FUZZY SETS AND OPERATIONS ON FUZZY SETS

**Concept of Fuzzy set :**

So many times we come across the situations where we define certain things vaguely. For example, instead of describing weather today in terms of the exact percentage of cloud cover , we could just say that it is sunny, which is more uncertain and less precise but more useful. In fact, it is imporrtant to realize that the imprecision or vagueness that is characteristic of natural language does not necessarily imply a loss of accuracy or meaingfulness. For example it is more accurate to say that it is usually warm in summer than to say that it is usually $45^{o}C$ in summer.

In order for a term such as SUNNY to accomplish the desired introduction of vagueness, however, we can not use it to mean precisely 0% cloud cover. It's meaning is not totally arbitrary. Definitely cloud cover of 100% is not sunny. Not only that if there is 80% cloud cover it is not sunny. We can accept certain intermediate states, such as 10 or 20% cloud cover is sunny. But where we draw a line ? If, for instance any cloud cover of 25% or less is considered as sunny, does this mean that cloud cover of 26% is not ? This is definitely unacceptable since 1% cloud cover hardly makes any difference. We could, therefore, add a qualification that any amount of cloud cover 1% greater than a cloud cover already considered to be sunny (i.e. 25% or less) will also be labeled as sunny. But, then this definition allows us to accept all degrees of cloud cover sunny, no matter how gloomy the weather looks . In order to resolve this gloomy situation, the term sunny may introduce vagueness by allowing some sort of gradual transition from degrees of cloud cover that are considered to be sunny and those that are not. This is, infact, precisely the basic concept of fuzzy set, a concept that is both simple and intuatively pleasing and it forms, a generalizattion of the classical or crisp set.

## 2 CRISP SET and Fuzzy SET

Crisp set has sharp boundaries. Elements of given universe of discourse are divided into two groups. Members (those that certainly belong to the set) and nonmembers (those that certainly do not). Many of the collections we think of do not exhibit this property. For example, when we think of collections such as class of tall people, expensive cars, Beautiful women, numbers much greater than 1, or sunny days, the characteristic of having sharp boundary is not seen. Instead their boundaries seem vague and transition from member to nonmember appears gradual rather than abrupt.

Fuzzy set introduces vagueness by eliminating the sharp boundary dividing members of the class from non members.

A Fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set.

This grade corrosponds to the degree to which that individual is similar or compatible with the concept represented by fuzzy set.

Thus, individuals may belong to a fuzzy set to a greater or lesser degree as indicated by a larger or smaller membership grade. These membership grades are very often represeented by real number values ranging in closed interval between 0 and 1.

Thus, a fuzzy set representing our concept of sunny might assign a degree of membership 1 to cloud cover at 0%, 0.4 to cloud cover of 30%, 0 to cloud cover of 75%.

Because full membership and full non-membership in the fuzzy set can still be indicated by the values of 1 and 0, respectively. We can consider the crisp set to be restricted case of more general fuzzy set for which only these two grades of membership are allowed.

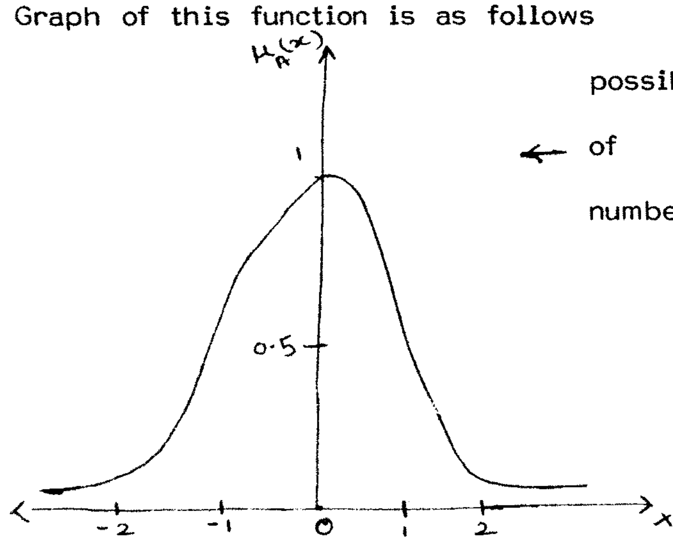
Let X denote a universal set. Then the membership function $\delta A$ by which fuzzy set A by usually defined has the form

$$\delta A : X \rightarrow [0,1]$$

For example, let A be a fuzzy set of real numbers closed to zero, then membership function can be defined as

$$\delta A\ (x) = \frac{1}{1 + 10\ x^2}$$

Graph of this function is as follows




possible membership function of fuzzy set of real numbers close to zero.

## 3 REPRESENTATION OF FUZZY SET :

To simplufy the representation of fuzzy set, following notation is used.

A non fuzzy finite set such as $X = \{x_1, x_2, \ldots x_n\}$ is expressed as

$$X = \sum_{j=1} x_j = x_1 + x_2 + x_3 \ldots + x_n$$

with the understanding that this is a representation of X as union of its constituent singltons, with the plus sign (+) playing role of "union"

Thus $x_j + x_k = x_k + x_j$ for $j,k = 1,2,3\ldots n$

and $x_j + x_j = x_j$

As a simple extension of this notation, a finite fuzzy set A on X is expressed as

$$A = \delta A(x_1)/x_1 + \delta A(x_2)/x_2 + \ldots + \delta A(x_n)/x_n$$

when x is not finite, we can use the notation

$$A = \int_X \delta A(x)/x$$

For example, In the universe of discourse comprising N= {positive integers} , the fuzzy set A, labelled "integers approximately equal to 5" may be defined as

$$A = 0.1/2 + 0.4/3 + 0.9/4 + 1.0/5 + 0.9/6 + 0.4/7 + 0.1/8$$

It is assumed here that remaining elements of N have grade of membership zero.

**4 OPERATIONS ON FUZZY SETS :**

1) Let A be a fuzzy subset of universal set X.

The support of A is the set of points in X at which $\delta A(x) > 0$. The height of A is the supremum of $\delta A(x)$ over X. crossover point of A is the point in X whose grade of membership in A is 0.5. We say that A is Normal if its height is 1 ; otherwise it is subnormal

For example

Let the universe be interval [0,120] with X interpreted as age. A fuzzy subset A of X labelled "old" may be defined by a grade of membership function such as

$$\delta A(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq 40 \\ \left(1 + \left(\frac{x-40}{5}\right)^{-2}\right)^{-1} & 40 < x < 120 \end{cases}$$

In this example, the support of "old" is the interval (40,120], Height of "old" is 1, and the cross over point of "old" is 45.

2) Two fuzzy sets A and B are said to be equal (A=B) iff $\delta A(x) = \delta B(x) \ \forall \ x \in X$.

3) A is contained in B ($A \subseteq B$) iff $\delta A(x) \leq \delta B(x)$

4) The union of fuzzy sets A and B is denoted by $A \cup B$ and is defined by

$$\delta A \cup B^{(x)} = \delta A(x) \vee \delta B(x) \quad \forall \; x \in X$$

where $\vee$ is the symbol for max.

5) The intersection of A and B is denoted by $A \cap B$ and is defined by

$$\delta(A \cap B)\ (x) = \delta A(x) \wedge \delta B(x) \; \forall \; x \in X$$

where $\wedge$ is the symbol for min.

6) The <u>complement</u> of A is denoted by $\bar{A}$ and is defined by

$$\delta \bar{A}\ (x) = 1 - \delta A\ (x) \quad \forall \; x \in X.$$

7) The <u>product</u> of A and B is denoted by AB and is defined by

$$\delta AB\ (x) = \delta A(x).\ \delta B(x) \; \forall \; x \in X$$

8) The <u>bounded</u> <u>sum</u> of A and B is denoted by $A \oplus B$ and is defined by

$$\delta(A \oplus B\ )\ (x) = 1 \wedge \delta A(x) + \delta B(x) \forall \; x \in X$$

a) If $\alpha$ is any non negative real number such that $\alpha \bullet \sup \delta A\ (x) < 1$ then

$$\delta_{\alpha A}(x) = \alpha \delta\ A\ (x) \quad \forall \; x \in X$$

10) As a special case, the operation of <u>CONCENTRATION</u> can be defined as

$$CON\ (A) = A^2$$

while that of <u>dilation</u> can be expressed as

$$DIL(A) = A^{0.5}$$

11) The <u>bounded</u> <u>difference</u> of A and B is denoted by $A \ominus B$ and is defined by

$$\delta_{A \ominus B}(x) = 0 \vee \delta_A(x) - \delta_B(x) \; \forall \; x \in X$$

12) The left square of A is denoted by $^2A$ and is defined by

$$^2A = \int_V \delta_A(x)/x^2$$

where $V = \{ x^2 / x \in X \}$

For example

Let $X = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$A = 0.8/3 + 1/5 + 0.6/6$

$B = 0.7/3 + 1/4 + 0.5/6$

then

$A \cup B = 0.8/3 + 1/4 + 1/5 + 0.6/6$

$A \cap B = 0.7/3 + 0.5/6$

$\bar{A} = 1/1 + 1/2 + 0.2/3 + 1/4 + 0.4/6 + 1/7$

$AB = 0.56/3 + 0.3/6$

$A^2 = 0.64/3 + 1/5 + 0.36/6$

$0.5A = 0.4/3 + 0.5/5 + 0.3/6$

$CON(B) = 0.49/3 + 1/4 + 0.25/6$

$DIL(B) = 0.84/3 + 1/4 + 0.7/6$

$A \oplus B = 1/3 + 1/4 + 1/5 + 1/6$

$A \ominus B = 0.1/3 + 1/5 + 0.1/6$

$^2A = 0.8/9 + 1/25 + 0.6/36$

$^3A = 0.8/27 + 1/125 + 0.6/216$

13) If $A_1$, $A_2$, ....$A_k$ are fuzzy subsets of $X_1$, $X_2$, ...$X_k$, respectively, the <u>Cartesian</u> <u>product</u> of $A_1$, ...$A_k$ is defined as a fuzzy subset of $X_1 \times X_2 \times x - - - \times X_k$ whose membership function is expressed by

$$\delta A_1 \times A_2 \times - - - \times A_k(x_1, x_2, - - - x_k)$$

$$= \delta A_1(x_1) \wedge \delta A_2(x_2) \wedge - - - \wedge \delta A_k(x_k)$$

$$(x_1 - - x_k) \varepsilon X_1 \times X_2 \times - - - \times X_k$$

For example

Let

$X_1 = \{1, 2, 3\}$

$X_2 = \{2, 4\}$

$A_1 = 0.5/1 + 1/2 + 0.6/3$

$A_2 = 1/2 + 0.6/4$

then

$$A_1 \times A_2 = 0.5/(1,2) + 0.5/(1,4) + 1/(2,2)$$
$$+ 0.6/(2,4) + 0.6/(3,2) + 0.6/(3,4)$$