let e and f be idempotents of S.

By $B(x)$ we denote principal bi-ideal generated by x in S .

$B(x) = \{x, x^2\} \cup xSx$

As S is regular $B(x) = xSx$

$\delta B(x)$ i.e. characteristic function of $B(x)$ is fuzzy bi-ideal

Since $f \varepsilon B(f)$

$\delta B(f)^{(e)} = \delta B(f)^{(f)} = 1$

$\Rightarrow e \varepsilon B(f) = fSf$

$\Rightarrow \exists x \varepsilon S$ s.t. $e = fxf$

Similarly, $\exists y \varepsilon S$ s.t. $f = eye$

Then ,we have

$e = fxf = fxff = ef = eeye = eye = f$

i.e. S is regular semigroup containing exactly one idempotent.

So by result proved in remark above

we have S is group.

So proved.

Theorem 9 :

For a semigroup S, following conditions are equivalent

1) S is intra-regular

2) For every fuzzy ideal δ of S , $\delta(a) = \delta(a^2)$ $\forall$ $a \varepsilon S$.

Proof :

Let (1) hold

Let δ be fuzzy ideal of S and $a \varepsilon S$, As S is intraregular, $\exists$ x and y in S, such that $a = xa^2y \; \forall a \varepsilon S$

As δ is fuzzy ideal of S

$\delta(a) = \delta(xa^2y) \geq \delta(a^2) \geq \delta(a)$

So $\delta(a) = \delta(a^2)\ \forall\ a \in S$

So (1) $\Rightarrow$ (2)

Conversly, assume (2) hold

$J(a^2)$ be principal ideal generated by $a^2 \in S$

$\delta J(a^2)$ is fuzzy ideal of S

$a^2 \in J(a^2)$

and $\delta J(a^2)^{(a^2)} = \delta J(a^2)^{(a)} = 1$

$\Rightarrow a \in \{a^2\} \cup Sa^2 \cup a^2 S \cup Sa^2 S$

consider different cases.

1) If $a = a^2 \Rightarrow a = a^4 \Rightarrow a = a.a^2.a$

So $x = y = a$

2) If $a \in Sa^2$ i.e. $\exists\ u \in S$ s.t. $a = ua^2$

$a = u.a.a = uua^2a = u^2a^2a$

So $x = u^2$ and $y = a$

3) If $a \in a^2S\ \exists\ v \in S$

s.t. $a = a^2v = a.av = a.a^2v.v = a.a^2.v^2$

So $x = a$ and $y = v^2$

4) If $a \in Sa^2S$, $\exists\ x, y \in S$ such that $a = xa^2y$

So in each case $\exists\ x, y \in S$ such that $a = xa^2y$

So s is intra-regular

Theorem 10 :

Let S be an intra regular semigroup. Then for every fuzzy ideal δ of S

$\delta(ab) = \delta(ba)$ holds for all a and b in S.

Proof :

By previous theorem, as S is intraregular and δ is fuzzy bi-ideal, we have $\delta(a) = \delta(a^2)$ $a \varepsilon S$.

Let a and $b \varepsilon S = ab \varepsilon S$

$= (ab)^2 \varepsilon S$

By above theorem

$\delta(ab) = \delta(ab)^2 = \delta(a(ba)b) \geqslant \delta(ba)$ .....(1)

But $\delta(ba) = \delta((ba)^2) = \delta(b(ab)a) \geqslant \delta(ab)$ .....(2)

So from (1) and (2)

$\delta(ab) = \delta(ba)$ Proved

Definition 3.2.3 :

A semigroup S is called <u>completely</u> <u>regular</u> if for each element a of S, $\exists$ elements x in S such that a=axa and ax = xa.

Theorem 11 :

For a semigroup S, the following conditions are equivalent.

1) S is left regular.

2) For every fuzzy left ideal δ of S, $\delta(a) = \delta(a^2)$ holds $\forall$ $a \varepsilon S$.

Proof : Assume (1) holds

i.e. S is left regular So $a = xa^2$ $a \varepsilon S$

Let δ be any fuzzy left ideal of S

Then $\delta(a) = \delta(xa^2) \geqslant \delta(a^2) \geqslant \delta(a)$

So $\delta(a) = \delta(a^2)$

So (1) $\Rightarrow$ (2)

Conversly, Let (2) hold.

Let a be any element of S

$L(a^2)$ be principal left ideal generated by $a^2 \varepsilon S$

$L(a^2) = \{ a^2 \} \cup Sa^2$

$\delta L(a^2)$ is fuzzy left ideal

$\delta L(a^2)\ (a) = \delta L(a^2)\ (a^2) = 1$

So $a \varepsilon L(a^2)$

i.e. $a \varepsilon \{ a^2 \} \cup Sa^2$

If $a = a^2 \Rightarrow a = a^3 \Rightarrow a = a.a^2$ So $x = a$

If $a \in Sa^2$, $\exists$ $x \varepsilon S$ such that $a = xa^2$

So in any case, S is left regular.

Theorem 12 :

For a left (Right) regular semigroup S, the following conditions are equivalent.

1) S is left (right) duo.

2) S is fuzzy left (right) duo.

Proof :

Let (1) hold.

i.e. S is left duo

i.e. Every left ideal of S is a two sided ideal.

Let δ be any fuzzy left ideal of S

and a and b be any elements of S

Then as left ideal Sa^2 is two sided ideal of S

and since S is left regular

$ab \varepsilon (Sa^2)\ b \subseteq (Sa^2)\ S \subseteq Sa^2$

So $ab \varepsilon Sa^2$

$\Rightarrow \exists x \in S$ such that $ab = xa^2$

$\delta(ab) = \delta(xa^2) \geq \delta(a^2) \geq \delta(a)$

i.e. $\delta(ab) \geq \delta(a)$

So δ is fuzzy right ideal of S

So S is fuzzy left duo.

Conversly,

Let (2) hold

i.e. S is fuzzy left duo.

A be any left ideal of S.

characteristic function of A i.e. δA is fuzzy left ideal

So By assumption

δA is fuzzy two sided ideal of S

=> A is two sided ideal of S.

So S is left duo.

i.e. (2) => (1) Hence proved.

Definiton 3.2.4 :

Semigroup S is called <u>fuzzy left (right) simple</u> iff every fuzzy left (right) ideal of S is constant function. Semigroup S is called <u>fuzzy simple</u> iff every fuzzy ideal of S is a constant function.

Theorem 13 :

For a semigropup S, the following conditions are equivalent.

1) s is left simple

2) S is fuzzy left simple.

Proof : Let (1) hold.

Let δ be any fuzzy left ideal of S.

Let a and b be any elements of S.

Since S is left simple, $\exists$ elements x and y in S such that b = xa and a = yb.

{ For, semigroup S is left simple iff Sa = S a ε S

i.e. a,b ε S, x,y ε S such that b=xa and a=yb }

As S is fuzzy left ideal of S

$\delta(a) = \delta(yb) \geqslant \delta(b) = \delta(xa) \geqslant \delta(a)$

So $\delta(a) = \delta(b)$

i.e. δ is a constant function.

Conversly, Let (2) hold.

To prove that S is left simple i.e. There exists no proper left ideal of S.

Let **A** be proper left ideal.

Consider characteristic function δ_A of A defined as

$\delta_A(x) = 1$ if $x \varepsilon A$

$= 0$ if $x \not\varepsilon A$

δ_A is fuzzy left ideal, But it is not constant function, which is contradiction.

So our assumption is wrong.

There is no proper left ideal of S

i.e. S is left simple.

So (2) $\Rightarrow$ (1) Proved.

Theorem 14 :

Let S be a left simple semigroup, then every fuzzy bi-ideal of S is fuzzy right ideal of S.

Proof :

Let δ be fuzzy bi-ideal of S.

a and b be any elements if S. Now since S is left simple, $\exists$ an element x in S such that b =xa .

As δ is fuzzy bi-ideal of S

$\delta(ab) = \delta(axa) \geq \min\{\delta(a), \delta(a)\} = \delta(a)$

i.e. $\delta(ab) \geq \delta(a)$

So δ is fuzzy right ideal of S.

Theorem 15 :

Let S be left simple semigroup. then every bi-ideal of S is a right ideal of S.

Proof : Let S be a left simple semigroup.

Let A be any bi-ideal of S.

$\Rightarrow \delta A$ is fuzzy bi-ideal of S.

$\Rightarrow \delta A$ is fuzzy right ideal of S.

$\Rightarrow$ A is right ideal of S.

Definition 3.2.5 :

1) A subsemigroup A of a semigroup S is called Interior Ideal of S iff SAS $\subseteq$ A.

2) By I(x), we denote the principal interior ideal of S, generated by x $\in$ S

i.e. $I(x) = \{x, x^2\} \cup SxS$

3) A fuzzy subsemigroup δ of a semigroup S is called fuzzy interior ideal of S, if

$\delta(xay) \geq \delta(a) \; \forall \; x,a,y \in S$.

Theorem 16 :

Let A be any non empty subset of a semigroup S, then following conditions are equivalent.

1) A is an interior ideal of S.

2) Characteristic function δA of A is fuzzy interior ideal of S.

Proof : Let (1) hold.

Let x,a,y be any elements of S.

If $a \in A$, then as A is an interior ideal of S

$xay \in SAS \subseteq A$

So $\delta A\,(xay) = 1 = \delta A(a)$

If $a \notin A$

Then $\delta A\,(xay) \geq 0 = \delta A(a)$

Also δA is fuzzy subsemigroup of S

$\Rightarrow \delta A$ is fuzzy interior ideal of S.

Conversly Let (2) hold.

Let x and $y \in S$. $a \in A$.

Since $a \in A$ $\delta A\,(a) = 1$

$\delta A(xay) \geq \delta A(a) = 1$

$\Rightarrow$ $xay \in A$

$\Rightarrow SAS \subseteq A$

Also A is subsemigroup of S.

Hence A is an interior ideal of S.

So proved.

Remarks

1) Any ideal of semigroup S is an interior ideal of S.

Let S be ideal of semigroup S.

i.e. $SA \subseteq A$ and $AS \subseteq A$

Now $SA \subseteq A = SAS \subseteq AS \subseteq A$

So A is an interior ideal of S.

2) Any fuzzy ideal of S is fuzzy interior ideal of S.

Let δ be any fuzzy ideal of S.

i..e. $\delta(xy) \geqslant \delta(x)$ and $\delta(xy) \geqslant \delta(y) \; \forall \; x, y \in S$.

To prove that δ is fuzzy interior ideal of S.

1) δ is fuzzy subsemigroup of S.

2) $\delta(xay) \geqslant \delta(xa) \geqslant \delta(a) \; \forall \; x, a, y \in S$

i.e. $\delta(xay) \geqslant \delta(a) \; \forall \; x, a, y \in S$

$\Rightarrow \delta$ is fuzzy interior ideal of S.

Theorem 17 :

Let δ be any fuzzy set in a regular semigroup S then following conditions are equivalent.

1) δ is fuzzy ideal of S.

2) δ is fuzzy interior ideal of S.

Proof : Let (1) hold.

Let δ be fuzzy ideal of S.

i.e. $\delta(xy) \geqslant \delta(x)$ and $\delta(xy) \geqslant \delta(y) \; \forall \; x, y \in S$

δ is trivially fuzzy subsemigroup of S.

To prove that δ is fuzzy interior ideal of S

i.e. To prove that $\delta(xay) \geqslant \delta(a) \; \forall \; x, a, y \in S$.

Consider

$\delta(xay) \geq \delta(ay) \geq \delta(a)$

So (1) $\Rightarrow$ (2).

Conversly, Let (2) hold.

i.e. δ is fuzzy interior ideal of S.

Let a and b be any elements of S.

Then as S is regular , $\exists$ x and y ε S

Such that a = axa and b = by b

As δ is fuzzy interior ideal of S

$\delta(ab) = \delta((axa)b) = \delta((ax)ab) \geq \delta(a)$

$\delta(ab) = \delta(a(byb)) = \delta(ab(yb)) \geq \delta(b) \quad \forall \ a,b, \varepsilon S.$

$\Rightarrow \delta$ is fuzzy ideal of S.

So (2) $\Rightarrow$ (1) Proved.

Theorem 18 :

A semigroup S is simple iff it is fuzzy simple.

Proof : Let semigroup S be simple.

i.e. it has no proper ideal.

Let A be fuzzy ideal of S

Let $A_\alpha = \{ x \ \varepsilon \ S/A(x) \geq \alpha \}$

A_α is ideal $\forall \ \alpha \in [0,1]$

Let $\alpha_1 < \alpha_2$

$\Rightarrow A\alpha_2 \subseteq A \ \alpha_1$

i.e. $A \ \alpha_2$ is proper ideal of S.

This is contradiction.

so $\alpha_1 = \alpha_2$

Hence S is fuzzy simple.

Conversly Let (2) hold.

To prove that S is simple.

Let if possible, S is not simple.

So, ∃ proper ideal S' of S

Define δ : S' → [0,1] by

$\delta(x) = 1$ if $x \varepsilon S'$

$= 0$ if $x \varepsilon S'$

i.e. δ is characteristic function of S'

So δ is fuzzy ideal which is not constant

This is contradiction.

So our assumption is wrong. There is no proper ideal.

S is simple.

Theorem 19 :

For a regular semigroup S, following conditions are equivalent.

1) S is simple

2) Every fuzzy interior ideal of S is a constant function.

Proof : Let (1) hold.

i.e. S is simple.

i.e. S has no proper ideal.

i.e. only ideal of S is S itself.

Let δ be any fuzzy interior ideal of S.

a and b be any elements of S.

Consider $S' = \{ xby \,/\, x,y \varepsilon S \}$

Let $p = x'by'$ $q \varepsilon S$

$pq = x'\,by'\,q = (x'b)\,(y'q) \varepsilon S'$ as $y'q \varepsilon S$

Similarly $qp \varepsilon S'$.

So S' is an ideal of S.

But as S is simple, we have S = S'

So, ∃ elements x and y in S such that a = xby

As δ is fuzzy interior ideal of S

$\delta(a) = \delta(xby) \geq \delta(b)$

Similarly $\delta(b) \geq \delta(a)$

So $\delta(a) = \delta(b)$

So (1) ⇒ (2).

Conversly, Let (2) hold.

i.e. Every fuzzy interior ideal of S is constant function.

But every fuzzy interior ideal of S is fuzzy ideal of S.

So S is fuzzy simple.

⇒ S is simple.

Theorem 20 :

For a fuzzy set δ of an intra-regular semigroups S, the following conditions are equivalent.

1) δ is fuzzy ideal of S

2) δ is fuzzy interior ideal of S.

Proof : Let (1) hold.

i.e. δ is fuzzy ideal of S.

⇒ δ is fuzzy subsemigroup of S.

Let $x, a, y \in S$ be any elements.

$\delta(xay) \geq \delta(ay) \geq \delta(a)$ Proved.

Conversly, Let (2) hold.

Let a and b be any two elements of S.

As S is intra-regular, ∃ $x, y, u, v \in S$

Such that $a = xa^2y$ and $b = ub^2v$

Then as δ is fuzzy interior ideal of S

$\delta(ab) = \delta(xa^2yb) = \delta((xa)a(yb)) \geq \delta(a)$

$\delta(ab) = \delta(aub^2v) = \delta((au)b(bv)) \geq \delta(b)$

i.e. $\delta(ab) \geq \delta(a)$ and $\delta(ab) \geq \delta(b) \forall a, b \in S$.

So δ is fuzzy ideal of S.

Hence proved.

Section 3 :

Prime Fuzzy Ideals and Semiprimality.

Definitons 3.3.1 :

A subset A of a semigroup S is called <u>semiprime</u> if $a^2 \varepsilon A$, $a \varepsilon S \Rightarrow a \varepsilon A$.

A fuzzy set δ in a semigroup S is called <u>fuzzy semiprime</u> if $\delta(a) \geq \delta(a^2) \forall a \varepsilon S$

A fuzzy ideal δ in a semigroup S is called <u>semiprime fuzzy ideal</u> if $\delta(a) = \delta(a^2) \forall a \varepsilon S$.

Theorem 21 :

For a non empty subset A of a semigroup S the following conditions are equivalent.

1) A is semiprime.

2) Characteristic function δA of A is fuzzy semiprime.

Proof : Let (1) hold.

i.e. A is semiprime.

i.e. $a^2 \varepsilon A$, $a \varepsilon S \Rightarrow a \varepsilon A$.

As $a \varepsilon A$, $\delta A(a) = 1 = \delta A(a^2)$

if $a^2 \not\varepsilon A$ we have

$\delta A(a) \geq 0 = \delta A(a^2)$

So we have $\delta A(a) \geq \delta A(a^2) \forall a \varepsilon S$

So (1) $\Rightarrow$ (2)

i.e. δA is fuzzy semiprime. So (1) $\Rightarrow$ (2) conversly

Let (2) hold.

i.e. δA is fuzzy semiprime.

i.e. $\delta A(a) \geq \delta A(a^2)$

Let $a^2 \in A$, $a \in S$

as δA is fuzzy semiprime

$\delta A(a) \geq \delta A(a^2) = 1$

$\Rightarrow \delta A(a) = 1 \Rightarrow a \in A \Rightarrow A$ is semiprime proved.

Theorem 22 :

For any fuzzy subsemigroup δ of a semigroup S the following conditions are equivalent.

1) δ is fuzzy semiprime

2) $\delta(a) = \delta(a^2) \forall a \in S$

Proof :

Clearly (2) $\Rightarrow$ (1)

As $\delta(a) = \delta(a^2) \forall a \in S$

$\Rightarrow \delta$ is fuzzy semiprime.

Conversly let (1) hold.

i.e. δ is fuzzy semiprime.

Let $a \in S$ be any element.

As δ is fuzzy subsemigroup of S, we have

$\delta(a) \geq \delta(a^2) \geq \min\{\delta(a), \delta(a)\} = \delta(a)$

So $\delta(a^2) = \delta(a)$

So (1) $\Rightarrow$ (2)

Theorem 23 :

For a semigroup S, following conditions are equivalent.

1) S is left regular.

2) Every fuzzy left ideal of S is fuzzy semiprime.

Proof :

Let (1) hold.

i.e. S is left regular i.e. $\forall$ aε S, $\exists$ xε S

suth that $a = xa^2$

Let δ be fuzzy left ideal of S and a be any element of S

$\delta(a) = \delta(xa^2) \geq \delta(a^2)$ {δ is left ideal}

So $\delta(a) \geq \delta(a^2)$

i.e. δ is fuzzy semiprime.

So (1) $\Rightarrow$ (2)

Conversly, Let (2) hold

i.e. every fuzzy left ideal of S is fuzzy semiprime.

To prove that S is left regular.

$L(a^2) = \{a^2\} \cup Sa^2$ is principal left ideal generated by $a^2 \varepsilon S$.

Then characteristic function of $L(a^2)$ i.e.

$\delta L(a^2)$ is fuzzy left ideal.

$\delta L(a^2)$ is fuzzy semiprime.

So $\delta L(a^2)^{(a)} \geq \delta L(a^2)^{(a^2)} = 1$

So $\delta L(a^2)^{(a)} = 1$

i.e $a \varepsilon L(a^2)$

i.e. $a \varepsilon \{a^2\} \cup Sa^2$

If $a = a^2 \Rightarrow a = a^3 = a.a^2$

If $a \varepsilon Sa^2$, $\exists x \varepsilon S$ such that $a = xa^2$

So in any case, S is left regular.

So (2) $\Rightarrow$ (1) Proved.

Definition 3.3.2

Prime Fuzzy Ideal

A fuzzy ideal δ of a semigroup S is called prime fuzzy ideal if $\delta(xy) = \delta(x)$ or $\delta(y) \forall x,y \in S$.

Theorem 24 :

Non empty subset A of a semigroup S is prime ideal iff characteristic function δA of A is prime fuzzy ideal.

Proof : Let A be prime ideal.

Then δA is fuzzy ideal

To show that δA is prime fuzzy ideal.

Let $\delta A(xy) = 0$ i.e. $xy \notin A \Rightarrow x \notin A, y \notin A$

i.e. $\delta A(x) = 0$ and $\delta A(y) = 0$

If $\delta A(xy) = 1 \Rightarrow xy \in A$

$\Rightarrow x \in A$ or $y \in A$

$\Rightarrow \delta A(x)=1$ or $\delta A(y) = 1$

So $\delta A(xy) = \delta A(x)$ or $\delta A(y)$

So δA is prime fuzzy ideal.

Conversly if δA is prime fuzzy ideal

To show that A is prime ideal.

As δA is fuzzy ideal $\Rightarrow$ A is ideal.

Let x and $y \in S \Rightarrow xy \in A$

So $\delta A(xy) = 1 = \delta A(x)$ or $\delta A(y)$

i.e. $\delta A(x)=1$ or $\delta A(y) = 1$

i.e. $x \in A$ or $y \in A$.

So A is prime ideal.

Theorem 25 :

If A is prime fuzzy ideal then $A\alpha$ is prime ideal

Proof :

Let A be prime fuzzy ideal.

$A_\alpha = \{ x \in S / A(x) \geqslant \alpha \}$

To prove that A_α is prime ideal.

Let x and $y \in S$ be such that $xy \in A_\alpha$

i.e. $A(xy) \geqslant \alpha$

But A is prime fuzzy ideal

So $A(xy) = A(x)$ or $A(y)$

i.e. $A(x) \geqslant \alpha$ or $A(y) \geqslant \alpha$

i.e. $x \in A_\alpha$ or $y \in A_\alpha$

So A_α is prime ideal.

Theorem 26 :

Let P be fuzzy ideal of semigroup S and every α cut of P is prime ideal of S, then P is prime fuzzy ideal.

Proof : As P is fuzzy ideal of S

As $P(xy) = \max \{ P(x), P(y)\} \; \forall \; x,y \in S$

$P_\alpha \{ x \quad S/P(x) \geqslant \alpha \}$ is prime ideal $\forall \; \alpha \in [0,1]$

To prove that P is prime fuzzy ideal

Let $P(xy) = \alpha \Rightarrow xy \in P_\alpha$

$\Rightarrow x \in P_\alpha$ or $y \in P_\alpha$ { As P_α is prime ideal}

i.e. $P(x) \geqslant P(xy)$

or $P(y) \geqslant P(xy)$

But as P is ideal of S

$P(xy) \geqslant P(x) \vee P(y)$

So $P(xy) = P(x)$ or $P(y)$

i.e. P is prime fuzzy ideal of S.

Theorem 27 :

Let P_1 and P_2 be two fuzzy semiprime ideals of S then $P_1 \cap P_2$ is fuzzy semiprime ideal

Let P_1 and P_2 be two fuzzy semiprime ideals of S

i.e. $P_1(x^2) = P_1(x)$ and $P_2(x^2) = P_2(x) \forall x \varepsilon S$

To prove that $P_1 \cap P_2$ is fuzzy semiprime ideal.

$P_1 \cap P_2$ is ideal trivially.

Now $(P_1 \cap P_2)(x^2) = \min\{P_1(x^2), P_2(x^2)\}$

$= \min\{(P_1(x), P_2(x)\}$

$= (P_1 \cap P_2)(x)$

So $(P_1 \cap P_2)(x^2) = (P_1 \cap P_2)(x)$

So $(P_1 \cap P_2)$ is fuzzy semiprime ideal.

Theorem 28 :

Let P_1 and P_2 be two fuzzy prime ideals of a semigroup S then $P_1 \cap P_2$ is fuzzy semiprime ideal of S.

Proof :

Let P_1 and P_2 be two prime fuzzy ideals of S

i.e. $P_1(xy) = P_1(x)$ or $P_1(y) \forall x,y \varepsilon S$

$P_2(xy) = P_2(x)$ or $P_2(y) \forall x,y \varepsilon S$

As P_1 and P_2 are fuzzy ideals of S

$P_1 \cap P_2$ is fuzzy ideal of S.

Now

$$(P_1 \cap P_2)\ (x^2) = \min\{P_1(x^2)\ ,\ P_2(x^2)\}$$
$$= \min\{\ P_1(xx),\ P_2(xx)\ \}$$
$$= \min\{\ P_1(x),\ P_2(x)\ \}$$

as P_1 and P_2 are prime fuzzy ideal of S.

$$= (P_1 \cap P_2)\ (x)$$

So $(P_1 \cap P_2)(x^2) = (P_1 \cap P_2)(x)$

$= (P_1 \cap P_2)$ is fuzzy semiprime ideal of S.

Theorem 29 :

For a semigroup S, following hold

1) Let : $S \to [0,1]$ be a fuzzy ideal. δ is semiprime. iff its level cuts $\delta\alpha = \{x \varepsilon\ S/\delta\ (x) \geq \alpha\}$ are semiprime ideals of S. $\forall\ \alpha\ \varepsilon\ [0,1]$

2) Let S' be an ideal of S. S' semiprime iff its characteristic function δ S' is semiprime fuzzy ideal of S.

3) Let S and S' be two semigroups. f : $S \to S'$ be a homomorphism. If $\delta : S' \to [0,1]$ is semiprime fuzzy ideals of S' then $f^{-1}(\delta)$ is a semiprime fuzzy ideal of S.

4) Every prime fuzzy ideal of S is a semiprime fuzzy ideal of S.

Proof :

1) If each level cut $\delta\alpha = \{\ x\ \varepsilon S/\delta\ (x) \geq \alpha\ \}$ of a fuzzy ideal δ of S is semiprime and $x \in S$, choose $\alpha = \delta\ (x^2)$

Then $x \varepsilon \delta\alpha$ and hence $x \varepsilon \delta\alpha$

Therefore $\delta(x) \geq \alpha = \delta(x^2) \geq \delta(x)$

i.e. $\delta(x) = \delta(x^2)$

$\Rightarrow \delta$ is fuzzy semiprime ideal.

Conversly

Let δ be fuzzy semiprime ideal of S

To prove that $\delta\alpha = \{ x \varepsilon S / \delta(x) \geq \alpha \}$ is semiprime ideal.

It follows by the fact that ideal A of S is semiprime

iff $x^2 \varepsilon A \Rightarrow x \varepsilon A$.

2) Let S' be an ideal of S

Let S' be semiprime ideal of S

To prove that $\delta S'$ is fuzzy semiprime ideal of S

Let $x \varepsilon S$ be any element.

As S' is semiprime ideal of S

$\Leftrightarrow x^2 \varepsilon S' \Rightarrow x \varepsilon S'$

So $\delta S'(x) \geq \delta S'(x^2)$

But as $\delta(S')$ is fuzzy ideal

$\delta S'(x^2) \geq \delta S'(x)$

So $\delta S'(x^2) = \delta S'(x)$

i.e. $\delta S'$ is fuzzy semiprime ideal.

Conversly

Let $\delta S'$ be fuzzy semiprime ideal

i.e. $\delta S'(x^2) = \delta S'(x) \forall x \varepsilon S$

i.e. $x^2 \varepsilon S' \Rightarrow x \varepsilon S'$

So S' is semiprime ideal.

3) Let S and S' be two semigroups.

$f : S \to S'$ be homomorphism.

Let $\delta : S' \to [0,1]$ be semiprime fuzzy ideal of S'

Then Let $x \in S$

$f^{-1}(\delta)(x^2) = \delta\,(f(x^2)$

$= \delta\,((f(x)^2)$

$= \delta\,(f(x))$

$= f^{-1}(\delta)\,(x)$

Then $f^{-1}(\delta)$ is semiprime

4) Let P be any prime fuzzy ideal of S

i.e. $P(xy) = P(x)$ or $(P(y)\ \forall\ x,y \in S$

To prove that P is fuzzy semiprime ideal of S

Let $x \in S$

Then we have

$P(x^2) = P\,(x.x)$

$= P(x)$ or $P(x)$

i.e. $P(x^2) = P(x)$

So P is fuzzy semiprime ideal of S.

Theorem 30 :

If $\delta : S \to [0,1]$ is fuzzy ideal of semigroup S then following are equivalent.

1) δ is semiprime

2) $\delta\,(x^n) = \delta\,(x)\ \forall\ n > 0$ and $x \in S$

Proof :

(2) $\Rightarrow$ (1) is obvious.

Conversly let (1) hold.

To prove that $\delta(x^n) = \delta(x) \forall n > 0$, $x \varepsilon S$

We prove this result by induction.

As δ is semiprime.

$\delta(x^2) = \delta(x) \forall x \varepsilon S$

So clearly result is true for n = 2.

Let k > 2, be any integer.

Let $\delta(x^n) = \delta(x)$ hold $\forall x \varepsilon S$ and $\forall$ n such that

$1 \leq n \leq k$

Claim : $\delta(x^{k+1}) = \delta(x)$

Case 1 :

If k is odd Let. k = 2m + 1

Then $\delta(x^{k+1}) = \delta(x^{2m+2})$

$= \delta((x^{m+1})^2)$

$= \delta(x^{m+1})$

Since m+1 < k, By induction hypothesis $\delta(x^{m+1}) = \delta(x)$

Case 2 :

If k is even Let k = 2m

Then by induction hypothesis

$\delta(x) \leq \delta(x^{k+1}) = \delta(x^{2m+1}) \leq \delta(x^{2m+2})$

$= \delta(x^{m+1}) = \delta(x)$

So in any case

$\delta(x^{k+1}) = \delta(x)$

Hence by induction result is proved.

Theorem 31 :

Let S be a semigroup

If $\delta : S \to [0,1]$ is a fuzzy ideal. then following are equivalent.

a) δ is semiprime

b) $A^2 \subseteq \delta \Rightarrow A \subseteq \delta$ for all fuzzy ideals $A : S \to [0,1]$

Proof :

$A^2 \subseteq \delta = A^2(x^2) \leq \delta(x^2) = \delta(x) \forall x \in S$

But $A^2(x^2) = \max_{x^2 = yz}\{ \min(A(y), A(z))\}$

$\geq A(x) \forall x \in S$

Hence $A \leq \delta$

{ As $A(x) \leq A^2(x^2) \leq \delta(x)$ }

Conversly

Let $x \in S$ and $\delta(x^2) = \alpha$

and $\langle x^2 \rangle$ be principal ideal of S generated by x^2

Then $x^2 \in \delta\alpha$

and hence $\langle x^2 \rangle \subseteq \delta\alpha$

Let A be fuzzy ideal defined as follows

$A(z) = \alpha$ if $z \in \langle x \rangle$

$= 0$ otherwise

Then

$A^2(z) = \max_{z=uv} \{ \min(A(u), A(v)) \} = \alpha$ if $z \in \langle x^2 \rangle$

$= 0$ if $z \notin \langle x^2 \rangle$

Therefore, $A^2 \subseteq \delta$ and hence by hypothesis $A \subseteq \delta$

But then

$\alpha = A(x) \leq \delta(x) \leq \delta(x^2) = \alpha$

So $\delta(x^2) = \delta(x)$

i.e. δ is fuzzy semiprime ideal of S proved.

Theorem 32 :

Let S be semigroup. $\delta : S \rightarrow [0,1]$ be a fuzzy ideal

Then following are equivalent

1) $A^2 \subseteq \delta \Rightarrow A \subseteq \delta$ for all fuzzy ideals $A:S \rightarrow [0,1]$

2) $A^n \subseteq \delta$ $n > 0$ = $A \subseteq \delta$ for all fuzzy ideals $A:S \rightarrow [0,1]$

Proof :

(2) $\Rightarrow$ (1) obvious

(1) $\Rightarrow$ (2)

We prove this result by induction.

Clearly result is true for n = 2.

Let k > 2 be any integer and let the result hold for each integer n, $1 \leq n \leq k$

Claim : $A^{k+1} \subseteq \delta = A \subseteq \delta$ and So (2) will be proved

If k is odd Let k = 2m + 1

Then $A^{k+1} = A^{2m+1+1} = A^{2(m+1)} = (A^{(m+1)})^2$

If k is even Let k = 2m

Then $A^{k+1} = A^{2m+1} \supseteq A^{2m+2} = (A^{m+1})^2$

So in any case

If $A^{k+1} \subseteq \delta$ then $A^{m+1} \subseteq \delta$

Since $m + 1 \leq k$

By induction hypothesis we get $A \subseteq \delta$.

Section 4 :

Fuzzy Indeals and Green's relations

We recall that

Green's relations are equivalence relations

R, L, J H, D defined for all $a,b \in S$

$a \, L \, b \Leftrightarrow S'a = S'b$

$a \, R \, b \Leftrightarrow aS' = bS'$

$a \, J \, b \Leftrightarrow S'aS' = S'bS'$

$H = L \cap R$

$J = L \vee R$

We shall also consider the relations

$\leq (R)$ $\leq (L)$, $\leq (J)$ defined $\forall \, a,b \in S$ by

$a \leq (R) \, b \Leftrightarrow aS' \subseteq bS'$

$a \leq (L) \, b \Leftrightarrow S'a \subseteq S'b$

$a \leq (J) \, b \Leftrightarrow S'aS' \subseteq S'bS'$

Note that X_I denotes characteristic function of I.

Proposition 33 :

If a and b are elements of semigroup S Then following are equivalent

(1) $a \leq (R) \, b$

(2) $a \in bS'$

(3) $X_B(a) = 1$ where $B = bS'$

(4) $X_I(a) \geq X_I(b)$ for all principal right ideals I of S.

(5) $X_I(a) \geq X_I(b)$ for all right ideals I of S

(6) $\delta(a) \geq \delta(b)$ for all fuzzy right ideals δ of S

Proof :

(6) $\Rightarrow$ (5)

Let $\delta(a) \geq \delta(b)$ $\forall$ fuzzy right ideals δ of S

Let I be any right ideal of S

X_I is fuzzy right ideal of S

So $X_I(a) \geq X_I(b)$

So (6) $\Rightarrow$ (5) Proved.

(5) $\Rightarrow$ (4)

As every principal right ideal I of S is right ideal of S, proof follows.

(4) $\Rightarrow$ (3)

Let $B = bS'$

The B is principal right ideal generated by $b \in S$

Now by (4)

$X_B(a) \geq X_B(b)$ .....(A)

But $b \in B = bS'$

$\Rightarrow X_B(b) = 1.$

So (A) $\Rightarrow X_B(a) = 1$

(3) $\Rightarrow$ (2)

Let $X_B(a) = 1$

$\Rightarrow a \in B = bS'$

(2) $\Rightarrow$ (1)

$a \in bS'$

So $aS' \subseteq bS'S'$

$= aS' \subseteq bS'$

$= a \leq (R)\ b$

(1) $\Rightarrow$ (2)

$a \leq (R)\ b \Rightarrow aS' \subseteq bS'$ ....(B)

as $a \in aS'$

So (B) $\Rightarrow a \in bS'$

So (1) $\Rightarrow$ (2)

(2) $\Rightarrow$ (5)

$a \in bS'$

and let I be right ideal of S

If $X_I\ (b) = 1$

then $b \in I$ So $a \in bS' \subseteq I$

giving $X_I\ (a) = 1$

Hence $X_I\ (a) \geq X_I\ (b)$

(5) $\Rightarrow$ (6)

Assume (5) holds

Let δ be any fuzzy right ideal

Then δ is convex combination of characteristics function of right ideals [from Fuzzy Ideals in semigroup by Mclean, Kummer]

$\Rightarrow$ For given $\varepsilon > 0$, $\exists\ \theta$ such that

$\theta\ (a) > \theta(b)$ by (5)

Such that $0 < \delta\ (x) - \theta(x) < \varepsilon\ \forall\ x \in S$

$\not\Rightarrow \delta\ (x) = \theta\ (x)$

$\Rightarrow \delta(a) > \delta\ (b)$

Proof over.

Proposition 34 :

Let δ be a map from semigroup S in [0,1]

Then δ is fuzzy right ideal iff for every x and y ε S

$x \leqslant (R)\ y \ = \delta(x) \geqslant \delta(y)$

If δ is fuzzy right ideal then δ is constant on R-classes

Proof :

Let δ be a right ideal of S

and $x \leqslant (R)\ y$ i.e. $xS' \subseteq yS'$

i.e. $\{x\} \cup \{xp/p \varepsilon s\} \subseteq \{y\} \cup \{yq/q \in S\}$

So $x = yr$ for some $r \varepsilon S$

Hence $\delta(x) = \delta(yr) \geqslant \delta(y)$

Conversly

Let $xS' \subseteq yS' \Rightarrow \delta(x) > \delta(y)$

Let a and b ε S $\Rightarrow$ b and ab ε S

Now $abS' \subseteq aS' \Rightarrow \delta(ab) > \delta(a)$

So δ is fuzzy right ideal

Let R_a be the R class of a

i.e. $R_a = \{ b \varepsilon S / a R b \}$

i.e. a R_b and b R_a

i.e. $a < (R) b \ = \delta(a) > \delta(b)$

and $b < (R) a \ = \delta(b) > \delta(a)$

i.e. $\delta(a) = \delta(b)\ \forall\ a,b \ \varepsilon\ S$

So δ is constant.

Corollary 35 :

If a and b are elements of a semigroup S then following conditions are equivalent

1) a R b

2) X_A (b) = 1 = X_B(a), A = aS' B= bS'

3) X_I (a) = X_I (b) $\forall$ Principal right ideals I of S

4) X_I (a) = X_I (b) $\forall$ right ideals I of S

5) δ(a) = δ (b) $\forall$ fuzzy right ideals δ of S

Proof :

(5) $\Rightarrow$ (4)

Let δ (a) = δ(b) fuzzy right ideals δ of S

Let I be any right ideal of S

Then X_I is fuzzy right ideal of S

$\Rightarrow$ X_I (a) = X_I (b)

(4) $\Rightarrow$ (5)

Let X_I (a) = X_I (b) $\forall$ right ideals I of S

$\Rightarrow$ X_I (a) = X_I (b) $\forall$ Principal right ideals I of S

(3) $\Rightarrow$ (2)

Let X_I (a) = X_I (b) Principal right ideals I of S

Let A = aS' and B = bS'

be principal right ideals of S

1 = X_A (a) = X_A(b)

and X_B(a) = X_B (b) = 1

So X_A (b) = 1 = X_B(a)

(2) $\Rightarrow$ (1)

A = aS' and B = bS'

X_A(b) = 1 = X_B(a)

X_B (a) = 1 $\Rightarrow$ a ε bS'

$\Rightarrow$ aS' $\subseteq$ bS'S' = bS'

$\Rightarrow$ aS' $\subseteq$ bS' .....(C)

$bS' \subseteq aS'S' = aS'$

$bS' \subseteq aS'$ .....(D)

From (C) and (D) we get $aS' = bS' \Rightarrow a\,R\,b$

(1) $\Rightarrow$ (2)

Let $a\,^{R}b$

i.e. $aS' = bS'$

Now $a \in aS' = bS' \Rightarrow X_{bS'}(a) = 1$

$b \in bS' = aS' \Rightarrow X_{aS'}(b) = 1$

(2) $\Rightarrow$ (4) Given that $X_{aS'}(b) = 1$ and $X_{bS'}(a) = 1$

Let I be any right ideal of S

If $X_I(b) = 1$

then $b \in I$ So $a \in bS' \subseteq I$

giving $X_I(a) = 1$

Hence $X_I(a) \geq X_I(b)$

Similarly it can be proved that

$X_I(b) \geq X_I(a)$

So $X_I(b) = X_I(a)$

(4) $\Rightarrow$ (5) Let δ be any fuzzy right ideal

$= \delta$ is convex combination of characteristic functions of right ideals

$\Rightarrow$ For given $\varepsilon > 0 \ \exists \ \theta$ such that

$\theta(a) = \theta(b)$ by (4)

Such that $0 \leq \delta(x) - \theta(x) < \varepsilon \ \forall x \in S$

$\Rightarrow \delta(x) = \theta(x)$

$\Rightarrow \delta(a) = \delta(b)$ Proved.